Kaluza-Klein Theory and Lorentz Force Geodesics

DRAFT – FEBRUARY 22, 2008

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Abstract:

We examine a general Kaluza-Klein theory of classical electrodynamics and gravitation in a five-dimensional Riemannian geometry. Based solely on the condition that the electrodynamic Lorentz force law must describe geodesic motion in this five-dimensional geometry, it appears possible to place all of Maxwell's electrodynamics, the theory of electrodynamic potentials, and the QED action on a solid geometrodynamic footing, *in vacuo*, for weak and strong electro-gravitational fields. We make no choice as between the fifth dimension being timelike or spacelike, but simply point out the impact in those places where this choice makes a difference. We also show, if the fifth dimension is chosen to be a compact, cylindrical spacelike dimension, that motion in this fifth dimension may be synonymous with intrinsic spin, and that the radius of the compact dimension, for a electromagnetic coupling on the order of unity, is equal to the Schwarzschild radius of the geometrodynamic vacuum first explored by Wheeler. We also show what may well be a completely natural solution to the Kaluza-Klein chirality problem.

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1. Introduction

The possibility of employing a fifth spacetime dimension to unite classical gravitation and electrodynamics has intrigued physicists for almost a century. [1], [2] Early theorists became perhaps overly-occupied with making assumptions about the scale or topology of the extra coordinate dimension. [3] Following the path of Wesson and other current-day theorists [4], we seek here to expose the main features of Kaluza- Klein theory irrespective of any particular model, and most importantly, to make the connection between Einstein's gravitation and Maxwell's electrodynamics which is offered by Kaluza-Klein theories as clear and solid as possible, and as independent as possible of the detailed choice of model.

Most fundamentally, we adopt the view of the above-noted theorists that matter and electrodynamic charge are "induced" in observed four dimensions of spacetime, from a vacuum in five dimensions, and so, in keeping with the spirit of Wheeler's program, [5] are of completely *geometrodynamic* origin. Particularly, we seek to show how classical electrodynamics emerges entirely from an Einstein-Hilbert Action of the general form $S = \frac{1}{2\kappa}\int R dV$ where R is a suitably-defined Ricci curvature scalar, integrated over a suitable multidimensional spacetime volume, and $\kappa = 8\pi G/c^4$ is the constant from Einstein's equation $-\kappa T^{\mu}{}_{\nu} = R^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}{}_{\nu}R$. The reader will observe that this omits any Lagrangian density $\mathfrak{L}_{\text{Matter}}$ of matter, i.e., that it is *not* of the form $S = \int\left(\frac{1}{2\kappa}R + \mathfrak{L}_{\text{Matter}}\right)dV$ and so is in the nature of a vacuum action equation.[6] In different terms, we seek to induce the entirely of Maxwell's electrodynamics with sources, particularly its Lagrangian density $\mathfrak{L}_{QED} = \left(F^{\sigma\tau}F_{\sigma\tau} - A_{\mu}j^{\mu}\right)$, $\hbar = c = 1$ out of a gravitationally-based vacuum.

The main line of development will be based on a single proposition: we shall require that *the Lorentz force of electrodynamics,* $m\dfrac{d^2x^{\mu}}{d\tau^2} = qF^{\mu}{}_{\tau}\dfrac{dx^{\tau}}{d\tau}$, *must be represented as fully geodesic motion in the five-dimensional geometry*.

In five dimensions, we shall employ $g_{\text{MN}} \equiv g_{\text{NM}}$ with $\text{M,N} = 0,1,2,3,5$ for the metric tensor, so $g_{\mu\nu}$ with $\mu,\nu = 0,1,2,3$ is the ordinary metric tensor in the spacetime subspace.

Inverses are defined in the usual manner according to $g^{\mathrm{M\Sigma}} g_{\Sigma\mathrm{N}} = \delta^{\mathrm{M}}{}_{\mathrm{N}}$ and so $g^{\mathrm{M\Sigma}}$ and $g_{\Sigma\mathrm{N}}$ raise and lower indexes in the customary manner.

While most authors treat the fifth dimension as spacelike and a few have considered this to be timelike, e.g., [7], [8], [9], we shall approach the fifth dimension as independently of this choice as possible. Where this choice does make a difference, we shall point this out. If we define $g_{\mathrm{MN}} \equiv \eta_{\mathrm{MN}} + \bar{\kappa} h_{\mathrm{MN}}$ in the usual manner with $\bar{\kappa} = \sqrt{16\pi G/\hbar c^5}$, then for the weak-field limit $g_{\mathrm{MN}} \to \eta_{\mathrm{MN}}$. If the fifth dimension is timelike, $\mathrm{diag}(\eta_{\mathrm{MN}}) = (+1,-1,-1,-1,+1)$; if it is spacelike, then $\mathrm{diag}(\eta_{\mathrm{MN}}) = (+1,-1,-1,-1,-1)$. In either case, $\eta_{\mathrm{MN}} = 0$ for $\mathrm{M} \neq \mathrm{N}$. Note that the constant κ in Einstein's equation $-\kappa T^{\mu}{}_{\nu} = R^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}{}_{\nu} R$ is related to the foregoing $\bar{\kappa}$, with fundamental constants restored, by $\kappa = \frac{1}{2}\hbar c \bar{\kappa}^2 = 8\pi G/c^4$, with the overbar used to distinguish these two constants $\kappa, \bar{\kappa}$. The constant $\bar{\kappa}$ will appear frequently in the various equations herein.

2. Geodesic Motion in Five Dimensions, and the Lorentz Force

We start by maintaining the usual interval in the 4-dimensional spacetime subspace, using $d\tau^2 = g_{\mu\nu} dx^{\mu} dx^{\nu}$, and define the five-space interval as:

$$\begin{aligned} d\mathrm{T}^2 \equiv g_{\mathrm{MN}} dx^{\mathrm{M}} dx^{\mathrm{N}} &= g_{\mu\nu} dx^{\mu} dx^{\nu} + g_{5\nu} dx^5 dx^{\nu} + g_{\mu 5} dx^{\mu} dx^5 + g_{55} dx^5 dx^5 \\ &= d\tau^2 + 2 g_{5\sigma} dx^5 dx^{\sigma} + g_{55} dx^5 dx^5 \end{aligned} . \tag{2.1}$$

The above is independent of whether the weak field $g_{55} \to \eta_{55} = \pm 1$, i.e., of whether the fifth dimension is timelike or spacelike, and is generally model-independent.

Like any metric equation, (2.1) can be algebraically-manipulated into:

$$1 = g_{\mathrm{MN}} \frac{dx^{\mathrm{M}}}{d\mathrm{T}} \frac{dx^{\mathrm{N}}}{d\mathrm{T}}, \tag{2.2}$$

which is the first integral of the equation of motion. In five dimensions, we specify the Christoffel connections in the usual manner, that is, $\Gamma^{\mathrm{M}}{}_{\Sigma\mathrm{T}} = \frac{1}{2} g^{\mathrm{MA}} \left(g_{\mathrm{A\Sigma,T}} + g_{\mathrm{TA},\Sigma} - g_{\Sigma\mathrm{T,A}} \right)$, hence $\Gamma^{\mathrm{M}}{}_{\Sigma\mathrm{T}} = \Gamma^{\mathrm{M}}{}_{\mathrm{T}\Sigma}$. We employ $g_{\mathrm{MN};\Sigma} = 0$ as usual, with the usual first rank covariant derivative $A^{\mathrm{M}}{}_{;\Sigma} = A^{\mathrm{M}}{}_{,\Sigma} + \Gamma^{\mathrm{M}}{}_{\mathrm{A}\Sigma} A^{\mathrm{A}}$. We then take the covariant derivative of each side of (2.2) above, and after the usual reductions employed in four dimensions, and multiplying the result through by $d\mathrm{T}^2 / d\tau^2$, we arrive at the five-dimensional geodesic equation:

$$\frac{d^2 x^{\mathrm{M}}}{d\tau^2} + \Gamma^{\mathrm{M}}{}_{\Sigma\mathrm{T}} \frac{dx^{\Sigma}}{d\tau}\frac{dx^{\mathrm{T}}}{d\tau} = 0 . \tag{2.3}$$

The above is five independent equations. We are interested for now in the four equations for which $\mathrm{M} = \mu$, which specify motion in ordinary spacetime:

$$\frac{d^2 x^{\mu}}{d\tau^2} + \Gamma^{\mu}{}_{\Sigma\mathrm{T}} \frac{dx^{\Sigma}}{d\tau}\frac{dx^{\mathrm{T}}}{d\tau} = 0 . \tag{2.4}$$

This expands, using the metric tensor symmetry $g_{\mathrm{MN}} = g_{\mathrm{NM}}$, to:

$$\frac{d^2 x^{\mu}}{d\tau^2} + \Gamma^{\mu}{}_{\sigma\tau} \frac{dx^{\sigma}}{d\tau}\frac{dx^{\tau}}{d\tau} + 2\Gamma^{\mu}{}_{5\sigma} \frac{dx^{5}}{d\tau}\frac{dx^{\sigma}}{d\tau} + \Gamma^{\mu}{}_{55} \frac{dx^{5}}{d\tau}\frac{dx^{5}}{d\tau} = 0 . \tag{2.5}$$

Now, let us contrast (2.5) above to the gravitational geodesic equation which includes the Lorentz force law, namely, equation (20.41) of [10]:

$$\frac{d^2 x^{\mu}}{d\tau^2} + \Gamma^{\mu}{}_{\sigma\tau} \frac{dx^{\sigma}}{d\tau}\frac{dx^{\tau}}{d\tau} - \frac{q}{m} F^{\mu}{}_{\sigma} \frac{dx^{\sigma}}{d\tau} = 0 . \tag{2.6}$$

We now take a critical step: *We require that the Lorentz force as expressed above, must be represented as nothing other than geodesic motion in the five-dimensional geometry*. The first two terms in (2.5) and (2.6) are identical, and they specify geodesic motion in an ordinary gravitational field absent any electrodynamic fields or sources. The absence of any mass or charge in the first two terms captures the Galilean principle of equivalence and expresses Newtonian inertial motion in a gravitational field via the Christoffel connections $\Gamma^{\mu}{}_{\sigma\tau}$.

If we require the Lorentz force to also be fashioned as geodesic motion through geometry, then we can do so by defining the third terms in (2.5) and (2.6) to be equivalent to one another, and the fourth term in (2.5) to be zero. Therefore, we now define:

$$2\Gamma^{\mu}{}_{5\sigma} \frac{dx^{5}}{d\tau}\frac{dx^{\sigma}}{d\tau} \equiv -\frac{q}{m} F^{\mu}{}_{\sigma} \frac{dx^{\sigma}}{d\tau} , \text{ and} \tag{2.7}$$

$$\Gamma^{\mu}{}_{55} \equiv 0 . \tag{2.8}$$

One might wish to consider $\Gamma^{\mu}{}_{55} \neq 0$, in which case $\Gamma^{\mu}{}_{55}\frac{dx^5}{d\tau}\frac{dx^5}{d\tau}$ in (2.5) would become an additional term in the Lorentz force law, but in the absence of experimental evidence for any deviations from the Lorentz force law, we shall proceed on the basis of (2.8).

The relationships (2.7) and (2.8), ensure that Lorentz force motion is in fact, no more and no less than geodesic motion in five dimensions. All else will be deduced from (2.7) and (2.8).

3. Placing the Lorentz Force on a Geometrodynamic Footing as Geodesic Motion

Now, let us focus on equation (2.7). We can divide out $dx^{\sigma}/d\tau$ from (2.7), and then write the remaining terms as.

$$2\Gamma^{\mu}{}_{5\sigma}\frac{dx^5}{d\tau} \equiv -\sqrt{\frac{1}{\hbar c^5}}F^{\mu}{}_{\sigma}\frac{q}{m}, \tag{3.1}$$

where we have explicitly restored $\hbar = c = 1$. Now, we separate the proportionalities $dx^5/d\tau \propto q/m$ and $2\Gamma^{\mu}{}_{5\sigma} \propto -F^{\mu}{}_{\sigma}$, and turn the proportionalities $\propto$ into equalities by restoring their dimensional and numeric constants, starting with the former proportionality.

Irrespective of whether the fifth dimension is timelike or spacelike, we take dx^5 to be given in dimensions of time, so that $dx^5/d\tau$ is a dimensionless ratio. In the event that the fifth dimension is spacelike, one need merely divide through by c. In rationalized Heaviside-Lorentz units, the electric charge strength q (for a unit charge such as the electron, muon and tauon) is related to the dimensionless (running) coupling $\alpha = q^2/4\pi\hbar c$ which approaches $\alpha \to 1/137.036$ at low energy. The value of α is the same in all systems of units but the numerical value of q is different, so it is imperative that the exact expression for $dx^5/d\tau \propto q/m$ be based on α rather than q, and be independent of where the 4π factor appears. Further, to match dimensions with $\sqrt{\hbar c}$ the mass m needs to be multiplied by a factor of $\sqrt{G}$. Taking all of this into account, we now define:

$$\frac{dx^5}{d\tau} \equiv -\frac{1}{b}\frac{\sqrt{\hbar c\alpha}}{\sqrt{G}m} = -\frac{1}{b}\frac{1}{\sqrt{4\pi G}}\frac{q}{m} = -\frac{1}{\sqrt{\hbar c^5}}\frac{2}{b\kappa}\frac{q}{m}. \tag{3.2}$$

where b is a dimensionless, numeric constant of proportionality that we are free at this moment to choose at will, which we will carry throughout development, and which will ultimately be fixed so as to yield an action of the general form $S = \frac{1}{2\kappa}\int RdV$, i.e., an Einstein-Hilbert action in which the constant of proportionality is $\frac{1}{2\kappa}$ so as to be compatible with Einstein's equation. The equivalence between the first two terms is independent of the system of units but the terms containing q are in Heaviside-Lorentz units.

Then, we substitute (3.2) into (3.1) to obtain:

$$\Gamma^{\mu}{}_{5\sigma} \equiv \tfrac{1}{4} b\overline{\kappa} F^{\mu}{}_{\sigma} \,. \tag{3.3}$$

The definitions (3.2) and (3.3), together with $\Gamma^{\mu}{}_{55} \equiv 0$ from (2.8), when substituted into (2.5), turn the five-dimensional geodesic equation (2.5) into the Lorentz force law, and places this electrodynamic motion onto a totally-geometrodynamic footing. Of course, (3.3) is of further value, because it also relates the mixed field strength tensor $F^{\mu}{}_{\sigma}$ to the axial connection components $\Gamma^{\mu}{}_{5\sigma}$, and this will lead to numerous other results. Although the $\Gamma^{\mathrm{M}}{}_{\Sigma\mathrm{T}}$ not themselves tensors in general, (3.3) does suggest that that particular components $\Gamma^{\mu}{}_{5\sigma}$ do transform in the same way as the mixed tensor $F^{\mu}{}_{\sigma}$, multiplied by a the constant factor $\overline{\kappa}$.

