Lab Note 2, Part 1: Rest Mass as Geometry

Jay R. Yablon, January 18, 2008

Physical science, which is an enterprise designed to make sense of what we measure when observing natural phenomena, has long rested on the three "elemental dimensions" of length, time, and mass. Various other physically-measurable quantities (velocity, acceleration, force, energy, momentum, power, etc.) are built out of well-known combinations of these three elemental dimensions. With the recognition in his 1908 paper *Space and Time* that the Lorentz transformations of special relativity signaled that "space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality," Minkowski began the integration of the space and time dimensions into what we now routinely think of as spacetime. Then, in 1915, in a stunning marriage of observational physics to pure Riemannian geometry, the general theory of relativity came to explain the dynamics of gravitational behavior solely on the basis of geodesic worldline motion through a curved generalization of Minkowski's spacetime geometry. The promise of general relativity, that we might one day be able to understand all of physics solely on the basis of pure geometry, and that general relativity itself might be merely the first glimpse of an elegant geometric structure underlying all of physical reality, was later coined by Wheeler as the "geometrodynamic" program. To date, the promise of this program is still largely unfulfilled, and one of the main reasons for this is that we still do not understand the rest mass of material bodies and elementary particles on a purely geometric foundation.

Notwithstanding the success of electroweak theory in predicting the $W^{\pm\mu}, Z^{\mu}$ masses via spontaneous symmetry breaking, rest mass is, for all intents and purposes, a foreign object introduced, ad hoc, "by hand," into spacetime. Mass still stands apart from Minkowski's spacetime. One can draw spacetime diagrams and worldlines which show the motion of a given massive body through spacetime, but to specify complete information about this massive body, we must also associate with that worldline, a "number" which represents the magnitude of that mass when viewed at rest. A worldline, by itself, omits this vital information about the material body. In a gravitational field, for example, the worldlines of a golf ball and a bowling ball starting out in the same place with the same velocity vectors will be identical due to the equivalence of gravitational and inertial mass, and just knowing the worldline of each will tell us nothing about their difference in mass. One must specify the mass as a separate parameter independent of the worldline. From a geometrodynamic viewpoint, this an unsatisfactory state of affairs, because we have to specify "worldline plus mass." It would be much preferred if we could speak merely of a body traveling along a worldline through spacetime, making no reference to its mass, and if we could deduce solely from our knowledge of this worldline, not only the path through spacetime, but also the mass of the body, and thereby the forces – if any – acting on this body, simply and solely by knowing the worldline of its travel. One would seek in this way to complete what Minkowski started, to arrive at a complete geometric union of the three elemental dimensions upon which we base all physical measurement: space and time *and mass*. We seek to go from having to know about "worldline plus mass," to having to know only about "worldline alone." We wish to understand particle worldlines in such a way that we can deduce the mass of a particle solely by knowing its worldline. In this way, we can move one step closer to Wheeler's dream of understanding all of physics as pure geometry – or to be precise – understanding all of the measurements we obtain in physics, including those of mass and energy – as measurements of geometric lengths and trajectories.

To pursue this course, we start with two well-established pillars of physics. First, we consider the Dirac relationship:

$$\tfrac{1}{2}\left\{\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\right\} \equiv \eta^{\mu\nu}, \tag{1}$$

whereby the Dirac γ^μ matrices, $\mu = 0,1,2,3$, are *defined* so as to reproduce the Minkowski metric tensor $\mathrm{diag}(\eta^{\mu\nu}) = (+1,-1,-1,-1)$ under anticommutation. It is this relationship (1) which not only underlies Dirac's equation, but which also ensures that the Klein-Gordon equation applies to fermions as well as bosons. Second, we consider the axial Dirac matrix first motivated by Weyl:

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3, \tag{2}$$

which is *defined* from matrix-multiplying the other four Dirac matrices, and which has a well-established and rigorously-observed physical meaning in relation to the left- and right-chiral handedness of elementary fermions. We know that when the γ^μ are sandwiched between Dirac spinors in the form $j^\mu = \overline{\psi}\gamma^\mu\psi$, the resulting current j^μ transforms as a four-vector in spacetime, and we also know that $j^5 \equiv \overline{\psi}\gamma^5\psi$ is taken to be a pseudo-scalar. We have, in this context, five Dirac gamma matrices, but only four of which are multiplicatively independent because any one of these five matrices can be formed by multiplying the remaining four, including a suitable factor of i.

