Dear Peter:

This will elaborate further in reply to your queries. Let's start with the identity I spoke about on my blog, namely:

$$\tfrac{1}{2}A^{\sigma\tau}\left(B_{\tau\sigma;\nu}+B_{\sigma\nu;\tau}+B_{\nu\tau;\sigma}\right)-{}^*A_{\nu\sigma}\,{}^*B^{\tau\sigma}{}_{;\tau}=0\,, \qquad (1)$$

This is an amazing little identity, and one can go in a number of different directions with this. Let's take one.

Keeping in mind that for second-rank duality, $**=-1$, Let's set $A\to F$ and $B\to F$. Let's separately set $A\to {}^*F$ and $B\to {}^*F$. This yields:

$$\tfrac{1}{2}F^{\sigma\tau}\left(F_{\tau\sigma;\nu}+F_{\sigma\nu;\tau}+F_{\nu\tau;\sigma}\right)-{}^*F_{\nu\sigma}\,{}^*F^{\tau\sigma}{}_{;\tau}=0\,, \qquad (2)$$

$$\tfrac{1}{2}{}^*F^{\sigma\tau}\left({}^*F_{\tau\sigma;\nu}+{}^*F_{\sigma\nu;\tau}+{}^*F_{\nu\tau;\sigma}\right)-F_{\nu\sigma}F^{\tau\sigma}{}_{;\tau}=0\,, \qquad (3)$$

One can also set up "duality-mixed" identities by setting $A\to {}^*F$; $B\to F$ as well as $A\to F$ and $B\to {}^*F$, but we'll not look at those yet.

Now, let's introduce the Abelian gauge potential $F=dA$ in forms language, or:

$$F_{\tau\sigma}=\partial_\tau A_\sigma-\partial_\sigma A_\tau\equiv\partial_{[\tau}A_{\sigma]} \qquad (4)$$

written full out. The dual of this is defined as:

$${}^*F_{\tau\sigma}=\tfrac{1}{2}\varepsilon_{\tau\sigma\alpha\beta}F^{\alpha\beta}=\tfrac{1}{2}\varepsilon_{\tau\sigma\alpha\beta}\left(\partial^\alpha A^\beta-\partial^\beta A^\alpha\right). \qquad (5)$$

Now, let's do some simple calculation. Substitute (4) into (2) to get:

$$\tfrac{1}{2}F^{\sigma\tau}\left(F_{\tau\sigma;\nu}+F_{\sigma\nu;\tau}+F_{\nu\tau;\sigma}\right)-{}^*F_{\nu\sigma}\,{}^*F^{\tau\sigma}{}_{;\tau}=\tfrac{1}{2}F^{\sigma\tau}(0)-{}^*F_{\nu\sigma}\,{}^*F^{\tau\sigma}{}_{;\tau}=0 \qquad (6)$$

in the usual way. Thus:

$$P_{\tau\sigma\nu}\equiv F_{\tau\sigma;\nu}+F_{\sigma\nu;\tau}+F_{\nu\tau;\sigma}=0\Rightarrow P^\sigma\equiv{}^*F^{\tau\sigma}{}_{;\tau}=0 \qquad (7)$$

This the Maxwell's usual magnetic equation, though it is important to think of $P_{\tau\sigma\nu}$ as a third rank source and P^σ (the magnetic charge) as a first rank source, each of which is zero. That is, although $P_{\tau\sigma\nu}$ and P^σ are themselves related via first/third rank duality transformations, these should be kept in one's mind as distinct source tensors. You know my fondness for the third rank sources, because in Yang-Mills, I believe these third-rank sources become baryons. But, let's not get ahead of ourselves yet.

Now, make use of (5) in (3). Contrary to what I said in 1984, this does not cause the electric charge $J^\sigma\equiv F^{\tau\sigma}{}_{;\tau}$ to be zero. Rather, the third-rank antisymmetric source:

$$J_{\tau\sigma\nu} \equiv {}^*F_{\tau\sigma;\nu} + {}^*F_{\sigma\nu;\tau} + {}^*F_{\nu\tau;\sigma} = \tfrac{1}{2}\left(\varepsilon_{\tau\sigma\alpha\beta}\partial_\nu + \varepsilon_{\sigma\nu\alpha\beta}\partial_\tau + \varepsilon_{\nu\tau\alpha\beta}\partial_\sigma\right)\partial^{[\alpha}A^{\beta]} \tag{8}$$