4. Timelike versus Spacelike for the Fifth Dimension, and a Possible Connection to Intrinsic Spin

The results above are independent of whether the extra dimension is timelike or spacelike. Transforming into an "at rest" frame, $dx^1 = dx^2 = dx^3 = 0$, the spacetime metric equation $d\tau^2 = g_{\mu\nu} dx^{\mu} dx^{\nu}$ reduces to $d\tau = \pm\sqrt{g_{00}}\, dx^0$, and (3.2) becomes:

$$\frac{dx^5}{dx^0} = \pm \frac{1}{b}\sqrt{\frac{g_{00}}{4\pi G}}\frac{q}{m} \,. \tag{4.1}$$

For a *timelike* fifth dimension, x^5 may be drawn as an "axial time" axis orthogonal to x^0, and the physics ratio q/m (which, by the way, results in the q/m material body in an electromagnetic field actually "feeling" a Newtonian force in the sense of $F = ma$) measures the "angle" at which the material body moves through the x^5, x^0 "time plane."

For a *spacelike* fifth dimension, where a compactified, hyper-cylindrical $x^5 \equiv R\phi$ (see [11], Figure 1) and R is a constant radius (distinguish from the Ricci scalar by context), $dx^5 \equiv Rd\phi$. Substituting this into (3.2), leaving in the $\pm$ ratio obtained in (4.1), and inserting c into the first term to maintain a dimensionless equation, then yields:

$$\frac{Rd\phi}{cd\tau} = \pm\frac{1}{b}\frac{\sqrt{\hbar c\alpha}}{\sqrt{G}m} = \pm\frac{1}{b}\frac{1}{\sqrt{4\pi G}}\frac{q}{m}. \tag{4.2}$$

We see that here, the physics ratio q/m measures an "angular frequency" of fifth-dimensional rotation. Interestingly, *this frequency runs inversely to the mass*, and by classical principles, this means that the angular momentum is independent of the mass, i.e., constant. If one doubles the mass, one halves the tangential velocity, while the radius stays constant. Together with the $\pm$ factor, one might suspect that this constant angular momentum is related to intrinsic spin. In fact, following this hunch, one can arrive at an exact expression for the compactification radius R, in the following manner:

Assume that x^5 is spacelike, casting one's lot with the preponderance of those who study Kaluza-Klein theory. In (4.2), move the c away from the first term and move the m over to the first term. Then, multiply all terms by another R. Everything is now dimensioned as an angular momentum, which we have just ascertained is constant irrespective of mass. So, set this all to $\pm\frac{1}{2}n\hbar$, which for $n=1$, represents intrinsic spin. The result is as follows:

$$m\frac{Rd\phi}{d\tau}R = \pm\frac{1}{b}\frac{\sqrt{\hbar c^3\alpha}}{\sqrt{G}}R = \pm\frac{1}{b}\frac{c}{\sqrt{4\pi G}}qR = \pm\frac{1}{2}n\hbar. \tag{4.3}$$

Now, take the second and fourth terms, and solve for R with $n=1$, to yield:

$$R = \frac{b}{2\sqrt{\alpha}}\sqrt{\frac{G\hbar}{c^3}} = \frac{b}{2\sqrt{\alpha}}L_P, \tag{4.4}$$

where $L_P = \sqrt{G\hbar/c^3}$ is the Planck length. *This gives a definitive size for the compactification radius, and it is very close to the Planck length.* What is of interest, is that α is a *running* coupling. At low probe energies, where $\alpha \to 1/137.036$, $R = 5.853 \cdot b \cdot L_P$. However, this is just the *apparent* radius relative to the low probe energy. If one were to probe to a regime where α

becomes large, say, of order unity, $\alpha = 1$ then $R = \frac{b}{2} L_P$ is, depending on the value of b, quite close to the Planck length of the geometrodynamic vacuum "foam." [10] at §43.4, [12]* Since we have based the foregoing on a unit charge with spin ½, and since this is independent of the mass, the foregoing would appear to characterize the compactification radius R for all of the charged leptons, and to provide a geometric foundation for intrinsic spin! This suggests that for $\alpha = 1$ or on the order of unity, the compactication radius of the fifth dimension may become synonymous with the Planck length itself, or the Schwarzschild radius of the vacuum.

5. Symmetric Gravitation and Antisymmetric Electrodynamics

Now, let us turn back to the association $\Gamma^{\mu}{}_{5\sigma} \equiv \frac{1}{4} b\overline{\kappa} F^{\mu}{}_{\sigma}$ in (3.3), which arises from the requirement that the Lorentz force be represented as geodesic motion in five dimensions. We know that $F^{\mu\nu} = -F^{\nu\mu}$ is an antisymmetric tensor. By virtue of (3.3), this will place certain constraints on the five-dimensional Christoffel connections $\Gamma^{\mathrm{M}}{}_{\Sigma\mathrm{T}} = \frac{1}{2} g^{\mathrm{MA}} \left(g_{\mathrm{A}\Sigma,\mathrm{T}} + g_{\mathrm{TA},\Sigma} - g_{\Sigma\mathrm{T},\mathrm{A}} \right)$, and it is important to find out what these are. These constraints, in the next section, will provide the basis for placing Maxwell's equations onto a purely geometrodynamic footing.

First, because we are working in five dimensions, we will find it desirable to generalize $F^{\mu\nu}$ to F^{MN}. We make no *a priori* supposition about the additional components in F^{MN}, other than to require that they be antisymmetric, $F^{\mathrm{MN}} \equiv -F^{\mathrm{NM}}$. *Any other information about these new components is to be deduced, not imposed.* Second, we generalize (3.3) into the full five dimensions, thus:

$$\Gamma^{\mathrm{M}}{}_{5\Sigma} = \frac{1}{4} b\overline{\kappa} F^{\mathrm{M}}{}_{\Sigma} . \qquad (5.1)$$

By virtue of (2.8), $\Gamma^{\mu}{}_{55} \equiv 0$, we may immediately deduce that:

$$\Gamma^{\mu}{}_{55} = \frac{1}{4} b\overline{\kappa} F^{\mu}{}_{5} = 0 . \qquad (5.2)$$

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* By way of review, the Planck mass, defined from the term atop Newton's law as a mass for which $G{M_P}^2 = \hbar c$, is thus $M_P = \sqrt{\hbar c / G}$. In the geometrodynamic vacuum, the negative gravitational energy between Planck masses separated by the Planck length $L_P = \sqrt{G\hbar / c^3}$ precisely counterbalances and cancels the positive energy of the Planck masses themselves. The Schwarzschild radius of a Plank mass $R_S = 2GM_P / c^2 = 2\sqrt{G\hbar / c^3} = 2L_P$.

As it stands, $F^{\mathrm{M}}{}_{\Sigma}$ is a mixed tensor, and it would be better to raise this into contravariant form where we can clearly examine the consequences of having an antisymmetric field strength $F^{\mathrm{MN}} \equiv -F^{\mathrm{NM}}$. Thus, let us now raise the lower index in (5.1), and at the same time equate this to the Christoffel connections, as such:

$$\tfrac{1}{4} b\overline{\kappa} F^{\mathrm{MN}} = \tfrac{1}{4} b\overline{\kappa} g^{\Sigma\mathrm{N}} F^{\mathrm{M}}{}_{\Sigma} = g^{\Sigma\mathrm{N}} \Gamma^{\mathrm{M}}{}_{5\Sigma} = \tfrac{1}{2} g^{\mathrm{MA}} g^{\Sigma\mathrm{N}} \left(g_{\mathrm{A}5,\Sigma} + g_{\Sigma\mathrm{A},5} - g_{5\Sigma,\mathrm{A}} \right). \tag{5.3}$$

Now, we use (5.3) to write $F^{\mathrm{MN}} = -F^{\mathrm{NM}}$ completely in terms of the metric tensor g_{MN} and its first derivatives, as:

$$\tfrac{1}{4} b\overline{\kappa} F^{\mathrm{MN}} = -\tfrac{1}{4} b\overline{\kappa} F^{\mathrm{NM}} = g^{\mathrm{MA}} g^{\Sigma\mathrm{N}} \left(g_{\mathrm{A}5,\Sigma} + g_{\Sigma\mathrm{A},5} - g_{5\Sigma,\mathrm{A}} \right) = -g^{\mathrm{NA}} g^{\Sigma\mathrm{M}} \left(g_{\mathrm{A}5,\Sigma} + g_{\Sigma\mathrm{A},5} - g_{5\Sigma,\mathrm{A}} \right). \tag{5.4}$$

Renaming indexes, and using the symmetry of the metric tensor, this is readily reduced to::

$$g^{\mathrm{M}\Sigma} g^{\mathrm{TN}} g_{\mathrm{T}\Sigma,5} = 0. \tag{5.5}$$

This is an alternative, geometric way of saying that $F^{\mathrm{MN}} = -F^{\mathrm{NM}}$.

We can further simplify this using the inverse relationship $g^{\mathrm{TN}} g_{\mathrm{T}\Sigma} = \delta^{\mathrm{N}}{}_{\Sigma}$, which we can differentiate to obtain $\left(g^{\mathrm{TN}} g_{\mathrm{T}\Sigma} \right)_{,\mathrm{A}} = g^{\mathrm{TN}}{}_{,\mathrm{A}} g_{\mathrm{T}\Sigma} + g^{\mathrm{TN}} g_{\mathrm{T}\Sigma,\mathrm{A}} = 0$, i.e., $g^{\mathrm{TN}} g_{\mathrm{T}\Sigma,\mathrm{A}} = -g^{\mathrm{TN}}{}_{,\mathrm{A}} g_{\mathrm{T}\Sigma}$. This can then be used with $\mathrm{A} = 5$ to reduce (5.4) to the very simple expressions, for both the covariant and contravariant metric tensor:

$$g^{\mathrm{MN}}{}_{,5} = 0; \; g_{\mathrm{MN},5} = 0. \tag{5.6}$$

All components of the metric tensor are constant over the variations taking place only through the fifth dimension.

Now, we return to write out $\Gamma^{\mu}{}_{55} = \tfrac{1}{2} g^{\mu\mathrm{A}} \left(g_{\mathrm{A}5,5} + g_{5\mathrm{A},5} - g_{55,\mathrm{A}} \right) = 0$ from (2.8), see also (5.2). Combined with $g_{\mathrm{MN},5} = 0$ above and $g^{\mathrm{TN}} g_{\mathrm{T}\Sigma,\mathrm{A}} = -g^{\mathrm{TN}}{}_{,\mathrm{A}} g_{\mathrm{T}\Sigma}$ we further deduce that:

$$g^{55}{}_{,\mathrm{A}} = 0; \; g_{55,\mathrm{A}} = 0 \tag{5.7}$$

This means, quite importantly, that $g_{55} = \text{constant}$ and $g^{55} = \text{constant}$, everywhere in the five-dimensional geometry.

To fix these constant values, consider the weak-field limit $g_{\mathrm{MN}} \to \eta_{\mathrm{MN}}$. If the fifth dimension is timelike, $\mathrm{diag}(\eta_{\mu\nu}) = (+1,-1,-1,-1,+1)$ and $g_{55} = g^{55} = +1$. If it is spacelike (leading to the intrinsic spin results of section 4), then $\mathrm{diag}(\eta_{\mathrm{MN}}) = (+1,-1,-1,-1,-1)$ and $g_{55} = g^{55} = -1$. But, by (5.7), if the above expressions for g_{55} and g^{55} are true *anywhere*, then $g_{55} = g^{55} = +1$ or $g_{55} = g^{55} = -1$ are true *everywhere*, respectively, for a timelike or spacelike fifth dimension. In either case, timelike or spacelike, $g^{55} g_{55} = 1$. As a result, the inverse relation $g^{\mathrm{T}5} g_{\mathrm{T}5} = g^{\tau 5} g_{\tau 5} + g^{55} g_{55} = g^{\tau 5} g_{\tau 5} + 1 = \delta^5{}_5 = 1$, leads also to the null condition:

$$g^{\tau 5} g_{\tau 5} = 0\,, \tag{5.8}$$

which applies *irrespective* of the timelike versus spacelike choice.

Finally, using (5.1) together with (5.6) and (5.7), we may deduce:

$$\tfrac{1}{4} b\overline{\kappa} F^5{}_5 = \Gamma^5{}_{55} = \tfrac{1}{2} g^{5\mathrm{A}} \left(g_{\mathrm{A}5,5} + g_{5\mathrm{A},5} - g_{55,\mathrm{A}} \right) = 0\,. \tag{5.9}$$

Taking this together with (5.2), $\Gamma^\mu{}_{55} = \tfrac{1}{4} b\overline{\kappa} F^\mu{}_5 = 0$, we have now deduced that all of the newly-introduced axial components for the mixed field tensor are zero, i.e.,

$$\tfrac{1}{4} b\overline{\kappa} F^{\mathrm{M}}{}_5 = \Gamma^{\mathrm{M}}{}_{55} = 0\,. \tag{5.10}$$

The free index above can easily be lowered to also find that the covariant:

$$F_{\mathrm{M}5} = -F_{5\mathrm{M}} = 0\,. \tag{5.11}$$

But, since the non-diagonal components of $F^\mu{}_\nu$ are non-zero, one should take care to ensure that the contravariant tensor components $F^{\mathrm{M}5} = -F^{5\mathrm{M}} = 0$ as well, that is, we want to make sure that the fixed index 5 in (5.10) can properly be raised. One can employ (5.1) together with the explicit components for $\Gamma^{\mathrm{M}}{}_{5\Sigma}$ to write:

$$F^{\mathrm{MN}} = g^{\Sigma\mathrm{N}} F^{\mathrm{M}}{}_{\Sigma} = g^{\Sigma\mathrm{N}} \Gamma^{\mathrm{M}}{}_{5\Sigma} = \tfrac{1}{2} g^{\Sigma\mathrm{N}} g^{\mathrm{MA}} \left(g_{\mathrm{A}5,\Sigma} + g_{\Sigma\mathrm{A},5} - g_{5\Sigma,\mathrm{A}} \right). \tag{5.12}$$

Expanding this to separate the μ from the 5 components, and applying (5.6), (5.7) and (5.8) as needed, together with $F^{\mathrm{MN}} = -F^{\mathrm{NM}}$ to eliminate the only term which (5.6), (5.7) and (5.8) cannot directly eliminate, one can indeed deduce that in addition to (5.10) and (5.11):

$$F^{M5} = -F^{5M} = 0. \tag{5.13}$$

Now, the free index can be easily lowered, referring also to (5.1), to find that:

$$\tfrac{1}{4} b\overline{\kappa} F^{5}{}_{M} = \Gamma^{5}{}_{5M} = \Gamma^{5}{}_{M5} = 0. \tag{5.14}$$

So, we find that all of the newly-introduced axial components of the field strength tensor, whether in raised, lowered, or mixed form, are equal to zero. Equations (5.10), $\Gamma^{M}{}_{55} = 0$, and (5.14), $\Gamma^{5}{}_{5M} = \Gamma^{5}{}_{M5} = 0$, taken together, tell us that as well that any Christoffel connection with two or more axial indexes, is also equal to zero.

Combining (5.1) with $F^{M5} = -F^{5M} = 0$ as well as $F_{M5} = -F_{5M} = 0$, we may deduce two further relationships:

$$g^{\Sigma M}\Gamma^{5}{}_{5\Sigma} = -g^{\Sigma 5}\Gamma^{M}{}_{5\Sigma} = 0 \text{ and } g_{TM}\Gamma^{M}{}_{55} = -g_{5M}\Gamma^{M}{}_{5T} = 0, \tag{5.15}$$

which are variations of the "two or more axial index" rule noted above.