Let us now consider a five-dimensional space defined such that j^μ and j^5, taken together, all transform as a five-vector $j^{\mathrm{M}} \equiv \left(j^\mu, j^5\right)$, with $\mathrm{M} = 0,1,2,3,5$. In this five-dimensional space, we promote lowercase Greek indexes to uppercase. Of course, these five dimensions are not completely independent of one another, due to the multiplicative relationship (2), as discussed. Notwithstanding this interdependence, one can still carry out some very simple calculations in these five dimensions which suggest that rest mass may arise out of this additional, interdependent dimension associated with γ^5. Here is how:

First, let us form the anticommutator (1) from all five of the γ^{M}. That is, we define a five-dimensional, 5x5 Minkowski metric tensor according to:

$$\eta^{\mathrm{MN}} \equiv \tfrac{1}{2}\left\{\gamma^{\mathrm{M}}\gamma^{\mathrm{N}} + \gamma^{\mathrm{N}}\gamma^{\mathrm{M}}\right\}. \tag{3}$$

Given the well-known anticommutation properties of the five γ^{M}, one can readily deduce that $\mathrm{diag}\left(\eta^{\mathrm{MN}}\right) = (+1,-1,-1,-1,+1)$, and that $\eta^{\mathrm{MN}} = 0$ for $\mathrm{M} \neq \mathrm{N}$. The usual Minkowski metric tensor $\eta^{\mu\nu}$ is of course preserved in the $16 = 4\mathrm{x}4$ $\mathrm{M},\mathrm{N} = 0,1,2,3$ components of η^{MN}, and because $\eta^{55} = +1$, we find that this fifth dimension has a *timelike*, rather than a spacelike signature. Put succinctly: this five-dimensional space consists of two timelike and three spacelike dimensions. Now, we use η^{MN} in (3) to define differential geometric intervals within this five-dimensional space.

We first define infinitesimal coordinate intervals in the usual way, including a fifth dx^5 interval, that is, $dx^{\mathrm{M}} \equiv \left(dx^0, dx^1, dx^2, dx^3, dx^5\right)$. Because γ^5 is known as the "axial" matrix and because it is associated with a timelike metric signature as noted just above, we shall refer to x^5 as the "axial time" dimension, and will continue to refer to x^0 as the "ordinary time" dimension. The x^1, x^2, x^3 coordinates of course retain their role as ordinary space coordinates.

Now, because $\eta^{\mathrm{MN}} = 0$ for $\mathrm{M} \neq \mathrm{N}$ (which together with $\mathrm{diag}(\eta^{\mathrm{MN}}) = (+1,-1,-1,-1,+1)$ makes this a "flat," non-gravitating five-dimensional space), we may define a metric interval $d\mathrm{T}$ in the five-dimensional space according to:

$$d\mathrm{T}^2 \equiv \eta^{\mathrm{MN}} dx_{\mathrm{M}} dx_{\mathrm{N}} = \eta^{\mu\nu} dx_\mu dx_\nu + \eta^{55} dx_5 dx_5 = d\tau^2 + \eta^{55} dx_5 dx_5 . \tag{4}$$

In the above, we employ the usual four-dimensional spacetime interval $d\tau^2 = \eta^{\mu\nu} dx_\mu dx_\nu$. We can also, in the usual way, define covariant and contravariant matrix inverses using $\eta^{\mathrm{MN}} \eta_{\mathrm{N}\Sigma} = \delta^{\mathrm{M}}{}_{\Sigma}$ where $\delta^{\mathrm{M}}{}_{\Sigma}$ is a five-dimensional Kronecker delta (unit matrix), and can then use $\eta^{\mathrm{MN}}, \eta_{\mathrm{N}\Sigma}$ to raise and lower indexes, e.g., $dx^{\mathrm{N}} = \eta^{\mathrm{MN}} dx_{\mathrm{M}}$, in the customary manner.