Then, still referring to (3), and using (8) and (5), we obtain the following (I'll show the full calculation, which makes use of $\varepsilon^{\mu\nu\delta\gamma}\varepsilon_{\tau\sigma\alpha\beta} = -\delta^{\mu\nu\delta\gamma}{}_{\tau\sigma\alpha\beta}$ and $\partial_\nu\left(F_{\alpha\beta}F^{\alpha\beta}\right) = 2F_{\alpha\beta}\partial_\nu F^{\alpha\beta}$, where $\mathfrak{L}_{free} = -\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}$ is the free-field Lagrangian density for QED):

$$\begin{aligned}
&\tfrac{1}{2}{}^*F^{\sigma\tau}\left({}^*F_{\tau\sigma;\nu} + {}^*F_{\sigma\nu;\tau} + {}^*F_{\nu\tau;\sigma}\right) = \tfrac{1}{8}\partial_{[\delta}A_{\gamma]}\left(\varepsilon^{\sigma\tau\delta\gamma}\varepsilon_{\tau\sigma\alpha\beta}\partial_\nu + \varepsilon^{\sigma\tau\delta\gamma}\varepsilon_{\sigma\nu\alpha\beta}\partial_\tau + \varepsilon^{\sigma\tau\delta\gamma}\varepsilon_{\nu\tau\alpha\beta}\partial_\sigma\right)\partial^{[\alpha}A^{\beta]} \\
&= -\tfrac{1}{8}\partial_{[\delta}A_{\gamma]}\left(\delta^{\sigma\tau\delta\gamma}{}_{\tau\sigma\alpha\beta}\partial_\nu + \delta^{\sigma\tau\delta\gamma}{}_{\sigma\nu\alpha\beta}\partial_\tau + \delta^{\sigma\tau\delta\gamma}{}_{\nu\tau\alpha\beta}\partial_\sigma\right)\partial^{[\alpha}A^{\beta]} = -\tfrac{1}{2}\partial_{[\delta}A_{\gamma]}\delta^{\delta\gamma}{}_{\alpha\beta}\partial_\nu\left(\partial^{[\alpha}A^{\beta]}\right) \quad . \\
&= -\tfrac{1}{2}F_{\alpha\beta}\partial_\nu F^{\alpha\beta} = \partial_\nu\left(-\tfrac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\right) = \partial_\nu\mathfrak{L}_{free}
\end{aligned} \tag{9}$$

Therefore, the identity (3), in total, becomes:

$$F_{\nu\sigma}J^\sigma \equiv F_{\nu\sigma}F^{\tau\sigma}{}_{;\tau} = \partial_\nu\mathfrak{L}_{free} = \tfrac{1}{2}{}^*F^{\sigma\tau}J_{\tau\sigma\nu}\,. \tag{10}$$

Of course, $F_{\nu\sigma}J^\sigma$ is a classical Lorentz-force term, written as a density. This source current J^σ is *not* forced to zero, contrary to what is stated in my 1984 paper. Kind of cool that the free-field Lagrangian emerges from identity (1) in this way. Thus is one of the many fascinating features of this identity. (By the way, if you look at (9) carefully, you will see that the result (10) is independent of (4); just use $F^{\alpha\beta}$ wherever there is a $\partial^{[\alpha}A^{\beta]}$)

Putting (10) and (7) together, gives us the usual Maxwell equations:

$$\begin{aligned}
&J^\sigma = F^{\tau\sigma}{}_{;\tau} \neq 0 \\
&F_{\tau\sigma;\nu} + F_{\sigma\nu;\tau} + F_{\nu\tau;\sigma} = 0
\end{aligned} \,. \tag{11}$$

This should respond to your proposal that I "start with the original set, show how to come to (8) and only then discuss the generalizations and specializations."

Now, let me give you the generalization of the 1984 paper in a nutshell (which survives the 1984 mis-statement above about (4) above forcing J^σ to zero). Take away equation (4) entirely. Then all we have to work with are (2) and (3). Consider what the universe would look like *if* there was an electric-magnetic duality symmetry in nature. Equations (2) and (3) differ from one another by a duality rotation. That is, if one set performs the transformation $F^{\sigma\tau} \to F^{\sigma\tau}{}' = {}^*F^{\sigma\tau}$ on (2), then one gets (3). If the universe were symmetric under duality transformations, then one would want to form equations which are invariant under the $F^{\sigma\tau} \to F^{\sigma\tau}{}' = {}^*F^{\sigma\tau}$ transformation. Neither (2), nor (3) is separately invariant under $F^{\sigma\tau} \to F^{\sigma\tau}{}' = {}^*F^{\sigma\tau}$. But, their sum:

$$\tfrac{1}{2}F^{\sigma\tau}\left(F_{\tau\sigma;\nu} + F_{\sigma\nu;\tau} + F_{\nu\tau;\sigma}\right) + \tfrac{1}{2}{}^*F^{\sigma\tau}\left({}^*F_{\tau\sigma;\nu} + {}^*F_{\sigma\nu;\tau} + {}^*F_{\nu\tau;\sigma}\right)_{;\tau} - {}^*F_{\nu\sigma}{}^*F^{\tau\sigma}{}_{;\tau} - F_{\nu\sigma}F^{\tau\sigma} = 0\,, \tag{12}$$

is invariant under $F^{\sigma\tau} \to F^{\sigma\tau}{}' = {}^*F^{\sigma\tau}$. (It is also invariant under continuous transformations through a real "complexion" angle α, that is, under $F^{\sigma\tau} \to F^{\sigma\tau}{}' = e^{*\alpha}F^{\sigma\tau}$. Remember that $** = -1$, so that * operates like $i = \sqrt{-1}$ and via a series expansion, $e^{*\alpha} = \cos\alpha + {}^*\sin\alpha$.)

Now, the main points of the 1984 paper are as follows:

1) Via energy conservation, it can be shown that equation (12) is absolutely identical to Einstein's equation for a gravitational field *in vacuo*:

$$T_{\mu\nu} = 0. \tag{13}$$

2) In the same manner, the "duality-mixed" identities formed from (1) by setting $A \to {}^*F$; $B \to F$ as well as $A \to F$ and $B \to {}^*F$, once also added together into a duality-invariant equation, are also identical to the gravitational equation $T_{\mu\nu} = 0$. Just for reference, this "duality-mixed" equation is:

$$\tfrac{1}{2} F^{\sigma\tau}\left({}^*F_{\tau\sigma;\nu} + {}^*F_{\sigma\nu;\tau} + {}^*F_{\nu\tau;\sigma}\right) + \tfrac{1}{2}\, {}^*F^{\sigma\tau}\left(F_{\tau\sigma;\nu} + F_{\sigma\nu;\tau} + F_{\nu\tau;\sigma}\right) + {}^*F_{\nu\sigma} F^{\tau\sigma}{}_{;\tau} + F_{\nu\sigma}\, {}^*F^{\tau\sigma}{}_{;\tau} = 0, \tag{14}$$

3) One then would be prone to ask, "are (12) and (14) really the same as $T_{\mu\nu} = 0$? And, what do we make of there being *two* equations? This is where the "field strength" calculation laid out by Einstein in "Relativistic Theory of the Non-Symmetric Field" comes in. Equations (12) and (14), *taken together*, have a "strength" that is absolutely identical to the equation $T_{\mu\nu} = 0$. Thus, $T_{\mu\nu} = 0$ is another way of saying "(12) and (14)." The equations are the same, the energy conservation relations are the same, and their strengths are the same. This is what Einstein said (see page 139 of "The Meaning of Relativity"): "It is surprising that the gravitational equations for empty space determine their field just as strongly as do Maxwell's equations in the case of the electromagnetic field." I believe he was leaving a very important clue here, for posterity.

4) And, here is the kicker. All of points 1, 2 and 3 above are based on taking away equation (4) entirely, that is, not using $F_{\tau\sigma} = \partial_\tau A_\sigma - \partial_\sigma A_\tau \equiv \partial_{[\tau} A_{\sigma]}$. We have only fields, not potentials. Now, impose the potential (4) on (12) and (14). Guess what? The duality symmetry get broken, out pop Maxwell's equations (11) as we showed earlier – and – as a consequence, $T_{\mu\nu}$ is no longer equal to zero. Now, we have broken the symmetry of the vacuum, and we have gravitating source matter.

That is more or less what I have been sitting on for 23 years, though it is only in the past 2-3 years that I have really honed in on the duality symmetry analysis. I am convinced that starting with duality symmetry, and then breaking that symmetry, will turn out to be an exceptionally-fruitful avenue of research. Especially, one should consider *local* duality symmetry, in which the complexion angle α of the continuous transformation $F^{\sigma\tau} \to F^{\sigma\tau}{}' = e^{*\alpha} F^{\sigma\tau}$, is taken to be a *local* angle $\alpha(x^\mu)$.

Hope that helps.

Jay.