It is also helpful as we shall soon see when we examine the Riemann tensor, to make note of the fact that:

$$\Gamma^{M}{}_{\Sigma T,5} = \tfrac{1}{2} g^{MA}{}_{,5}\left(g_{A\Sigma,T} + g_{TA,\Sigma} - g_{\Sigma T,A}\right) + \tfrac{1}{2} g^{MA}\left(g_{A\Sigma,T,5} + g_{TA,\Sigma,5} - g_{\Sigma T,A,5}\right) = 0. \tag{5.16}$$

This makes use of (5.6) and the fact that ordinary derivatives commute. A further variation of (5.16) employs (5.1) to also write, for the field strength tensor:

$$\Gamma^{M}{}_{5\Sigma,5} = \tfrac{1}{4} b\overline{\kappa} F^{M}{}_{\Sigma,5} = 0. \tag{5.17}$$

Again, at bottom, every result in this section is a consequence of relationship (5.1), taken in combination with the antisymmetric field strength $F^{MN} \equiv -F^{NM}$. Now, we turn to the Riemann tensor, and Maxwell's equations.

6. Maxwell's Equations as Pure Geometry

We have shown how Lorentz force motion might be described as simple geodesic motion in a five-dimensional Kaluza-Klein spacetime. But equations of motion are only one part of a complete field theory. The other part is a specification of how the "sources" of that theory influence the "fields" originating from those sources. In a complete theory, the equations of motion then describe motion through the fields originating from the sources. It is now time to place Maxwell's equations on a firm geometric footing.

In five dimensions, we specify the Riemann tensor in the usual way, albeit with an extra axial index. That is:

$$R^{\mathrm{A}}{}_{\mathrm{BMN}} = -\Gamma^{\mathrm{A}}{}_{\mathrm{BM,N}} + \Gamma^{\mathrm{A}}{}_{\mathrm{BN,M}} + \Gamma^{\Sigma}{}_{\mathrm{BN}}\Gamma^{\mathrm{A}}{}_{\Sigma\mathrm{M}} - \Gamma^{\Sigma}{}_{\mathrm{BM}}\Gamma^{\mathrm{A}}{}_{\Sigma\mathrm{N}} . \tag{6.1}$$

Now, let's consider the $\mathrm{M}=5$ component of this equation, that is:

$$R^{\mathrm{A}}{}_{\mathrm{B5N}} = -\Gamma^{\mathrm{A}}{}_{\mathrm{B5,N}} + \Gamma^{\mathrm{A}}{}_{\mathrm{BN,5}} + \Gamma^{\Sigma}{}_{\mathrm{BN}}\Gamma^{\mathrm{A}}{}_{\Sigma 5} - \Gamma^{\Sigma}{}_{\mathrm{B5}}\Gamma^{\mathrm{A}}{}_{\Sigma\mathrm{N}} . \tag{6.2}$$

By virtue of $\Gamma^{\mathrm{M}}{}_{\Sigma\mathrm{T},5} = 0$, equation (5.16), which is in turn a consequence of $g_{\mathrm{MN},5} = 0$, which is in turn a consequence of $F^{\mathrm{MN}} \equiv -F^{\mathrm{NM}}$, the second term zeros out, and (6.2) becomes:

$$R^{\mathrm{A}}{}_{\mathrm{B5N}} = -\Gamma^{\mathrm{A}}{}_{\mathrm{B5,N}} + \Gamma^{\Sigma}{}_{\mathrm{BN}}\Gamma^{\mathrm{A}}{}_{\Sigma 5} - \Gamma^{\Sigma}{}_{\mathrm{B5}}\Gamma^{\mathrm{A}}{}_{\Sigma\mathrm{N}} . \tag{6.3}$$

Substituting (5.1), i.e., $\Gamma^{\mathrm{M}}{}_{5\Sigma} = \frac{1}{4} b\overline{\kappa} F^{\mathrm{M}}{}_{\Sigma}$ into the above, and with some minor term rearrangement, we immediately arrive at the *very critical expression*:

$$R^{\mathrm{A}}{}_{\mathrm{B5N}} = -\tfrac{1}{4} b\overline{\kappa}\left(F^{\mathrm{A}}{}_{\mathrm{B,N}} + \Gamma^{\mathrm{A}}{}_{\Sigma\mathrm{N}} F^{\Sigma}{}_{\mathrm{B}} - \Gamma^{\Sigma}{}_{\mathrm{BN}} F^{\mathrm{A}}{}_{\Sigma}\right) = -\tfrac{1}{4} b\overline{\kappa} F^{\mathrm{A}}{}_{\mathrm{B;N}} . \tag{6.4}$$

In particular, these three remaining terms of $R^{\mathrm{A}}{}_{\mathrm{B5N}}$ *turn out to be identical with the expression for the gravitationally-covariant derivative* $F^{\mathrm{A}}{}_{\mathrm{B;N}}$ *of the mixed field strength tensor*, times the constant factor $-\frac{1}{4} b\overline{\kappa}$. This will lead us immediately to a geometric foundation for Maxwell's equations in the following way:

As regards *Maxwell's electric charge equation*, we contract (6.4) down to its Ricci tensor component R_{B5} and define a five-current J_{B} with covariant 5-space index:

$$R_{\mathrm{B5}} = R^{\Sigma}{}_{\mathrm{B5}\Sigma} = -\tfrac{1}{4} b\overline{\kappa}\left(F^{\Sigma}{}_{\mathrm{B},\Sigma} + \Gamma^{\Sigma}{}_{\mathrm{T}\Sigma} F^{\mathrm{T}}{}_{\mathrm{B}} - \Gamma^{\mathrm{T}}{}_{\mathrm{B}\Sigma} F^{\Sigma}{}_{\mathrm{T}}\right) = -\tfrac{1}{4} b\overline{\kappa} F^{\Sigma}{}_{\mathrm{B};\Sigma} \equiv -\tfrac{1}{4} b\overline{\kappa} J_{\mathrm{B}} . \tag{6.5}$$

We will now want to see how J_{B} relates to the observed four-current $j_\beta = F^{\sigma}{}_{\beta;\sigma}$ of electrodynamics. We first expand the Σ and T indexes into spacetime and axial parts, and use $\Gamma^{5}{}_{\mathrm{T5}} = 0$ and $F^{5}{}_{\mathrm{T}} = 0$ from (5.14) to zero out some terms (but *not* any of the covariant derivatives, for reasons to soon become apparent), to obtain:

$$R_{\mathrm{B5}} = -\tfrac{1}{4} b\overline{\kappa}\left(F^{\sigma}{}_{\mathrm{B},\sigma} + \Gamma^{\sigma}{}_{\tau\sigma} F^{\tau}{}_{\mathrm{B}} - \Gamma^{\tau}{}_{\mathrm{B}\sigma} F^{\sigma}{}_{\tau}\right) = -\tfrac{1}{4} b\overline{\kappa}\left(F^{\sigma}{}_{\mathrm{B};\sigma} + F^{5}{}_{\mathrm{B};5}\right) \equiv -\tfrac{1}{4} b\overline{\kappa} J_{\mathrm{B}} \equiv -\tfrac{1}{4} b\overline{\kappa}\left(j_{\mathrm{B}} + j_{(5)\mathrm{B}}\right) . \tag{6.6}$$

The definition $J_{\mathrm{B}} \equiv j_{\mathrm{B}} + j_{(5)\mathrm{B}}$ separates the five-component current J_{B} into an "ordinary" part j_{B} and what we shall, for now, simply refer to as an "extra" part $j_{(5)\mathrm{B}}$, and lays the foundation for a separation that will soon become important. This means that there are actually four currents we will want to consider: $j_\beta \equiv F^{\sigma}{}_{\beta;\sigma}$ (four components), $j_5 \equiv F^{\sigma}{}_{5;\sigma}$ (one component), $j_{(5)\beta} \equiv F^{5}{}_{\beta;5}$ (four components) and $j_{(5)5} \equiv F^{5}{}_{5;5}$ (one component).

Now, we split the above into two equations, namely:

$$R_{\beta 5} = -\tfrac{1}{4} b\overline{\kappa}\left(F^{\sigma}{}_{\beta,\sigma} + \Gamma^{\sigma}{}_{\tau\sigma}F^{\tau}{}_{\beta} - \Gamma^{\tau}{}_{\beta\sigma}F^{\sigma}{}_{\tau}\right) = -\tfrac{1}{4} b\overline{\kappa}\left(F^{\sigma}{}_{\beta;\sigma} + F^{5}{}_{\beta;5}\right) \equiv -\tfrac{1}{4} b\overline{\kappa} J_{\beta} \equiv -\tfrac{1}{4} b\overline{\kappa}\left(j_\beta + j_{(5)\beta}\right), \quad (6.7)$$

$$R_{55} = -\tfrac{1}{4} b\overline{\kappa}\left(F^{\sigma}{}_{5,\sigma} + \Gamma^{\sigma}{}_{\tau\sigma}F^{\tau}{}_{5} - \Gamma^{\tau}{}_{5\sigma}F^{\sigma}{}_{\tau}\right) = -\tfrac{1}{4} b\overline{\kappa}\left(F^{\sigma}{}_{5;\sigma} + F^{5}{}_{5;5}\right) \equiv -\tfrac{1}{4} b\overline{\kappa} J_{5} \equiv -\tfrac{1}{4} b\overline{\kappa}\left(j_5 + j_{(5)5}\right). \quad (6.8)$$

In (6.7), we discern the four-covariant derivative $F^{\sigma}{}_{\beta;\sigma} = F^{\sigma}{}_{\beta,\sigma} + \Gamma^{\sigma}{}_{\tau\sigma}F^{\tau}{}_{\beta} - \Gamma^{\tau}{}_{\beta\sigma}F^{\sigma}{}_{\tau}$, which means that $j_{(5)\beta} = F^{5}{}_{\beta;5} = 0$ and that $J_\beta = j_\beta$ is the observed electromagnetic current source density. We may therefore reduce (6.7) to:

$$R_{\beta 5} = -\tfrac{1}{4} b\overline{\kappa} F^{\sigma}{}_{\beta;\sigma} \equiv -\tfrac{1}{4} b\overline{\kappa} j_\beta\,; \quad (6.9)$$

This is Maxwell's electric charge equation, on a geometric foundation.

For the axial equation (6.8), we use $F^{\Sigma}{}_{5} = 0$ to reduce terms as before, but we also employ the substitution $\Gamma^{\mathrm{M}}{}_{5\Sigma} = \tfrac{1}{4} b\overline{\kappa} F^{\mathrm{M}}{}_{\Sigma}$ from (5.1). Thus:

$$R_{55} = -\tfrac{1}{16} b^2 \overline{\kappa}^2 F^{\tau}{}_{\sigma}F^{\sigma}{}_{\tau} = -\tfrac{1}{4} b\overline{\kappa}\left(F^{\sigma}{}_{5;\sigma} + F^{5}{}_{5;5}\right) \equiv -\tfrac{1}{4} b\overline{\kappa} J_5 \equiv -\tfrac{1}{4} b\overline{\kappa}\left(j_5 + j_{(5)5}\right) \neq 0\,. \quad (6.10)$$

Now, we begin to notice an interesting result that will have great significance in the development to follow: Despite the $F^{\Sigma}{}_{5;\Sigma} = F^{\sigma}{}_{5;\sigma} + F^{5}{}_{5;5}$ term containing components of a mixed tensor which vanish in their own right, namely $F^{\Sigma}{}_{5} = 0$, this term for R_{55} is *not* equal to zero. Rather, we find that the covariant derivative term $F^{\Sigma}{}_{5;\Sigma} = F^{\sigma}{}_{5;\sigma} + F^{5}{}_{5;5} \neq 0$ *does not vanish*, and in fact, leaves a very central term $F^{\sigma\tau}F_{\sigma\tau}$ found in the QED free-field Lagrangian $\mathfrak{L}_{QCD\,(Free)} = -\tfrac{1}{4} F^{\sigma\tau}F_{\sigma\tau}$ and in $T^{\mu}{}_{\nu\,Maxwell} = -\left(F^{\mu\sigma}F_{\nu\sigma} - \tfrac{1}{4}\delta^{\mu}{}_{\nu}F^{\sigma\tau}F_{\sigma\tau}\right)$, the Maxwell stress-energy tensor in Heaviside-Lorentz units. This is the first of several instances where we will find that a covariant derivative of $F^{\Sigma}{}_{5} = 0$ or its covariant and contravariant relatives, leaves a non-zero term containing $F^{\sigma\tau}F_{\sigma\tau}$. One may think of $F^{\Sigma}{}_{5;\Sigma} = \frac{1}{4}b\overline{\kappa}F^{\sigma\tau}F_{\sigma\tau} \neq 0$ as being *"gravitationally induced"* out of $F^{\Sigma}{}_{5} = 0$, solely as a non-linear gravitational effect, because in the absence of gravitation, covariant derivatives approach ordinary derivatives and so $F^{\sigma}{}_{5;\sigma} \to F^{\sigma}{}_{5,\sigma} = 0$. This induced term originates from the final term $-\Gamma^{\Sigma}{}_{\mathrm{BM}}\Gamma^{\mathrm{A}}{}_{\Sigma\mathrm{N}}$ of the Riemann tensor $R^{\mathrm{A}}{}_{\mathrm{BMN}}$, see (6.1), via the progression $+\Gamma^{\Sigma}{}_{\mathrm{BN}}\Gamma^{\mathrm{A}}{}_{\Sigma\mathrm{M}} \to +\Gamma^{\Sigma}{}_{5\mathrm{T}}\Gamma^{\mathrm{T}}{}_{\Sigma 5} = +\frac{1}{16}b^2\overline{\kappa}^2 F^{\Sigma}{}_{\mathrm{T}}F^{\mathrm{T}}{}_{\Sigma}$.