Now, we perform a simple three-step calculation. First, we algebraically rewrite (4) as:

$$\frac{d\tau^2}{d\mathrm{T}^2} = 1 - \frac{dx^5 dx_5}{d\mathrm{T}^2} \quad \text{i.e.,} \quad \frac{d\tau}{d\mathrm{T}} = \pm\sqrt{1 - \frac{dx^5 dx_5}{d\mathrm{T}^2}} . \tag{5}$$

Second, we define the four-velocity vector $u^\mu \equiv dx^\mu / d\tau$ in the same way as always, and use this to alternatively rewrite (4), again algebraically, as:

$$1 = \frac{dx^\mu dx_\mu}{d\mathrm{T}^2} + \frac{dx^5 dx_5}{d\mathrm{T}^2} = \frac{dx^\mu dx_\mu}{d\tau^2} \frac{d\tau^2}{d\mathrm{T}^2} + \frac{dx^5 dx_5}{d\mathrm{T}^2} = u^\mu u_\mu \frac{d\tau^2}{d\mathrm{T}^2} + \frac{dx^5 dx_5}{d\mathrm{T}^2} . \tag{6}$$

Finally, we isolate a 0 on the left side of (6), and then use (5) to rewrite (6) as:

$$0 = u^\mu u_\mu \frac{d\tau^2}{d\mathrm{T}^2} - \left(1 - \frac{dx^5 dx_5}{d\mathrm{T}^2}\right) = u^\mu u_\mu \frac{d\tau^2}{d\mathrm{T}^2} - \frac{d\tau^2}{d\mathrm{T}^2} = \frac{d\tau^2}{d\mathrm{T}^2}\left(u^\mu u_\mu - 1\right). \tag{7}$$

The key result in (7) is the equation $\frac{d\tau^2}{d\mathrm{T}^2}\left(u^\mu u_\mu - 1\right) = 0$. All explicit traces of the fifth dimension dx^5 have dropped out of (7), which is desirable because only four of the five γ^{M} are multiplicatively independent anyway. The only residue of the fifth dimension is contained in the ratio $d\tau / d\mathrm{T}$ of the usual spacetime invariant interval $d\tau$ to the five-dimensional interval $d\mathrm{T}$. If we contrast this to the equation $p^\mu p_\mu - m^2 = m^2\left(u^\mu u_\mu - 1\right) = 0$ for the energy-momentum of an on-shell mass m, defining a momentum four-vector $p^\mu \equiv m u^\mu$ in the usual way, we see that $d\tau / d\mathrm{T}$ in (7) plays a role identical to that of the mass m in $m^2\left(u^\mu u_\mu - 1\right) = 0$. This leads us to suspect that the ratio $d\tau / d\mathrm{T}$ might perhaps be used to develop a totally geometric understanding of rest mass.

(Note, by way of review, that $p^{\mu} p_{\mu} - m^2 = m^2 \left(u^{\mu} u_{\mu} - 1\right) = 0$ arises by starting with the usual metric equation $d\tau^2 = \eta^{\mu\nu} dx_{\mu} dx_{\nu}$ in spacetime, algebraically turning this into $\frac{dx^{\mu} dx_{\mu}}{d\tau^2} - 1 = u^{\mu} u_{\mu} - 1 = 0$, introducing the rest mass m of a material body into spacetime *"by hand" as a "foreign" object*, and then multiplying through by m^2 to obtain $m^2 \left(u^{\mu} u_{\mu} - 1\right) = 0$. Note also, via the quantum-operator substitution $p^{\mu} \to i\partial^{\mu}$, that this then readily migrates over to the Klein-Gordon equation $\left(\partial^{\mu} \partial_{\mu} + m^2\right)\phi = 0$, with Lagrangian density $\mathfrak{L} = \frac{1}{2}\left(\partial^{\mu}\phi \partial_{\mu}\phi - m^2 \phi^2\right)$.)

