Lab Note 1, Part 2: Einstein's 1915 General Relativity Paper, §20: Maxwell's Electromagnetic Field Equations for Free Space

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In the first part of this lab note, I talked about how, in my view, the identity

$$\tfrac{1}{2}A^{\sigma\tau}\left(B_{\tau\sigma;\nu}+B_{\sigma\nu;\tau}+B_{\nu\tau;\sigma}\right)-{}^{*}A_{\nu\sigma}\,{}^{*}B^{\tau\sigma}{}_{;\tau}=0\,, \tag{1}$$

is one of the most important, yet least-utilized identities in (classical) physics. Here, to lay the foundation for a classical unification of electromagnetism in the absence of potentials, with gravitation, I will show how this identity is at the heart of §20 of Einstein's 1915 paper on General Relativity. To do this, let us use the version of (1) above which is obtained by setting $A\rightarrow F$ and $B\rightarrow F$ to obtain:

$$\tfrac{1}{2}F^{\sigma\tau}\left(F_{\tau\sigma;\nu}+F_{\sigma\nu;\tau}+F_{\nu\tau;\sigma}\right)-{}^{*}F_{\nu\sigma}\,{}^{*}F^{\tau\sigma}{}_{;\tau}=0\,, \tag{2}$$

Now, showing the full calculation, it is possible to directly recast identity (2) into:

$$\begin{aligned}
0&=\tfrac{1}{2}F^{\sigma\tau}\left(F_{\tau\sigma;\nu}+F_{\sigma\nu;\tau}+F_{\nu\tau;\sigma}\right)-{}^{*}F_{\nu\sigma}\,{}^{*}F^{\tau\sigma}{}_{;\tau}\\
&=\left(F^{\sigma\tau}F_{\sigma\nu;\tau}+\tfrac{1}{2}F^{\sigma\tau}F_{\tau\sigma;\nu}\right)-{}^{*}F_{\nu\sigma}\,{}^{*}F^{\tau\sigma}{}_{;\tau}\\
&=\left(\left(F^{\sigma\tau}F_{\sigma\nu}\right)_{;\tau}-\tfrac{1}{4}\left(F^{\sigma\tau}F_{\sigma\tau}\right)_{;\nu}-F_{\nu\sigma}F^{\tau\sigma}{}_{;\tau}\right)-{}^{*}F_{\nu\sigma}\,{}^{*}F^{\tau\sigma}{}_{;\tau}\\
&=\left(\left(F^{\sigma\mu}F_{\sigma\nu}\right)_{;\mu}-\tfrac{1}{4}\delta^{\mu}{}_{\nu}\left(F^{\sigma\tau}F_{\sigma\tau}\right)_{;\mu}-F_{\nu\sigma}F^{\tau\sigma}{}_{;\tau}\right)-{}^{*}F_{\nu\sigma}\,{}^{*}F^{\tau\sigma}{}_{;\tau}\,,\\
&=\left(\left[F^{\sigma\mu}F_{\sigma\nu}-\tfrac{1}{4}\delta^{\mu}{}_{\nu}F^{\sigma\tau}F_{\sigma\tau}\right]_{\mu}-F_{\nu\sigma}F^{\tau\sigma}{}_{;\tau}\right)-{}^{*}F_{\nu\sigma}\,{}^{*}F^{\tau\sigma}{}_{;\tau}\\
&=4\pi\left[\left(-T^{\mu}{}_{\nu(Maxwell)}{}_{;\mu}-\kappa_{\nu}\right)-{}^{*}F_{\nu\sigma}P^{\sigma}\right]
\end{aligned} \tag{3}$$

by simply renaming and reordering indexes and by applying the product rule, and, in the final line, employing the Maxwell energy tensor $T^{\mu}{}_{\nu(Maxwell)}=-\frac{1}{4\pi}\left[F^{\sigma\mu}F_{\sigma\nu}-\frac{1}{4}\delta^{\mu}{}_{\nu}F^{\sigma\tau}F_{\sigma\tau}\right]$, the expression $\kappa_{\nu}\equiv F_{\nu\sigma}J^{\sigma}=\frac{1}{4\pi}F_{\nu\sigma}F^{\tau\sigma}{}_{;\tau}$ which represents the energy-momentum transfer between the electric masses and the electromagnetic field per unit of time, and a magnetic monopole charge density specified by $P^{\sigma}\equiv\frac{1}{4\pi}{}^{*}F^{\tau\sigma}{}_{;\tau}$.

Now, at the moment, of course, the magnetic charge need not be zero, because we have not yet introduced a potential. But, as soon as we introduce the Abelian potential:

$$F_{\sigma\tau}=A_{\sigma;\tau}-A_{\tau;\sigma}\,, \tag{4}$$

equation (2) immediately decouples into $P_{\tau\sigma\nu}=\frac{1}{4\pi}F_{\tau\sigma;\nu}+F_{\sigma\nu;\tau}+F_{\nu\tau;\sigma}=0$ and $P^{\sigma}=\frac{1}{4\pi}{}^{*}F^{\tau\sigma}{}_{;\tau}=0$, which means that the magnetic monopoles are zeroed out, and equation (3) simplifies to:

$$-T^{\mu}{}_{\nu(Maxwell)}{}_{;\mu}-\kappa_{\nu}=0\,. \tag{5}$$

This expresses the exchange of energy-momentum between the electric masses and the electromagnetic field. When $\kappa_\nu = 0$, then so too, $T^{\mu}{}_{\nu(Maxwell);\mu} = 0$, that is, the Maxwell energy tensor is separately conserved, i.e., does not exchange energy with the electric charges. In this manner, §20 of Einstein's 1915 paper on General Relativity is easily re-cast on the basis of particular version of identity (1) given by identity (2). In particular, when one employs a potential (4), then one immediately and directly obtains (5) from identity (2).

First, it is important to note from equation (5) that identity (2) contains a term $T^{\mu}{}_{\nu(Maxwell);\mu}$, which expresses energy conservation / transfer. More to the point, identity (2) specifically, and identity (1) more generally, is of the same differential order as an equation for conservation of energy, and the single free vector index in (1) and (2) corresponds with the single free vector index of an energy conservation equation.

Second, equation (3) *is identically equal to zero whether or not there exists a potential at all, whether the magnetic charges are zero or non-zero, whether or not the potential should it exist is Abelian or non-Abelian, and whether or not energy is exchanged between the electric charges and the electromagnetic field.*

Therefore, it becomes quite useful to think of equation (1) as providing the mathematical basis for energy conservation in the electromagnetism, in very much the same manner that $\left(R^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}{}_{\nu}R\right)_{;\mu} = 0$ provides the mathematical and geometric basis for energy conservation in gravitational theory. It is out of this observation that *classical* gravitation, and *classical* electromagnetic theory *in the absence of a potential*, may be fully unified into the single, simple electro-gravitational equation

$$T^{\mu}{}_{\nu} = 0 . \tag{6}$$

This will be demonstrated in Part 3 of Lab Note 1.