Now a question arises which will also become quite important in the later discussion: if $-\frac{1}{16}b^2\overline{\kappa}^2 F^{\tau}{}_{\sigma}F^{\sigma}{}_{\tau} = -\frac{1}{4}b\overline{\kappa}\left(F^{\sigma}{}_{5;\sigma} + F^{5}{}_{5;5}\right) \neq 0$, to which of the covariant derivatives $j_5 \equiv F^{\sigma}{}_{5;\sigma}$, or $j_{(5)5} \equiv F^{5}{}_{5;5}$, do we attribute the emergence of this term $F^{\tau}{}_{\sigma}F^{\sigma}{}_{\tau}$ which is so central to QED? We shall, for reasons that will become apparent in the next section when we consider the mixed Ricci tensor $R^{\mathrm{M}}{}_{5} = g^{\mathrm{MB}}R_{\mathrm{B}5}$, attribute the $F^{\tau}{}_{\sigma}F^{\sigma}{}_{\tau}$ term to $j_{(5)5} \equiv F^{5}{}_{5;5}$. By this attribution, which will be justified rigorously in the next section, we set $j_5 = F^{\sigma}{}_{5;\sigma} = 0$ so write (6.10) as:

$$R_{55} = -\frac{1}{16}b^2\overline{\kappa}^2 F^{\tau}{}_{\sigma}F^{\sigma}{}_{\tau} = -\frac{1}{4}b\overline{\kappa}F^{5}{}_{5;5} \equiv -\frac{1}{4}b\overline{\kappa}J_5 \equiv -\frac{1}{4}b\overline{\kappa}j_{(5)5} \neq 0. \qquad (6.11)$$

We make a special point of this, because in the next section, when we consider the Ricci tensor in mixed form, e.g., $R^{\mathrm{B}}{}_{5}$, we will find that *all* of the contravariant current terms become non-zero, i.e., that all of $j^{\beta} \equiv F^{\sigma\beta}{}_{;\sigma}$, $j^{5} \equiv F^{\sigma 5}{}_{;\sigma}$, $j_{(5)}{}^{\beta} \equiv F^{5\beta}{}_{;5}$ and $j_{(5)}{}^{5} \equiv F^{55}{}_{;5}$ are non-zero, and in particular, that $j_{(5)}{}^{\mathrm{M}} = F^{5\mathrm{M}}{}_{;5} = \overline{\kappa}g^{\mathrm{M}5}F^{\sigma\tau}F_{\sigma\tau}$ contains an induced $F^{\sigma\tau}F_{\sigma\tau}$ term in all of its components. Therefore:

$$g_{\mathrm{BM}}j_{(5)}{}^{\mathrm{M}} = j_{(5)\mathrm{B}} = g_{\mathrm{BM}}F^{5\mathrm{M}}{}_{;5} = F^{5}{}_{\mathrm{B};5} = \frac{1}{4}b\overline{\kappa}g_{\mathrm{BM}}g^{\mathrm{M}5}F^{\sigma\tau}F_{\sigma\tau} = \frac{1}{4}b\overline{\kappa}\delta_{\mathrm{B}}{}^{5}F^{\sigma\tau}F_{\sigma\tau}, \qquad (6.12)$$

which explains $j_{(5)\beta} = F^{5}{}_{\beta;5} = \frac{1}{4}b\overline{\kappa}\delta_{\beta}{}^{5}F^{\sigma\tau}F_{\sigma\tau} = 0$ and $j_{(5)5} = F^{5}{}_{5;5} = \frac{1}{4}b\overline{\kappa}\delta_{5}{}^{5}F^{\sigma\tau}F_{\sigma\tau} = \frac{1}{4}b\overline{\kappa}F^{\sigma\tau}F_{\sigma\tau}$ above, and is why we attributed the $F^{\tau}{}_{\sigma}F^{\sigma}{}_{\tau}$ term to $j_{(5)5} \equiv F^{5}{}_{5;5}$ rather than to $j_5 \equiv F^{\sigma}{}_{5;\sigma}$. Consolidating (6.9), (6.10) and (6.12) together for contrast, we may summarize all of the foregoing results in the following manner:

$$\begin{cases} R_{\beta 5} = -\tfrac{1}{4} b\overline{\kappa} j_\beta = -\tfrac{1}{4} b\overline{\kappa} F^{\sigma}{}_{\beta;\sigma} = -\tfrac{1}{4} b\overline{\kappa}\left(F^{\sigma}{}_{\beta,\sigma} + \Gamma^{\sigma}{}_{\tau\sigma} F^{\tau}{}_{\beta} - \Gamma^{\tau}{}_{\beta\sigma} F^{\sigma}{}_{\tau}\right) \\ \qquad -\tfrac{1}{4} b\overline{\kappa} j_5 = -\tfrac{1}{4} b\overline{\kappa} F^{\sigma}{}_{5;\sigma} = 0 \\ \qquad -\tfrac{1}{4} b\overline{\kappa} j_{(5)\beta} = -\tfrac{1}{4} b\overline{\kappa} F^{5}{}_{\beta;5} = -\tfrac{1}{16} b^2 \overline{\kappa}^2 \delta_\beta{}^5 F^{\sigma\tau} F_{\sigma\tau} = 0 \\ R_{55} = -\tfrac{1}{4} b\overline{\kappa} j_{(5)5} = -\tfrac{1}{4} b\overline{\kappa} F^{5}{}_{5;5} = -\tfrac{1}{16} b^2 \overline{\kappa}^2 \delta_5{}^5 F^{\sigma\tau} F_{\sigma\tau} = -\tfrac{1}{16} b^2 \overline{\kappa}^2 F^{\sigma\tau} F_{\sigma\tau} \end{cases}, \tag{6.13}$$

Turning now to Maxwell's magnetic equation, we first lower the A index in (6.4), $R_{5\mathrm{NMB}} = g_{\mathrm{MA}} R^{\mathrm{A}}{}_{\mathrm{B5N}}$, and use $R_{\mathrm{ABMN}} = R_{\mathrm{MNAB}}$ to write:

$$R_{5\mathrm{NMB}} = -\tfrac{1}{4} b\overline{\kappa}\left(g_{\mathrm{MA}} F^{\mathrm{A}}{}_{\mathrm{B,N}} + g_{\mathrm{MA}} \Gamma^{\mathrm{A}}{}_{\Sigma\mathrm{N}} F^{\Sigma}{}_{\mathrm{B}} - g_{\mathrm{MA}} \Gamma^{\Sigma}{}_{\mathrm{BN}} F^{\mathrm{A}}{}_{\Sigma}\right) = -\tfrac{1}{4} b\overline{\kappa} g_{\mathrm{MA}} F^{\mathrm{A}}{}_{\mathrm{B;N}} = -\tfrac{1}{4} b\overline{\kappa} F_{\mathrm{MB;N}} . \tag{6.14}$$

Maxwell's magnetic equation then arises straight from the 5-dimensional rendition of the "first" Bianchi identity:

$$R_{\mathrm{MNAB}} + R_{\mathrm{MABN}} + R_{\mathrm{MBNA}} = 0 . \tag{6.15}$$

Making use of (6.14), the $\mathrm{M} = 5$ component of this is:

$$R_{5\mathrm{NAB}} + R_{5\mathrm{ABN}} + R_{5\mathrm{BNA}} = -\tfrac{1}{4} b\overline{\kappa}\left(F_{\mathrm{AB;N}} + F_{\mathrm{BN;A}} + F_{\mathrm{NA;B}}\right) = -\tfrac{1}{4} b\overline{\kappa}\left(F_{\mathrm{AB,N}} + F_{\mathrm{BN,A}} + F_{\mathrm{NA,B}}\right) = 0 , \tag{6.16}$$

where we account for the well-known fact that in the cyclic combination of (6.16) with antisymmetric tensors, the Christoffel terms in the covariant derivatives cancel identically, so the covariant derivatives becomes ordinary derivatives. In the $\mathrm{NAB} = \nu\alpha\beta$ subset of this, we immediately obtain Maxwell's magnetic equation

$$F_{\alpha\beta,\nu} + F_{\beta\nu,\alpha} + F_{\nu\alpha,\beta} = 0 . \tag{6.17}$$

In light of our earlier discovery of some new terms in Maxwell's electric charge equation arising from the fifth dimension, see, e.g., the R_{55} equation in (6.13), one may ask whether there are any additional electrodynamic terms in the (6.16) above, in the circumstance where more than a single axial index is employed. Because $R_{\mathrm{ABMN}} = R_{\mathrm{MNAB}} = -R_{\mathrm{BAMN}}$, it is clear that with more than two axial indexes, i.e., $R_{555\mu}$, (6.16) will identically reduce to zero. But we should explore whether there is any additional electrodynamic information to be gleaned when exactly two axial indexes are used in (6.16). Thus, we may examine, say:

$$R_{55\mathrm{AB}} + R_{5\mathrm{AB5}} + R_{5\mathrm{B5A}} = -\tfrac{1}{4} b\overline{\kappa}\left(F_{\mathrm{AB;5}} + F_{\mathrm{B5;A}} + F_{\mathrm{5A;B}}\right) = -\tfrac{1}{4} b\overline{\kappa}\left(F_{\mathrm{AB,5}} + F_{\mathrm{B5,A}} + F_{\mathrm{5A,B}}\right) = 0 . \tag{6.18}$$

We learn from (6.13), especially $R_{55} = -\frac{1}{4} b\overline{\kappa} F^{\sigma}{}_{5;\sigma} = -\frac{1}{16} b^2 \overline{\kappa}^2 F^{\sigma\tau} F_{\sigma\tau}$, not to automatically eliminate a field strength term such as $F^{\sigma}{}_5$ when it appears in a *covariant* derivative, i.e., $F^{\sigma}{}_{5;\sigma}$. However, the migration of covariant to ordinary derivatives in the cyclic combination of (6.16) removes this complication. We know from (5.11) that $F_{\mathrm{B}5} = F_{5\mathrm{A}} = 0$, so their *ordinary* derivatives will vanish as well. The remaining $F_{\mathrm{AB},5} = \left(g_{\mathrm{A}\Sigma} F^{\Sigma}{}_{\mathrm{B}}\right)_{,5} = g_{\mathrm{A}\Sigma,5} F^{\Sigma}{}_{\mathrm{B}} + g_{\mathrm{A}\Sigma} F^{\Sigma}{}_{\mathrm{B},5} = 0$ in (6.18), by virtue of (5.6) and (5.17). Thus, (6.18) is identically equal to zero, not only because of the Bianchi identity, but because of the inherent properties of the F_{AB} and g_{AB} developed in section 5. Thus, there is no additional electrodynamic information to be gleaned from (6.18).

We have now placed each of Maxwell's equations on a solely geometric footing. Maxwell's source equation in covariant (lower index) form is specified by (6.9), namely, $R_{\beta 5} = -\frac{1}{4} b\overline{\kappa} j_{\beta} = -\frac{1}{4} b\overline{\kappa} F^{\sigma}{}_{\beta;\sigma}$, and there is an additional component in the 5-dimensional space given by the latter of (6.13), namely, $R_{55} = -\frac{1}{4} b\overline{\kappa} j_5 = -\frac{1}{4} b\overline{\kappa} F^{5}{}_{5;5} = -\frac{1}{16} b^2 \overline{\kappa}^2 F^{\sigma\tau} F_{\sigma\tau}$, which contains the very central term $\mathfrak{L}_{QCD\,(Free)} = -\frac{1}{4} F^{\sigma\tau} F_{\sigma\tau}$, and which will be of great interest in the discussion to follow. Maxwell's magnetic equation is simply an axial component (6.16) of the first Bianchi identity $R_{\mathrm{MNAB}} + R_{\mathrm{MABN}} + R_{\mathrm{MBNA}} = 0$. And, the Lorentz force equation (2.6), upon which the foregoing geometrization of Maxwell's equations is based, is merely the equation for four-space geodesic motion in the five-dimensional geometry, $\dfrac{d^2 x^{\mu}}{d\tau^2} + \Gamma^{\mu}{}_{\Sigma\mathrm{T}} \dfrac{dx^{\Sigma}}{d\tau} \dfrac{dx^{\mathrm{T}}}{d\tau} = 0$, (2.4). With source equations producing fields and with material bodies in those fields moving over geodesics that are identical to and synonymous with the Lorentz force, Maxwell's electrodynamics now rests on the firm geometrodynamic footing of a five-dimensional Kaluza-Klein geometry.

7. Calculation of the 5-Dimensional Curvature Scalar, and the Fifth-Dimensional Components of the Einstein Equation.

Especially in light of the gravitationally-induced $R_{55} = -\frac{1}{4} b\overline{\kappa}^2 F^{\sigma\tau} F_{\sigma\tau}$, see (6.13), we now turn our attention to the QED Lagrangian density $\hbar c^2 \mathfrak{L}_{QED} = \left(-F^{\sigma\tau} F_{\sigma\tau} - A_{\mu} j^{\mu}\right)$, and in particular, to seeing if we can place this entire $\mathfrak{L}_{QED}$ *with sources*, on a purely geometric footing, *in vacuo*.

In other words, we now take aim, as discussed in the introduction, at using an Einstein-Hilbert Action of the general form $S = \frac{1}{2\kappa}\int R dV$ to specify $\mathcal{L}_{QED}$ with sources, without explicitly adding an $\mathcal{L}_{\mathrm{Matter}}$. We begin discussion here by deriving the five-dimensional Ricci curvature scalar $R_{(5)} \equiv R^{\Sigma}{}_{\Sigma} = R + R^{5}{}_{5}$, taking the four dimensional curvature scalar to be $R = R^{\sigma}{}_{\sigma}$, since these are clear candidates for inclusion in such an action. In addition, we need $R_{(5)}$ if we wish to consider the five-dimensional extensions of Einstein's equation, i.e., $-\kappa T^{\mathrm{M}}{}_{\mathrm{N}} = R^{\mathrm{M}}{}_{\mathrm{N}} - \frac{1}{2}\delta^{\mathrm{M}}{}_{\mathrm{N}} R_{(5)}$. Let's begin with $R^{5}{}_{5}$.

There are two ways to calculate $R^{5}{}_{5}$ which lead to alternative, but equivalent expressions, each useful in different contexts. First, in (6.5), we have already found $R_{\mathrm{B}5}$. So, all we need do is raise the index using $R^{\mathrm{M}}{}_{5} = g^{\mathrm{MB}} R_{\mathrm{B}5}$, i.e.,

$$R^{\mathrm{M}}{}_{5} = -\frac{1}{4} b\overline{\kappa}\left(g^{\mathrm{MB}} F^{\Sigma}{}_{\mathrm{B},\Sigma} + \Gamma^{\Sigma}{}_{\mathrm{T}\Sigma} F^{\mathrm{TM}} - g^{\mathrm{MB}} \Gamma^{\mathrm{T}}{}_{\mathrm{B}\Sigma} F^{\Sigma}{}_{\mathrm{T}}\right) = -\frac{1}{4} b\overline{\kappa} g F^{\Sigma\mathrm{M}}{}_{;\Sigma} \equiv -\frac{1}{4} b\overline{\kappa} J^{\mathrm{M}} \equiv -\frac{1}{4} b\overline{\kappa}\left(j^{\mathrm{M}} + j_{(5)}{}^{\mathrm{M}}\right), \tag{7.1}$$

and then take the $\mathrm{M} = 5$ component. Second, alternatively, we can write the covariant Ricci tensor as $R_{\mathrm{MN}} = g_{\mathrm{M}\Sigma} R^{\Sigma}{}_{\mathrm{N}} = g_{\mathrm{M}\sigma} R^{\sigma}{}_{\mathrm{N}} + g_{\mathrm{M}5} R^{5}{}_{\mathrm{N}}$, then take the $R_{55} = g_{5\sigma} R^{\sigma}{}_{5} + g_{55} R^{5}{}_{5}$ component, which we rewrite as:

$$g_{55} R^{5}{}_{5} = R_{55} - g_{5\sigma} R^{\sigma}{}_{5}. \tag{7.2}$$

In this latter approach, we can take advantage of the fact that $g_{55} = \pm 1 = \text{constant}$ depending on whether the fifth dimension is timelike (+1) or spacelike (-1), see the discussion following (5.7), and can make use of $R_{55} = -\frac{1}{16} b^2 \overline{\kappa}^2 F^{\sigma\tau} F_{\sigma\tau}$, as already found in (6.10). In either approach, since the unknowns in (7.2) are $R^{\sigma}{}_{5}$ and $g_{5\sigma}$, the first step is to deduce $R^{\mathrm{M}}{}_{5}$ in (7.1).