So, when we contrast the two equations:

$$\frac{d\tau^2}{d\mathrm{T}^2}\left(u^{\mu} u_{\mu} - 1\right) = 0 \quad \text{versus} \quad m^2 \left(u^{\mu} u_{\mu} - 1\right) = 0, \tag{8}$$

we see that $d\tau / d\mathrm{T}$ transforms in four-dimensional spacetime in exactly the same manner as the rest mass m, and that $d\tau / d\mathrm{T}$ appears to serve exactly the same role as m. So the question arises: might the magnitude of the rest mass of a material body be determined simply by the ratio $d\tau / d\mathrm{T}$, which is a totally *geometric* ratio of one length/time interval $d\tau$ to a second length/time interval $d\mathrm{T}$? If so, the prospect is raised, in five dimensions, of arriving at a totally geometric understanding of rest mass, in contrast to the practice of having to introduce rest mass *by hand* when one considers only four dimensions. That is, as noted at the outset, in four dimensional spacetime, we must specify the dynamical behavior of a material body by specifying its "worldline plus mass." The prospect we wish to consider here, is that when using the fifth (not multiplicatively independent) dimension originating in Weyl's axial γ^5, we need only specify "worldline alone," for the mass itself might then be deduced from the geometric ratio $d\tau / d\mathrm{T}$. Let us now see if we can make this connection a little more firmly.

The ratio $d\tau / d\mathrm{T}$ is a dimensionless ratio of two length/time intervals, while m has dimensions of mass. So, it is clear that if this connection is to be developed further, we will need to introduce some sort of mass scale, so that $d\tau / d\mathrm{T}$ then specifies the rest mass of any particular material body in relation to this introduced mass scale. Fortunately, we know that it is reasonable to employ a vacuum expectation value (vev), designated v, against which to establish mass scales. For example, in the Higgs mechanism, a field is typically expanded about a vev, in the form $\phi(x) = v + h(x)$. We also know that such vevs are inversely related to the square root of a coupling constant G, that is, that $v \sim 1/\sqrt{G}$ (using $\hbar = c = 1$). And, we know that the Newton gravitational constant, and the Fermi coupling constant, are the two experimentally-based coupling constants known to exist in nature. Finally, we know that the rest masses of the $W^{\pm\mu}, Z^{\mu}$ vector bosons are indeed established in relation to the Fermi vev, and it is widely suspected that the same is true of the rest masses of the fermions even thought the detailed mechanism for achieving this has not yet been discovered.

Therefore, without getting too pedantic about the exact numeric value of the vev, let us simply posit *some* vev $v \sim 1/\sqrt{G}$ which is based on a coupling constant G, which may, or may not, be either the Newton or Fermi coupling constant. Then, let us *define* the rest mass m of any material body geometrically in relation to this vev, according to:

$$m \equiv v\frac{d\tau}{d\mathrm{T}}. \tag{9}$$

This is a wholly-geometric way to specify a rest mass, but for a single mass scale v which applies to all rest masses, with $d\tau/d\mathrm{T}$ responsible for variations in magnitude from one material body (or elementary particle) to the next. Viewed in this light, $d\tau/d\mathrm{T}$ is in the nature of a dimensionless coupling α, such as the low-energy $\alpha = 1/137.036$ of electromagnetic interactions. With all of this in mind, we then multiply (7) through by the vev v^2, to obtain:

$$v^2\frac{d\tau^2}{d\mathrm{T}^2}\left(u^{\mu}u_{\mu} - 1\right) = m^2\left(u^{\mu}u_{\mu} - 1\right) = p^{\mu}p_{\mu} - m^2 = 0. \tag{10}$$