Starting from (7.1), we separate all contracted indexes into their spacetime and axial components. We can reduce many terms throughout making use of $F^{\Sigma}{}_{5} = 0$ and its raised and lowered variants, as well as $\Gamma^{5}{}_{\tau 5} = 0$, see (5.14). Along the way, we also use (5.1) to substitute $\Gamma^{\mathrm{T}}{}_{5\Sigma} = \frac{1}{4} b\overline{\kappa} F^{\mathrm{T}}{}_{\Sigma}$. This introduces another "gravitationally-induced" term $-\frac{1}{16} b^2 \overline{\kappa}^2 g^{\mathrm{M}5} F^{\mathrm{T}}{}_{\Sigma} F^{\Sigma}{}_{\mathrm{T}}$ as in the fourth equation (6.13). Because $F^{\mathrm{T}}{}_{\Sigma} F^{\Sigma}{}_{\mathrm{T}}$ is summed over all five dimensions, we can readjust the indexes according to $F^{\mathrm{T}}{}_{\Sigma} F^{\Sigma}{}_{\mathrm{T}} = F^{\mathrm{T}\Sigma} F_{\Sigma\mathrm{T}} = -F^{\Sigma\mathrm{T}} F_{\Sigma\mathrm{T}}$. Finally, recalling the "gravitationally-induced" term $-\frac{1}{4}b\overline{\kappa}F^5{}_{5;5} = -\frac{1}{16}b^2\overline{\kappa}^2 F^{\sigma\tau}F_{\sigma\tau} \neq 0$ from (6.13), we use $F^\Sigma{}_5 = 0$ to eliminate *only ordinary derivatives* such as $F^\Sigma{}_{5,\mathrm{T}} = 0$, but *not the covariant derivatives* $F^\Sigma{}_{5;\mathrm{T}}$. The net result of all of this, is that (7.1) reduces to:

$$\begin{aligned} R^{\mathrm{M}}{}_5 &= -\tfrac{1}{4}b\overline{\kappa}\left(g^{\mathrm{M}\beta}F^\sigma{}_{\beta,\sigma} + \Gamma^\sigma{}_{\tau\sigma}F^{\tau\mathrm{M}} - g^{\mathrm{M}\beta}\Gamma^\tau{}_{\beta\sigma}F^\sigma{}_\tau + \tfrac{1}{4}b\overline{\kappa}g^{\mathrm{M}5}F^{\sigma\tau}F_{\sigma\tau}\right) \\ &= -\tfrac{1}{4}b\overline{\kappa}F^{\Sigma\mathrm{M}}{}_{;\Sigma} = -\tfrac{1}{4}b\overline{\kappa}\left(F^{\sigma\mathrm{M}}{}_{;\sigma} + F^{5\mathrm{M}}{}_{;5}\right) \equiv -\tfrac{1}{4}b\overline{\kappa}J^{\mathrm{M}} \equiv -\tfrac{1}{4}b\overline{\kappa}\left(j^{\mathrm{M}} + j_{(5)}{}^{\mathrm{M}}\right) \end{aligned} \tag{7.3}$$

In the above, we define an "ordinary" 5-dimensional current

$$j^{\mathrm{M}} \equiv F^{\sigma\mathrm{M}}{}_{;\sigma} = g^{\mathrm{M}\beta}F^\sigma{}_{\beta,\sigma} + \Gamma^\sigma{}_{\tau\sigma}F^{\tau\mathrm{M}} - g^{\mathrm{M}\beta}\Gamma^\tau{}_{\beta\sigma}F^\sigma{}_\tau\,, \tag{7.4}$$

as well as a "gravitationally-induced" "extra" five-dimensional current:

$$j_{(5)}{}^{\mathrm{M}} \equiv F^{5\mathrm{M}}{}_{;5} = \overline{\kappa}g^{\mathrm{M}5}F^{\sigma\tau}F_{\sigma\tau} \neq 0\,. \tag{7.5}$$

If we multiply the above through by g_{BM} to lower the free index, we obtain (6.12), with accounts for the way in which we wrote (6.13) and rigorously justifies that earlier attribution of the $F^\tau{}_\sigma F^\sigma{}_\tau$ term to $j_{(5)5} \equiv F^5{}_{5;5}$.

Pausing to examine (7.5) more closely, we can now formalize the "extra" current term in (7.5). This is certainly in the nature of a source current, because it is of the form $j_{(5)}{}^{\mathrm{M}} \equiv F^{5\mathrm{M}}{}_{;5}$. Also, as in (7.3), we see that it is naturally added to the "ordinary" current in the form $J^{\mathrm{M}} \equiv -j^{\mathrm{M}} + j_{(5)}{}^{\mathrm{M}}$ by the five-dimensional geometry, because of its origin in $F^{\sigma\mathrm{M}}{}_{;\sigma} + F^{5\mathrm{M}}{}_{;5}$. Further, it contains what is for many reasons a very desirable term, $F^{\sigma\tau}F_{\sigma\tau}$. However, from (7.5), we see that $j_{(5)}{}^{\mathrm{M}}$ is actually a scalar, $F^{\sigma\tau}F_{\sigma\tau}$. multiplied by the $\mathrm{M}5$ components of second-rank, contravariant symmetric tensor $g^{\mathrm{M}5}$, times a constant $\overline{\kappa}$. This can be generalized so that $j_{(5)}{}^{\mathrm{M}}$ itself also consists of the $\mathrm{M}5$ components of a second-rank, contravariant symmetric tensor, i.e., $j_{(5)}{}^{\mathrm{M}} \equiv j^{\mathrm{M}5} \equiv \frac{1}{4}b\overline{\kappa}g^{\mathrm{M}5}F^{\sigma\tau}F_{\sigma\tau}$, and our "extra" current can then be generalized into a full second-rank, contravariant, symmetric tensor defined as:

$$j^{\mathrm{MN}} \equiv \tfrac{1}{4}b\overline{\kappa}g^{\mathrm{MN}}F^{\sigma\tau}F_{\sigma\tau}\,. \tag{7.6}$$

With this new definition (7.6), we return to rewrite (7.3), with a change only in the very last term, as:

$$
\begin{aligned}
R^{\mathrm{M}}{}_5 &= -\tfrac{1}{4}b\overline{\kappa}\left(g^{\mathrm{M}\beta}F^{\sigma}{}_{\beta,\sigma} + \Gamma^{\sigma}{}_{\tau\sigma}F^{\tau\mathrm{M}} - g^{\mathrm{M}\beta}\Gamma^{\tau}{}_{\beta\sigma}F^{\sigma}{}_{\tau} + \tfrac{1}{4}b\overline{\kappa}g^{\mathrm{M}5}F^{\sigma\tau}F_{\sigma\tau}\right) \\
&= -\tfrac{1}{4}b\overline{\kappa}F^{\Sigma\mathrm{M}}{}_{;\Sigma} = -\tfrac{1}{4}b\overline{\kappa}\left(F^{\sigma\mathrm{M}}{}_{;\sigma} + F^{5\mathrm{M}}{}_{;5}\right) \equiv -\tfrac{1}{4}b\overline{\kappa}J^{\mathrm{M}} \equiv -\tfrac{1}{4}b\overline{\kappa}\left(j^{\mathrm{M}} + j^{\mathrm{M}5}\right)
\end{aligned}
\qquad (7.7)
$$

Though seemingly-innocuous, this change to $j^{\mathrm{M}} + j^{\mathrm{M}5}$ may help to resolve the so-called "chirality problem" which plagues many Kaluza-Klein theories, and this resolution rests, in particular, upon the appearance of g^{MN} in (7.6), which we will wish to represent in a 5-D Clifford algebra. [11], section 3. However that is for separate, later consideration, in section 10 of this paper.

Returning to our present task, which is to calculate $R_{(5)}$ in two alternative ways, let's now separate (7.3) into two separate equations:

$$
\begin{aligned}
R^{\mu}{}_5 &= -\tfrac{1}{4}b\overline{\kappa}\left(j^{\mu} + j^{\mu5}\right) = -\tfrac{1}{4}b\overline{\kappa}\left(F^{\sigma\mu}{}_{;\sigma} + F^{5\mu}{}_{;5}\right) \\
&= -\tfrac{1}{4}b\overline{\kappa}\left(g^{\mu\beta}F^{\sigma}{}_{\beta,\sigma} + \Gamma^{\sigma}{}_{\tau\sigma}F^{\tau\mu} - g^{\mu\beta}\Gamma^{\tau}{}_{\beta\sigma}F^{\sigma}{}_{\tau} + \tfrac{1}{4}b\overline{\kappa}g^{\mu5}F^{\sigma\tau}F_{\sigma\tau}\right)
\end{aligned}
, \text{ and} \qquad (7.8)
$$

$$
\begin{aligned}
R^{5}{}_5 &= -\tfrac{1}{4}b\overline{\kappa}\left(j^{5} + j^{55}\right) = -\tfrac{1}{4}b\overline{\kappa}\left(F^{\sigma5}{}_{;\sigma} + F^{55}{}_{;5}\right) \\
&= -\tfrac{1}{4}b\overline{\kappa}\left(g^{5\beta}F^{\sigma}{}_{\beta,\sigma} - g^{5\beta}\Gamma^{\tau}{}_{\beta\sigma}F^{\sigma}{}_{\tau} + \tfrac{1}{4}b\overline{\kappa}g^{55}F^{\sigma\tau}F_{\sigma\tau}\right)
\end{aligned}
, \qquad (7.9)
$$

where we use $F^{\tau5} = 0$ to eliminate one term from (7.9). The foregoing contains four distinct current types, which, in contrast to (6.13), are specified by:

$$
\begin{cases}
j^{\mu} = F^{\sigma\mu}{}_{;\sigma} = g^{\mu\beta}F^{\sigma}{}_{\beta,\sigma} + \Gamma^{\sigma}{}_{\tau\sigma}F^{\tau\mu} - g^{\mu\beta}\Gamma^{\tau}{}_{\beta\sigma}F^{\sigma}{}_{\tau} \\
j^{5} = F^{\sigma5}{}_{;\sigma} = g^{5\beta}F^{\sigma}{}_{\beta,\sigma} - g^{5\beta}\Gamma^{\tau}{}_{\beta\sigma}F^{\sigma} \\
j^{\mu5} = F^{5\mu}{}_{;5} = \tfrac{1}{4}b\overline{\kappa}g^{\mu5}F^{\sigma\tau}F_{\sigma\tau} \\
j^{55} = F^{55}{}_{;5} = \tfrac{1}{4}b\overline{\kappa}g^{55}F^{\sigma\tau}F_{\sigma\tau}
\end{cases}
, \qquad (7.10)
$$

and in thinking about chirality, it is worth noting that $g^{\mu5}$, in a Clifford algebra, associates with $\tfrac{1}{2}\left\{\Gamma^{\mu},\Gamma^{5}\right\}$ in the chosen representation, and that this makes the current term $j^{\mu} + j^{\mu5}$ of particular interest.

So, now we can write out the five-dimensional curvature scalar $R_{(5)} = R + R^5{}_5$, leaving R as a remaining unknown still to be deduced. The first way to do this, from (7.9), rearranged as $R^5{}_5 = -\frac{1}{16}b^2\overline{\kappa}^2 g^{55}F^{\sigma\tau}F_{\sigma\tau} - \frac{1}{4}b\overline{\kappa}j^5$, is to write:

$$R_{(5)} = R - \frac{1}{16}b^2\overline{\kappa}^2 g^{55}F^{\sigma\tau}F_{\sigma\tau} - \frac{1}{4}b\overline{\kappa}j^5 . \tag{7.11}$$

That is the easy approach.

The second way to do this, more roundaboutly, is to multiply $R_{(5)} = R + R^5{}_5$ through into $g_{55}R_{(5)} = g_{55}R + g_{55}R^5{}_5$. Then, we use (7.2) to turn this into $g_{55}R_{(5)} = g_{55}R + R_{55} - g_{5\mu}R^\mu{}_5$. Then, we substitute $R_{55} = -\frac{1}{16}b^2\overline{\kappa}^2 F^{\sigma\tau}F_{\sigma\tau}$ and $R^\mu{}_5 = -\frac{1}{4}b\overline{\kappa}\left(j^\mu + \frac{1}{4}b\overline{\kappa}g^{\mu 5}F^{\sigma\tau}F_{\sigma\tau}\right)$ from (6.10) and (7.8) respectively, to write:

$$g_{55}R_{(5)} = g_{55}R + R_{55} - g_{5\sigma}R^\sigma{}_5 = g_{55}R - \frac{1}{16}b^2\overline{\kappa}^2 F^{\sigma\tau}F_{\sigma\tau} + \frac{1}{4}b\overline{\kappa}g_{5\mu}\left(j^\mu + \frac{1}{4}b\overline{\kappa}g^{\mu 5}F^{\sigma\tau}F_{\sigma\tau}\right) \tag{7.12}$$

This last expression can, however, be reduced using $g_{5\mu}g^{\mu 5} = 0$, see (5.8), down to:

$$g_{55}R_{(5)} = g_{55}R - \frac{1}{16}b^2\overline{\kappa}^2 F^{\sigma\tau}F_{\sigma\tau} + \frac{1}{4}b\overline{\kappa}g_{5\mu}j^\mu . \tag{7.13}$$

Keep in mind that $g_{55} = \pm 1$, depending on whether the fifth dimension is timelike or spacelike.

Equations (7.11) and (7.13) are totally-equivalent expressions, and they are each of interest in different circumstances. If we multiply (7.11) through by g_{55} and then equate this to (7.13), we find a new relationship of interest, $-g_{55}j^5 = g_{5\mu}j^\mu$, or, in five-covariant form:

$$g_{5\Sigma}j^\Sigma = 0 . \tag{7.14}$$

Equation (7.13), though more labor-intensive to derive, is of keen interest, because it appears to resemble the QED Lagrangian $\hbar c^2 \mathfrak{L}_{QED} = \left(-\frac{1}{4}F^{\sigma\tau}F_{\sigma\tau} - A_\mu j^\mu\right)$, and may provide a direct basis for geometrically representing $\mathfrak{L}_{QED}$. But to do so, we must make a suitable association between $g_{5\mu}$ and A_μ to use in the term $g_{5\mu}j^\mu$. So, in the next section, we will explicitly explore the connection between the gravitational potentials $g_{5\mu}$ and the electrodynamic potentials A_μ.

Further, as Witten points out, ([13] at pg. 28) the vector potential A^μ is essential to the quantum mechanical treatment of electromagnetism quantum-mechanically.

However, first, it behooves us to calculate the axial components of Einstein's equation generalized to five dimensions, $-\kappa T^{\mathrm{M}}{}_{\mathrm{N}} = R^{\mathrm{M}}{}_{\mathrm{N}} - \frac{1}{2}\delta^{\mathrm{M}}{}_{\mathrm{N}} R_{(5)}$, and here, (7.11) is the preferred expression. For this task, we use (7.7) and (7.11) to write:

$$-\kappa T^{\mathrm{M}}{}_5 = R^{\mathrm{M}}{}_5 - \tfrac{1}{2}\delta^{\mathrm{M}}{}_5 R_{(5)} = -\tfrac{1}{4} b\overline{\kappa}\left(j^{\mathrm{M}} + \tfrac{1}{4} b\overline{\kappa} g^{\mathrm{M}5} F^{\sigma\tau} F_{\sigma\tau}\right) - \tfrac{1}{2}\delta^{\mathrm{M}}{}_5\left(R - \tfrac{1}{16} b^2 \overline{\kappa}^2 g^{55} F^{\sigma\tau} F_{\sigma\tau} - \tfrac{1}{4} b\overline{\kappa} j^5\right). \quad (7.15)$$

This splits into two equations:

$$-\kappa T^{\mu}{}_5 = R^{\mu}{}_5 = -\tfrac{1}{4} b\overline{\kappa}\left(j^\mu + j^{5\mu}\right) = -\tfrac{1}{4} b\overline{\kappa}\left(j^\mu + \tfrac{1}{4} b\overline{\kappa} g^{\mu 5} F^{\sigma\tau} F_{\sigma\tau}\right); \text{ and} \quad (7.16)$$

$$-\kappa T^5{}_5 = R^5{}_5 - \tfrac{1}{2} R_{(5)} = -\tfrac{1}{8} b\overline{\kappa}\left(j^5 + j^{55}\right) - \tfrac{1}{2} R = -\tfrac{1}{8} b\overline{\kappa}\left(j^5 + \tfrac{1}{4} b\overline{\kappa} g^{55} F^{\sigma\tau} F_{\sigma\tau}\right) - \tfrac{1}{2} R, \quad (7.17)$$

where we have also employed (7.10). The four-dimensional Ricci scalar R is still an unknown in (7.17) and elsewhere. Deducing $R = R^\sigma{}_\sigma$, is part of the work cut out for the next section.

8. The Vector Potential, the Gravitational Potential, and the Exact QED Lagrangian

We now formally introduce the four-vector potential $A^\mu \equiv (\phi, A_1, A_2, A_3)$, related to the field strength tensor according to $F^{\mu\nu} = A^{\mu;\nu} - A^{\nu;\mu} = A^{\mu,\nu} - A^{\nu,\mu}$, where the covariant derivatives become ordinary derivatives in the particular combination used to form $F^{\mu\nu}$.

Once again, we start with (5.1), written out as (recall $g_{\Sigma\mathrm{T},5} = 0$, see (5.6)):

$$\tfrac{1}{4} b\overline{\kappa} F^{\mathrm{M}}{}_{\mathrm{T}} = \Gamma^{\mathrm{M}}{}_{\mathrm{T}5} = \tfrac{1}{2} g^{\mathrm{MA}}\left(g_{\mathrm{AT},5} + g_{5\mathrm{A},\mathrm{T}} - g_{\mathrm{T}5,\mathrm{A}}\right) = \tfrac{1}{2} g^{\mathrm{MA}}\left(g_{5\mathrm{A},\mathrm{T}} - g_{5\mathrm{T},\mathrm{A}}\right). \quad (8.1)$$

It is helpful to lower the indexes in field strength tensor and connect this to the covariant vector potentials A_μ, generalized into 5-dimensions as A_{M} via $F_{\Sigma\mathrm{T}} \equiv A_{\Sigma;\mathrm{T}} - A_{\mathrm{T};\Sigma} = A_{\Sigma,\mathrm{T}} - A_{\mathrm{T},\Sigma}$, as such:

$$\tfrac{1}{4} b\overline{\kappa}\left(A_{\Sigma;\mathrm{T}} - A_{\mathrm{T};\Sigma}\right) = \tfrac{1}{4} b\overline{\kappa} F_{\Sigma\mathrm{T}} = \tfrac{1}{4} b\overline{\kappa} g_{\Sigma\mathrm{M}} F^{\mathrm{M}}{}_{\mathrm{T}} = \tfrac{1}{2} g_{\Sigma\mathrm{M}} g^{\mathrm{MA}}\left(g_{5\mathrm{A},\mathrm{T}} - g_{5\mathrm{T},\mathrm{A}}\right) = \tfrac{1}{2}\left(g_{5\Sigma,\mathrm{T}} - g_{5\mathrm{T},\Sigma}\right). \quad (8.2)$$

The relationship $\tfrac{1}{4} b\overline{\kappa} F_{\Sigma\mathrm{T}} = \tfrac{1}{4} b\overline{\kappa}\left(A_{\Sigma;\mathrm{T}} - A_{\mathrm{T};\Sigma}\right) = \tfrac{1}{2}\left(g_{5\Sigma,\mathrm{T}} - g_{5\mathrm{T},\Sigma}\right)$ expresses clearly, the antisymmetry of $F_{\Sigma\mathrm{T}}$ in terms of the remaining connection terms involving the gravitational potential. Of particular interest, is that we may extract from (8.2), the relation:

$$\tfrac{1}{4}b\overline{\kappa}A_{\Sigma;\mathrm{T}} = \tfrac{1}{2}g_{5\Sigma,\mathrm{T}} = \tfrac{1}{8}b\overline{\kappa}h_{5\Sigma,\mathrm{T}}, \tag{8.3}$$

using also $g_{\mathrm{MN}} = \eta_{\mathrm{MN}} + \overline{\kappa}h_{\mathrm{MN}}$ for the gravitational potential energy h_{MN}. This makes perfect sense: after all, the oft-employed $\Gamma^{\mathrm{M}}{}_{5\Sigma} = \tfrac{1}{4}b\overline{\kappa}F^{\mathrm{M}}{}_{\Sigma}$ of (5.1) is simply a first order differential equation between the vector potential and the gravitational field, and each is a dynamical field. If one forms $A_{\Sigma;\mathrm{T}} - A_{\mathrm{T};\Sigma}$ from (8.3) and then renames indexes and uses $g_{\mathrm{MN}} = g_{\mathrm{NM}}$, one arrives back at (8.2). So (8.3) is just the explicit form of that equation. The reason we did not remove the covariant derivative via $F_{\Sigma\mathrm{T}} \equiv A_{\Sigma;\mathrm{T}} - A_{\mathrm{T};\Sigma} = A_{\Sigma,\mathrm{T}} - A_{\mathrm{T},\Sigma}$, is that in (8.3), $A_{\Sigma;\mathrm{T}}$ is considered distinctly from $-A_{\mathrm{T};\Sigma}$, and so the covariant derivatives do not become ordinary unless and until one forms $F_{\Sigma\mathrm{T}} \equiv A_{\Sigma;\mathrm{T}} - A_{\mathrm{T};\Sigma} = A_{\Sigma,\mathrm{T}} - A_{\mathrm{T},\Sigma}$ and must, in (8.3), be left intact.

Equation (8.3) is a first order differential equation which tells us that the *covariant* derivative of the electrodynamic potential A_{Σ} is equal to the *ordinary* derivative of the gravitational potential $h_{5\Sigma}$. In the weak field limit, where covariant derivatives become *approximately* equal to ordinary derivatives, $\tfrac{1}{4}b\overline{\kappa}A_{\Sigma;\mathrm{T}} = \tfrac{1}{2}g_{5\Sigma,\mathrm{T}} = \tfrac{1}{8}b\overline{\kappa}h_{5\Sigma,\mathrm{T}} \approx \tfrac{1}{4}b\overline{\kappa}A_{\Sigma,\mathrm{T}}$, and so, integrating based on this approximation, we obtain:

$$g_{5\Sigma} = \eta_{5\Sigma} + \overline{\kappa}h_{5\Sigma} \approx \tfrac{1}{2}b\overline{\kappa}A_{\Sigma}. \tag{8.4}$$

Keep in mind, (8.3) is exact; (8.4) only applies to the weak-field approximation $A_{\Sigma;\mathrm{T}} \approx A_{\Sigma,\mathrm{T}}$.

Now, we return to examine (7.13) *in this weak field limit*, $A_{\Sigma;\mathrm{T}} \approx A_{\Sigma,\mathrm{T}}$. Most importantly, referring to (8.4), the final term in (7.13) becomes $\overline{\kappa}g_{5\mu}j^{\mu} \approx 2\overline{\kappa}^{2}A_{\mu}j^{\mu}$. Thus, substituting from (8.4) into (7.13) yields: $\tfrac{1}{4}b\overline{\kappa}g_{5\mu}j^{\mu} \approx \tfrac{1}{8}b^{2}\overline{\kappa}^{2}A_{\mu}j^{\mu}$

$$g_{55}R_{(5)} \approx g_{55}R + \tfrac{1}{16}b^{2}\overline{\kappa}^{2}\left(-F^{\sigma\tau}F_{\sigma\tau} + 2A_{\mu}j^{\mu}\right). \tag{8.5}$$

Our goal now it to connect this weak field approximation to the QED Lagrangian density $\hbar c^{2}\mathfrak{L}_{QED} = \left(-\tfrac{1}{4}F^{\sigma\tau}F_{\sigma\tau} - A_{\mu}j^{\mu}\right)$, and then to generalize back out of the weak field limit to the non-linear theory for fields of any strength.

First, we seek to deduce the four-dimensional Ricci scalar $R = R^{\sigma}{}_{\sigma}$ which is still an unknown. Because (8.5) contains $g_{55}R_{(5)}$, the exact expression for $R_{(5)}$ depends upon whether $g_{55} = +1$ (timelike) or $g_{55} = -1$, spacelike.

Using $g_{55} = \pm 1$ to consolidate the timelike and spacelike results into a single expression, (8.5) becomes:

$$g_{55}R_{(5)} \approx g_{55}R + \tfrac{1}{16} b^2 \overline{\kappa}^2 \left(-F^{\sigma\tau}F_{\sigma\tau} + 2A_{\mu} j^{\mu} \right). \tag{8.6}$$

Now, we are at a juncture: Up until this point, all of the development has been based on a single supposition introduced just after (2.6): the requirement that the Lorentz force must be represented as nothing other than geodesic motion in a five-dimensional geometry, as expressed in (2.7) and (2.8). Other than perhaps our imposing the requirement that $F^{\mathrm{MN}} \equiv -F^{\mathrm{NM}}$, every step taken since then has been fully deductive, with no other assumptions. We have even left open the question of whether the fifth dimension is timelike or spacelike, simply exploring the consequences in the alternative, as pertinent. This has enabled us to place Maxwell's equations, deductively, on a fully geometric footing, fully specify the axial components of the energy tensor $T^{\mathrm{M}}{}_5$, and obtain the five dimensional Ricci scalar $R_{(5)}$, but only up to the four-dimensional scalar $R = R^{\sigma}{}_{\sigma}$, which still stands out as undetermined. Determining R, would give us a window into $R^{\mu}{}_{\nu}$, and this in turn into the remaining $T^{\mu}{}_{\nu}$ components, among which, one would expect to find the Maxwell stress energy tensor. So, we need to find R. To deduce R, we must now, finally, make a new supposition beyond that of Lorentz force geodesics, which we do as follows:

Many authors write the QED Lagrangian density as $\hbar c^2 \mathfrak{L}_{QED} = \left(-\tfrac{1}{4} F^{\sigma\tau}F_{\sigma\tau} - A_{\mu} j^{\mu} \right)$ (with $\hbar = c = 1$). However, by rescaling the sign of the source current density, it is equally proper to use the convention $\hbar c^2 \mathfrak{L}_{QED} = \left(-\tfrac{1}{4} F^{\sigma\tau}F_{\sigma\tau} + A_{\mu} j^{\mu} \right)$, see, e.g., [14], page 30. The key feature of either form of the Lagrangian density, is the $-1 : \pm 4$ ratio between the constant factors multiplying $F^{\sigma\tau}F_{\sigma\tau}$ and $A_{\mu} j^{\mu}$. By virtue of the opposite signs as between $F^{\sigma\tau}F_{\sigma\tau}$ and $2A_{\mu} j^{\mu}$ in (8.6) and (8.7), and given that there is no choice of the constant factors back in (3.2) and (3.3) which would have reversed this, we shall use $\hbar c^2 \mathfrak{L}_{QED} = \left(-\tfrac{1}{4} F^{\sigma\tau}F_{\sigma\tau} + A_{\mu} j^{\mu} \right)$, the latter convention, to write $\mathfrak{L}_{QED}$. Nor would any choice, by the way, have altered the ratio of $-1:2$ between the constant factors multiplying $F^{\sigma\tau}F_{\sigma\tau}$ and $A_\mu j^\mu$, into the $-1:4$ ratio in $\mathfrak{L}_{QED}$. We will need to rely on the still-undetermined R to properly fix this ratio.

Now, using this $\hbar c^2 \mathfrak{L}_{QED} = \left(-\frac{1}{4}F^{\sigma\tau}F_{\sigma\tau} + A_\mu j^\mu\right)$, the action is formed according to $S(A_\mu) = \int \mathfrak{L}_{QED}\sqrt{-g}\,d^4x$. If, however, we can turn (8.6) and (8.7) into expressions in which the ratio of the constant factors multiplying $F^{\sigma\tau}F_{\sigma\tau}$ and $A_\mu j^\mu$ is $-1:4$ rather than $-1:2$, then we could use these expressions to write QED in terms of a gravitational action, *in vacuo*, of the form $S = \int kR dV$. Because $R = R^\sigma{}_\sigma$ is still an unknown, we shall now use all of the foregoing observations to deduce $R = R^\sigma{}_\sigma$ as such:

We shall deduce $R = R^\sigma{}_\sigma$ in (8.6) and (8.7) <u>such that</u> the ratio of the constant factors multiplying $F^{\sigma\tau}F_{\sigma\tau}$ and $A_\mu j^\mu$ changes from $-1:2$, to $-1:4$, and also, <u>such that</u> R only contains $F^{\sigma\tau}F_{\sigma\tau}$, and not $A_\mu j^\mu$. Less technocratically, the requirement we are about to impose means that we shall require the full five-dimensional curvature scalar $R_{(5)}$ to itself be identical with the QED Lagrangian density, up to a constant factor. We may impose this requirements by rewriting (8.6) and (8.7) as:

$$R_{(5)} \approx R + \tfrac{1}{16}b^2\overline{\kappa}^2\left(-F^{\sigma\tau}F_{\sigma\tau} + 2A_\mu j^\mu\right) \equiv \tfrac{1}{16}b^2\overline{\kappa}^2\left(-\tfrac{1}{2}F^{\sigma\tau}F_{\sigma\tau} + 2A_\mu j^\mu\right), \text{ and} \tag{8.7}$$

$$-R_{(5)} \approx -R + \tfrac{1}{16}b^2\overline{\kappa}^2\left(-F^{\sigma\tau}F_{\sigma\tau} + 2A_\mu j^\mu\right) \equiv \tfrac{1}{16}b^2\overline{\kappa}^2\left(-\tfrac{1}{2}F^{\sigma\tau}F_{\sigma\tau} + 2A_\mu j^\mu\right). \tag{8.8}$$

These amount to a *definition* of R. Again, these are affirmative requirements, not deductions. The reader may wish to explore this same juncture with a different set of suppositions, which would then yield a different expression for R.