What does all of this suggest? From a purely formal standpoint, let us now think about all of the above development in the reverse. Start with $m\left(u^{\mu}u_{\mu} - 1\right) = p^{\mu}p_{\mu} - m^2 = 0$ which we know is the foundation of a great deal of real, validated physics. Define $m \equiv v\frac{d\tau}{d\mathrm{T}}$. Then, the fundamental physics equation:

$$p^{\mu}p_{\mu} - m^2 = 0 \quad \text{a.k.a.} \quad \mathfrak{L} = \tfrac{1}{2}\left(\partial^{\mu}\phi\partial_{\mu}\phi - m^2\phi^2\right). \tag{11}$$

is *completely equivalent* to the equation set:

$$\begin{cases} dT^2 \equiv \eta^{MN} dx_M dx_N \\ d\tau^2 = \eta^{\mu\nu} dx_\mu dx_\nu \\ \eta^{MN} \equiv \tfrac{1}{2}\left\{\gamma^M\gamma^N + \gamma^N\gamma^M\right\} \\ m = v\dfrac{d\tau}{dT} \end{cases}. \tag{12}$$

These are two different mathematical ways of saying exactly the same thing. However, in equation (11), mass is inserted *by hand* into four dimensional spacetime. In equation set (12), the rest mass of a material body is a measure – in relation to the vev v – of how rapidly its worldline moves through the four-dimensional length/time interval $d\tau$, in relation to its motion through the five-dimensional length/time interval dT. Particle rest mass now becomes geometry in a vacuum, and the scale of mass is established by the physical energy of the vacuum. The traditional dimensional pillars of physics, namely, length, time, and mass, are now seen as descriptors of worldline motion, in a physically-energetic vacuum. One mass becomes less than or greater than another mass, solely by virtue of its $d\tau / dT$ ratio, which is a purely geometric idea that can be descried by worldlines alone. For example, the muon is 105.658 MeV, the electron is .511 MeV, and so this means that $d\tau / dT$ for the muon is 206.767 times as great as $d\tau / dT$ for the electron. This statement about the relative masses of these two electrons is now turned into a statement about the worldlines of these electrons in a five-dimensional spacetime with both ordinary time and axial time, *with no reference at all to the concept of mass*.

Of equally-fundamental interest, the $d\tau / dT$ for any given material body, is alternatively a statement of how large a component of that body's worldline is oriented through ordinary time, and how large a component is oriented through axial time. If one draws x^0 vertically and x^5 horizontally along orthogonal axes to specify a time *plane* rather than a time *line*, then Feynman's original idea of a particle moving forward or backward in time, is extended to the possibility of a particle moving *sideways* through time, and mass is simply an indicator of the magnitude of this sideways – versus forward or backward – movement through time.

One final point should be made, which will be expanded in part 2 of this lab note. When talking about elementary particles, quantum field theory informs us that we can no longer speak strictly of of worldlines, because one cannot with certainty specify an exact path taken by, say, a particular electron, to get from "point A" to "point B." Rather, one employs a path integral, which accounts for an infinite array of possible paths, and in which the expected paths emerge from interference among all the possible paths. Yet, even in quantum field theory, one thing is definite: the rest mass of a given elementary particle is certain and unchanging. The rest mass of an electron, for example, will *always* be .511 MeV. It will not be .8 MeV in one experiment, .2 MeV in the next experiment, and only average out to .511 MeV after a large number of experiments. This means that for all of the "uncertainty" in the path by which an electron goes from point A to point B, there is one aspect of its motion which can be represented by an invariant, certain, unchanging worldline: namely, the value of the electron's $d\tau / d\mathrm{T}$ path. This provides a point of geometric certainty, amidst all the other uncertainties of the spacetime position and energy momentum of elementary particles.

Now, the union of space and time first uncovered by Minkowski, might finally be extended to rest mass, and all three of the "elemental dimensions" of physics measurements might be unified into one, all described with reference to worldlines through a five-dimensional spacetime geometry including sideways, axial time.