It is then easy to deduce from these, respectively, also using $\frac{\kappa}{\hbar c} = \frac{1}{2}\overline{\kappa}^2 = \frac{8\pi G}{\hbar c^5}$, and noting that the trace of Einstein's equation is $\kappa T = \kappa T^\sigma{}_\sigma = R = R^\sigma{}_\sigma$, that:

$$g_{55}\kappa T = g_{55}R = \tfrac{1}{32}b^2\overline{\kappa}^2 F^{\sigma\tau}F_{\sigma\tau} = \tfrac{1}{16}b^2\frac{\kappa}{\hbar c}F^{\sigma\tau}F_{\sigma\tau}, \tag{8.9}$$

where $g_{55} = \pm 1$ for a timelike (+) and spacelike (-) fifth dimension, respectively. The choice of timelike versus spacelike, merely flips the sign of the (four-dimensional) Ricci scalar. In units of $\hbar = c = 1$, is simply $g_{55} T = \frac{1}{16} b^2 F^{\sigma\tau} F_{\sigma\tau}$.

With $R = R^{\sigma}{}_{\sigma}$ now established, we can use $g_{55} = \pm 1$ to consolidate the five-dimensional Ricci scalars in (8.7) and (8.8), respectively, as:

$$g_{55} R_{(5)} \approx \frac{1}{16} b^2 \overline{\kappa}^2 \left(-\frac{1}{2} F^{\sigma\tau} F_{\sigma\tau} + 2 A_\mu j^\mu \right) = \frac{1}{4} b^2 \frac{\kappa}{\hbar c} \left(-\frac{1}{4} F^{\sigma\tau} F_{\sigma\tau} + A_\mu j^\mu \right) = \frac{1}{4} b^2 \kappa c \mathfrak{L}_{QED} . \tag{8.10}$$

The respective timelike versus spacelike choice also flips the sign for $R_{(5)}$.

Now, with (8.10), we may rewrite the QED action $S(A_\mu) = \int \mathfrak{L}_{QED} \sqrt{-g}\, d^4 x$ as:

$$S(A_\mu) = \int \mathfrak{L}_{QED} \sqrt{-g}\, d^4 x \approx g_{55} \frac{4}{b^2 \kappa c} \int R_{(5)} \sqrt{-g}\, d^4 x . \tag{8.11}$$

Finally, we are in a position to fix b. As stated in the introduction, for compatibility with Einstein's equation, we require (8.11) to be in the form of the Einstein-Hilbert action $S = \frac{1}{2\kappa c} \int R dV$, with a $c = 1$ restored. Comparing, $\frac{1}{2\kappa c} = \frac{4}{b^2 \kappa c}$, and so, at long last, we find:

$$b^2 = 8 . \tag{8.12}$$

This is the Lagrangian (action) *for the vacuum*, because it does not contain any explicit matter terms, but only contains $R_{(5)}$. We can put this into words by saying that the QED action is equal to the five-dimensional Ricci scalar, integrated over the four-volume of spacetime. A Ricci scalar derived from all five dimensions, integrated over only the four dimensions of ordinary spacetime, results in Quantum Electrodynamics. QED is the four-dimensional manifestation of a five-dimensional universe! This achieves the goal set out in the introduction, of generating QED out of an *in vacuo* action of the general form $S = \frac{1}{2\kappa} \int R dV$, and (8.11) is the explicit, "suitable" form of this action.

Making a quick trip back to (4.4), this means that if the fifth dimension is spacelike, that the exact dimension for the compactification radius is given by:

$$R = \sqrt{\frac{2}{a}} \sqrt{\frac{G\hbar}{c^3}} = \sqrt{\frac{2}{a}} L_P , \tag{8.13}$$

Keep in mind, however, that is a *weak-field* limit, because it is based on the approximation $A_{\Sigma;\mathrm{T}} \approx A_{\Sigma,\mathrm{T}}$, hence $g_{5\Sigma} = \overline{\kappa} h_{5\Sigma} \approx \frac{1}{2} b \overline{\kappa} A_\Sigma$, see (8.4). If one thinks carefully about this approximation, it becomes clear that the term $g_{5\Sigma}$, and *not* A_Σ, is to be is associated with the *exact* $\mathfrak{L}_{QED}$. And, in this light, it should become clear that *the usual* $\mathfrak{L}_{QED}$ *is itself the weak field limit*, that is, if $\mathfrak{L}_{QED}$ is to designate the *exact* QED Lagrangian no matter how strong the fields, then $\hbar c^2 \mathfrak{L}_{QED} \approx \left(-\frac{1}{4} F^{\sigma\tau} F_{\sigma\tau} + A_\mu j^\mu\right)$ for weak fields. If we carefully backtrack, we can now deduce find the exact $\mathfrak{L}_{QED}$, for all field strengths, as follows:

First, since $g_{55} = g^{55} = \pm 1$, for a timelike (+) and spacelike (-) fifth dimension, respectively, we can rewrite (8.9) with our newly-deduced $b^2 = 8$ as $R = \frac{1}{4} \overline{\kappa}^2 g_{55} F^{\sigma\tau} F_{\sigma\tau}$. Now, we employ this expression as well as $b^2 = 8$ in (7.13), to obtain:

$$g_{55} R_{(5)} = -\tfrac{1}{4} \overline{\kappa}^2 F^{\sigma\tau} F_{\sigma\tau} + \tfrac{\sqrt{2}}{2} \overline{\kappa} g_{5\mu} j^\mu = -\tfrac{1}{4} \overline{\kappa}^2 F^{\sigma\tau} F_{\sigma\tau} - \tfrac{\sqrt{2}}{2} \overline{\kappa} g_{55} j^5 . \tag{8.14}$$

where we have also employed (7.14).

Next, we return to (8.10). The terms with $A_\mu j^\mu$ were based on the $A_{\Sigma;\mathrm{T}} \approx A_{\Sigma,\mathrm{T}}$ approximation. The exact relation in (8.10), is that given geometrically by $2\kappa c \mathfrak{L}_{QED} = g_{55} R_{(5)}$, with $b^2 = 8$. Converting back via $\kappa = \frac{1}{2} \hbar c \overline{\kappa}^2 = 8\pi G/c^4$, and using (8.14) with some term manipulation, we can now write the *exact* QED Lagrangian for all fields weak and strong, as:

$$\hbar c^2 \overline{\kappa}^2 \mathfrak{L}_{QED} = -\tfrac{1}{4} \overline{\kappa}^2 F^{\sigma\tau} F_{\sigma\tau} + \tfrac{\sqrt{2}}{2} \overline{\kappa} g_{5\mu} j^\mu = -\tfrac{1}{4} \overline{\kappa}^2 F^{\sigma\tau} F_{\sigma\tau} - \tfrac{\sqrt{2}}{2} \overline{\kappa} g_{55} j^5 = g_{55} R_{(5)} \tag{8.15}$$

This is the exact expression for $\mathfrak{L}_{QED}$*!* In the weak field limit, where $A_{\Sigma;\mathrm{T}} \approx A_{\Sigma,\mathrm{T}}$, we may make the approximate substitution $g_{5\Sigma} \approx \sqrt{2} \overline{\kappa} A_\Sigma$ of (8.4) with $b^2 = 8$ into the above, as such:

$$\hbar c^2 \overline{\kappa}^2 \mathfrak{L}_{QED} = -\tfrac{1}{4} \overline{\kappa}^2 F^{\sigma\tau} F_{\sigma\tau} + \tfrac{\sqrt{2}}{2} \overline{\kappa} g_{5\mu} j^\mu \approx -\tfrac{1}{4} \overline{\kappa}^2 F^{\sigma\tau} F_{\sigma\tau} + \overline{\kappa}^2 A_\mu j^\mu \tag{8.16}$$

which recovers the customary, familiar QED Lagrangian.

Now lets return to the action (8.11). We now understand that this contains an approximation symbol, *not because of the term with* $R_{(5)}$, but because the $\mathfrak{L}_{QED}$ was taken to be the weak field $\hbar c^2 \mathfrak{L}_{QED} \approx \left(-\frac{1}{4}F^{\sigma\tau}F_{\sigma\tau} + A_\mu j^\mu\right)$. If we now use (8.16), then the action (8.11) now can be rewritten in an exact expression, with $\hbar = c = 1$, namely:

$$
\begin{aligned}
S(A_\mu) &= g_{55}\frac{1}{2\overline{\kappa}}\int R_{(5)}\sqrt{-g}\,d^4x = \int \mathfrak{L}_{QED}\sqrt{-g}\,d^4x \\
&= \int\left(-\frac{1}{4}F^{\sigma\tau}F_{\sigma\tau} + \frac{\sqrt{2}}{2\overline{\kappa}}g_{5\mu}j^\mu\right)\sqrt{-g}\,d^4x \approx \int\left(-\frac{1}{4}F^{\sigma\tau}F_{\sigma\tau} + A_\mu j^\mu\right)\sqrt{-g}\,d^4x
\end{aligned}
\quad (8.17)
$$

where in the final set of terms, we have employed the weak field $g_{5\Sigma} \approx \sqrt{2\overline{\kappa}}A_\Sigma$. The $g_{55} = \pm 1$ simply represents the sign to be used depending on whether the fifth dimension is timelike or spacelike. Once that choice is, made, this is either a plus or a minus sign.

Finally, one may also wish to fashion this action completely as a function of the g_{MN}, that is, to write $S(g_{\mathrm{MN}})$. One may thereby replace $F^{\sigma\tau}F_{\sigma\tau}$ and $j^\mu = F^{\sigma\mu}{}_{;\sigma}$ with the Christoffels with which they are related, particularly using (5.1) and (6.3), (6.4). The net result of this is:

$$
S(g_{\mathrm{MN}}) = g_{55}\frac{1}{2\overline{\kappa}}\int R_{(5)}\sqrt{-g}\,d^4x = \int\left(-\frac{1}{4\overline{\kappa}}g_{\sigma\beta}g^{\tau\alpha}\Gamma^\sigma{}_{5\alpha}\Gamma^\beta{}_{5\tau} - \frac{1}{2\overline{\kappa}}\Gamma^\sigma{}_{5\alpha}\Gamma^\alpha{}_{5\sigma}\right)\sqrt{-g}\,d^4x\,. \quad (8.18)
$$

This is the exact, non-linear action for electrodynamics, for fields of any strength, written wholly as a gravitational entity.

9. Electrodynamic Energy Tensors, including the Maxwell Stress Energy

Now, let consider the ordinary components $T^\mu{}_\nu$ of the energy tensor, and see how they interrelate with the $T^{\mathrm{M}}{}_5$ components found in (7.15). First, as of equation (7.15), we had not yet found R or b, and therefore did not have a complete expression for $R_{(5)}$. Therefore, as a housekeeping matter, we insert (8.9), rewritten as $R = \frac{1}{4}\overline{\kappa}^2 g^{55}F^{\sigma\tau}F_{\sigma\tau}$, into (7.15) to yield:

$$
-\kappa T^{\mathrm{M}}{}_5 = R^{\mathrm{M}}{}_5 - \tfrac{1}{2}\delta^{\mathrm{M}}{}_5 R_{(5)} = -\tfrac{\sqrt{2}}{2}\overline{\kappa}\left(j^{\mathrm{M}} + \tfrac{\sqrt{2}}{2}\overline{\kappa}g^{\mathrm{M}5}F^{\sigma\tau}F_{\sigma\tau}\right) - \tfrac{1}{2}\delta^{\mathrm{M}}{}_5\left(-\tfrac{1}{4}\overline{\kappa}^2 g^{55}F^{\sigma\tau}F_{\sigma\tau} - \tfrac{\sqrt{2}}{2}\overline{\kappa}j^5\right). \quad (9.1)
$$

Also using (7.6) which is now $j^{\mathrm{MN}} \equiv \frac{\sqrt{2}}{2}\overline{\kappa}g^{\mathrm{MN}}F^{\sigma\tau}F_{\sigma\tau}$, this splits into:

$$
-\kappa T^\mu{}_5 = R^\mu{}_5 = -\tfrac{\sqrt{2}}{2}\overline{\kappa}\left(j^\mu + \tfrac{\sqrt{2}}{2}\overline{\kappa}g^{\mu 5}F^{\sigma\tau}F_{\sigma\tau}\right) = -\tfrac{\sqrt{2}}{2}\overline{\kappa}\left(j^\mu + j^{\mu 5}\right). \quad (9.2)
$$

$$-\kappa T^5{}_5 = R^5{}_5 - \tfrac{1}{2}R_{(5)} = -\tfrac{\sqrt{2}}{2}\bar{\kappa}\left(\tfrac{1}{2}j^5 + \tfrac{3}{4}\tfrac{\sqrt{2}}{2}\bar{\kappa}g^{55}F^{\sigma\tau}F_{\sigma\tau}\right) = -\tfrac{\sqrt{2}}{2}\bar{\kappa}\left(\tfrac{1}{2}j^5 + \tfrac{3}{4}j^{55}\right). \tag{9.3}$$

Now, let turn to $T^\mu{}_\nu$. From (8.9), we know that $\kappa T = R = \tfrac{1}{4}\bar{\kappa}^2 g^{55}F^{\sigma\tau}F_{\sigma\tau}$. From (9.3), we can use this to derive the five-dimensional trace energy:

$$\kappa T_{(5)} \equiv \kappa\left(T + T^5{}_5\right) = \tfrac{5}{8}\bar{\kappa}^2 g^{55}F^{\sigma\tau}F_{\sigma\tau} + \tfrac{1}{2}\tfrac{\sqrt{2}}{2}\bar{\kappa}j^5 = \tfrac{\sqrt{2}}{2}\bar{\kappa}\left(\tfrac{1}{2}j^5 + \tfrac{5}{4}j^{\mathrm{MN}}\right). \tag{9.4}$$

Additionally, $R = \tfrac{1}{4}\bar{\kappa}^2 g^{55}F^{\sigma\tau}F_{\sigma\tau}$ is a familiar trace expression, and we know that it is compatible with a Ricci tensor:

$$R^\mu{}_\nu = \tfrac{1}{4}\bar{\kappa}^2 g^{55}F^{\mu\tau}F_{\nu\tau}. \tag{9.5}$$

Of course, there are other possibilities, for example, $R^\mu{}_\nu = \delta^\mu{}_\nu R = \tfrac{1}{4}\bar{\kappa}^2 g^{55}\delta^\mu{}_\nu F^{\sigma\tau}F_{\sigma\tau}$, and the reader may wish to explore these, but (9.5), while not uniquely determined from R, certainly is a sensible choice consistent with our knowledge of electrodynamics.

With this choice of $R^\mu{}_\nu$ compatible with R we may now combine $R^\mu{}_\nu$ and R together, making use of $\kappa = \tfrac{1}{2}\hbar c\bar{\kappa}^2$, and with $\hbar = c = 1$, to obtain:

$$-\kappa T^\mu{}_\nu = R^\mu{}_\nu - \tfrac{1}{2}\delta^\mu{}_\nu R = \tfrac{1}{2}\kappa g^{55}F^{\mu\tau}F_{\nu\tau} - \tfrac{1}{4}\kappa\delta^\mu{}_\nu g^{55}F^{\sigma\tau}F_{\sigma\tau}, \tag{9.6}$$

which easily reduces to:

$$T^\mu{}_\nu = -g^{55}\left(\tfrac{1}{2}F^{\mu\tau}F_{\nu\tau} - \tfrac{1}{4}\delta^\mu{}_\nu F^{\sigma\tau}F_{\sigma\tau}\right), \tag{9.7}$$

where $g^{55} = \pm 1$ in this context simply switches the sign based on whether the fifth dimension is timelike or spacelike.

Now, we know that Maxwell's stress energy tensor in Heaviside-Lorentz units is given by $T^\mu{}_{\nu\,Maxwell} = -\left(F^{\mu\tau}F_{\nu\tau} - \tfrac{1}{4}\delta^\mu{}_\nu F^{\sigma\tau}F_{\sigma\tau}\right)$, and is traceless, so this means that:

$$T^\mu{}_\nu = g_{55}\left(T^\mu{}_{\nu\,Maxwell} + \tfrac{1}{2}F^{\mu\tau}F_{\nu\tau}\right). \tag{9.8}$$

10. The "Chirality Problem"

One of the most important connections in all of physics is given by the Dirac relationship:

$$\tfrac{1}{2}\{\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu\}\equiv\eta^{\mu\nu}, \tag{10.1}$$

whereby the Dirac γ^μ matrices, are *defined* so as to reproduce the Minkowski metric tensor $\mathrm{diag}(\eta^{\mu\nu})=(+1,-1,-1,-1)$ under anticommutation. This relationship not only underlies Dirac's equation, but also ensures that the Klein-Gordon equation applies to fermions as well as bosons.

A fifth matrix, the axial Dirac matrix first motivated by Weyl:

$$\gamma^5\equiv i\gamma^0\gamma^1\gamma^2\gamma^3, \tag{10.2}$$

is *defined* from matrix-multiplying the other four Dirac matrices, and has a well-established and rigorously-observed physical meaning in relation to the left- and right-chiral handedness of elementary fermions, and which is used in the chiral projection operators $P_R\equiv\frac{1}{2}(1+\gamma^5)$ and $P_L\equiv\frac{1}{2}(1-\gamma^5)$.

In a five-dimensional spacetime, such that that which has been explored here, we need to similarly choose a representation of the Clifford Algebra so as to produce η^{MN}. In keeping with our desire to consider a timelike fifth dimension alongside of a spacelike fifth dimension which as we have seen is vastly aided by our finding that $g^{55}=\pm1=\text{constant}$ for timelike versus spacelike character respectively, we may, for example, choose:

$$\Gamma^{\mathrm{M}}=\left(\gamma^\mu,-\sqrt{g^{55}}\gamma^5\right) \tag{10.3}$$

which, it will be noted, will yield the correct metric signature when employed in:

$$\tfrac{1}{2}\{\Gamma^{\mathrm{M}}\Gamma^{\mathrm{N}}+\Gamma^{\mathrm{N}}\Gamma^{\mathrm{M}}\}\equiv\eta^{\mathrm{MN}} \tag{10.4}$$

for either a timelike or a spacelike fifth dimension. (See [11], section 3)

However, as soon as one chooses a Clifford algebra such as (10.4), the γ^5 becomes part of the Dirac algebra, and one needs, it is thought, to obtain yet another matrix – a multiplicative combination of all five of the Γ^{M} – which can be used as a chiral projection operator. But, there is nothing left. For a timelike fifth dimension, $g_{55}=1$, $\Gamma^5=\gamma^5$, and so the axial matrix (call it γ^6 just for discussion) would be defined by $\gamma^6\equiv\gamma^0\gamma^1\gamma^2\gamma^3\gamma^5=-i$, so that $P_R\equiv\frac{1}{2}(1-i)$, while

$P_L \equiv \frac{1}{2}(1+i)$, or vice versa if $\gamma^6 \equiv -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^5 = +i$. For a spacelike fifth dimension, the factor of i moves over, so that $\gamma^6 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^5 = +1$, and then $P_R \equiv \frac{1}{2}(1+\gamma^6) = 1$ and $P_L \equiv \frac{1}{2}(1-\gamma^6) = 0$ or vice versa if we define $\gamma^6 \equiv -i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^5 = -1$. In all events, this is a disaster if one is looking for a good projection operator. [15]

Against this backdrop, let us return to a closer look at the symmetric second-rank current which naturally emerged, and was defined in (7.6) as $j^{\mathrm{MN}} \equiv \frac{\sqrt{2}}{2}\overline{\kappa} g^{\mathrm{MN}} F^{\sigma\tau} F_{\sigma\tau}$, including the $b^2 = 8$ determined in (8.12). Let us now return to (7.6) an make a slight modification to the definition of j^{MN}, and instead define it as:

$$\sqrt{g^{55}}\, j^{\mathrm{MN}} \equiv \frac{\sqrt{2}}{2}\overline{\kappa} g^{\mathrm{MN}} F^{\sigma\tau} F_{\sigma\tau}. \tag{10.5}$$

which is no change for a timelike fifth dimension and which places an i multiplier for a spacelike fifth dimension. The, let's return to take a loser look at the four-vector $R^{\mu}{}_5$ in (7.8), which we reproduce below with the new definition (10.5) and $b^2 = 8$, and to keep things simple, with $g_{\mu\nu} = \eta_{\mu\nu}$.

$$R^{\mu}{}_5 = -\frac{\sqrt{2}}{2}\overline{\kappa} J^{\mu} = -\frac{\sqrt{2}}{2}\overline{\kappa}\left(j^{\mu} + \sqrt{g^{55}}\, j^{\mu 5}\right) = -\frac{\sqrt{2}}{2}\overline{\kappa} j^{\mu} - \frac{1}{2}\overline{\kappa}^2 \eta^{\mu 5} F^{\sigma\tau} F_{\sigma\tau}. \tag{10.6}$$

We now seek a way to *define* a four-component Dirac wavefunction ψ and its adjoint such that $j^{\mu} + \sqrt{g^{55}}\, j^{\mu 5}$ in (10.6) will represent a chiral electric current $J^{\mu} = \overline{\psi}_R \gamma^{\mu} \psi_R + \overline{\psi}_L \gamma^{\mu} \psi_L$, with proper projection operators, independently of whether the fifth dimension is timelike or spacelike.

Now, from (10.3) and (10.4), we can obtain $\eta^{\mu 5} = -\sqrt{g^{55}}\,\frac{1}{2}\left\{\gamma^{\mu}\gamma^5 + \gamma^5\gamma^{\mu}\right\}$. We use this in (10.5) to *define* a the Dirac wavefunction ψ and its adjoint $\overline{\psi} = \psi^{\dagger}\gamma^0$ according to:

$$\sqrt{g^{55}}\, j^{\mu 5} = \frac{\sqrt{2}}{2}\overline{\kappa}\eta^{\mu 5} F^{\sigma\tau} F_{\sigma\tau} \equiv \sqrt{g^{55}}\,\overline{\psi}\eta^{\mu 5}\psi = -g^{55}\overline{\psi}\frac{1}{2}\left\{\gamma^{\mu}\gamma^5 + \gamma^5\gamma^{\mu}\right\}\psi \tag{10.7}$$

As always, we employ:

$$j^{\mu} = \overline{\psi}\gamma^{\mu}\psi \tag{10.8}$$

With the foregoing definitions, we may make use of (10.7) and (10.8) and the $\{\gamma^\mu, \gamma^5\} = 0$ to find, showing the full calculation, that:

$$\begin{aligned} J^\mu &= j^\mu + \sqrt{g^{55}}\, j^{\mu 5} = \overline{\psi}\gamma^\mu\psi - g^{55}\overline{\psi}\tfrac{1}{2}\{\gamma^\mu\gamma^5 + \gamma^5\gamma^\mu\}\psi \\ &= \overline{\psi}\tfrac{1}{2}\gamma^\mu\psi - g^{55}\overline{\psi}\tfrac{1}{2}\gamma^\mu\gamma^5\psi + \overline{\psi}\tfrac{1}{2}\gamma^\mu\psi + g^{55}\overline{\psi}\tfrac{1}{2}\gamma^\mu\gamma^5\psi \\ &= \overline{\psi}\gamma^\mu\tfrac{1}{2}(1 - g^{55}\gamma^5)\psi + \overline{\psi}\gamma^\mu\tfrac{1}{2}(1 + g^{55}\gamma^5)\psi = \overline{\psi}_R\gamma^\mu\psi_R + \overline{\psi}_L\gamma^\mu\psi_L \end{aligned} \quad . \tag{10.9}$$

We find, then, that $J^\mu = j^\mu + \sqrt{g^{55}}\, j^{\mu 5}$ is a chiral current, and that this is so whether the fifth dimension is timelike or spacelike, because $g^{55} = \pm 1 = \text{constant}$. Therefore, the representation of a five-dimensional Clifford Algebra chosen in (10.3), (10.4), together with the definition of j^{MN} in (10.5) and the definition of the Dirac spinor wavefunctions in (10.7) and (10.8), leads to good chiral projection operators $P \equiv \tfrac{1}{2}(1 \pm g^{55}\gamma^5)$, and a vector along the fifth component of the Mixed Ricci tensor which is given by:

$$R^\mu{}_5 = -\tfrac{\sqrt{2}}{2}\overline{\kappa}J^\mu = -\tfrac{\sqrt{2}}{2}\overline{\kappa}\left(\overline{\psi}_R\gamma^\mu\psi_R + \overline{\psi}_L\gamma^\mu\psi_L\right). \tag{10.10}$$

What is pleasing about this result, aside from the fact that it offers a possible solution to the chirality problem confronted by 5-dimensional Kaluza-Klein theory, is that one does not have to go out of one's way to artificially construct so-called V+A and V-A currents *ad hoc*. The very seeds of these currents, naturally exist in the five-dimensional geometry, already planted in the $R^\mu{}_5 = -\tfrac{\sqrt{2}}{2}\overline{\kappa}J^\mu = -\tfrac{\sqrt{2}}{2}\overline{\kappa}\left(j^\mu + \sqrt{g^{55}}\, j^{\mu 5}\right)$ components of the mixed Ricci Tensor.

11. Summary and Conclusion

We have attempted throughout the discussion to specify all of classical electrodynamics on the basis of a five-dimensional Kaluza-Klein type theory, built on the bedrock of Riemannian geometry, and independently of whether this fifth dimension is timelike or spacelike. The fact that it turns out that $g_{55} = g^{55} = \pm 1 = \text{constant}$ for respective timelike and spacelike extensions is very helpful in this endeavor, because one can capture the $\pm$ character of this contrast in fully covariant fashion.

The assumptions used in this exposition were minimal, conservative, and based on what we observe to be true in the natural world. Our mathematical foundation was a five-dimensional

Riemannian geometry, without any changes or enhancements, but merely extending the entire apparatus of gravitational theory into one more dimension. The key assumption, which drove almost all of the mathematical development, was the requirement effectively imposed by equations (2.7) and (2.8), that the Lorentz force as it is experimentally observed, must be represented as nothing other than geodesic motion in the five-dimensional Riemannian geometry. In section 3, where we started to implement this requirement, there was a freedom with respect to choosing the constant labeled b throughout. We carried this constant throughout the development, and only finally fixed its value in (8.12), when we mandated that the action of quantum electrodynamics be made identical with a gravitational action *in vacuo*, and that this action be fully compatible with Einstein's equation, see (8.17) and (8.18).

In section 4, we showed that, *if* the fifth dimension is taken to be spacelike and compactified, *then* angular frequency movements (4.2) through this dimension, may well be the foundation of intrinsic spin, and that if this is so, then one can actually calculate the compactification radius with precision, and that this radius is quite close to the Planck length, as finally given in (8.13).

Section 5 introduced but one further assumption, as conservative as can be, that the field strength tensor, when extended to five dimensions, must continue to be fully antisymmetric, $F^{\mathrm{MN}} \equiv -F^{\mathrm{NM}}$. All else was deductive, and in the end, we found that all of these new components were equal to zero. We also found other helpful relations involving the metric tensor, the most important of which is that $g_{55} = g^{55} = \pm 1 = \text{constant}$, for a timelike and spacelike fifth dimension, respectively.

In section 6, we used the tools developed in section 5 to examine the Riemann tensor. Almost immediately, in (6.4), we deduced a critical, central relation which laid the foundation for casting both of Maxwell's equations, including sources, on a totally geometric foundation, see (6.9) and (6.17). We also, in the course of this development, demonstrated how one of the central terms of QED, $F^{\sigma\tau}F_{\sigma\tau} = \mathbf{B}^2 - \mathbf{E}^2$ in Minowski space, is induced out of the five-dimensional gravitational interaction, see (6.10). Not only was this the start of the path to placing the QED Lagrangian onto a fully-geometric footing, but it was the first inkling of a symmetric tensor current term later given definition in (7.6), which then, with a further redefinition in (10.5), provided the basis for a possible solution to the so-called "chirality problem" which has plagued many Kaluza-Klein efforts.

Section 7 laid some further deductive groundwork, but most importantly, enabled us to determine the exact expression for the five-dimensional Ricci scalar $R_{(5)} = R^{\sigma}{}_{\sigma} + R^{5}{}_{5}$, up to the four dimensional scalar $R = R^{\sigma}{}_{\sigma}$, which was still left unknown.

Then, in section 8, we started a close examination of the vector four-potential A^{μ} which is at the heart of QED, and the differential equations through which this relates to the gravitational potentials $h_{\mu\nu}$. But the requirement that the Lorentz force must be represented as simple geodesic motion in the five-dimensional Riemannian geometry – our primary assumption for development – had been pushed as far as possible, and a second assumption was needed to deduce the ordinary Ricci scalar $R = R^{\sigma}{}_{\sigma}$ which the primary assumption left undetermined. Here, at this juncture, we imposed the requirement that the full five-dimensional curvature scalar $R_{(5)}$ must itself be identical with the QED Lagrangian density, up to a constant factor. This led us in (8.11), to express the weak-field QED action solely in terms of integration of the curvature scalar $R_{(5)}$ over the four dimensions of spacetime, $S = g_{55}\dfrac{1}{2\kappa}\int R_{(5)}\sqrt{-g}\,d^4x = \int \mathfrak{L}_{QED}\sqrt{-g}\,d^4x$, effectively extracting QED from the geometrodynamic vacuum; fashioning light from the void. Proceeding even further, in (8.17) we were able to specify that exact *non-linear* QED action for any field strength weak or strong, and in (8.18), we came to express this QED action solely as a function of the metric tensor g_{MN}. Quantum electrodynamics was now a fully gravitational phenomenon.

In section 9 we turned to examine the ordinary components $T^{\mu}{}_{\nu}$ of the energy tensor suggested by all of the foregoing, and its relation to the Maxwell stress energy tensor. This is work still in progress, for the struggle which Einstein faced in trying to reconcile a field equation with trace, to an energy tensor without trace, [16] still warrants consideration, perhaps in the context of section 9. Finally, in section 10, we offered a possible resolution to the chirality problem of five-dimensional theories.

Ever since Galileo's refutation of Aristotle which legend situates at the Leaning Tower of Pisa, it has been understood that heavier and lighter gravitational masses similarly-disposed in a gravitational field will accelerate at the same rate and reach the ground after identical times have elapsed, because of the so-called "weak equivalence" of gravitational and inertial mass. As a material body becomes more massive and so more-susceptible to the pull of a gravitational field (back when gravitation was viewed as action at a distance), so too this increase in massiveness causes the material body in equal measure to resist the gravitational pull. Along his path to developing the General Theory of Relativity (GTR), Albert Einstein made a brief stop in 1911 in an imaginary elevator, to conduct a *gedanken* in which he concluded that the physical experience of an observer falling freely in a gravitational field before terminally hitting the ground is no different from what was commonly thought of as Newton's inertial motion in which a body in motion remained in motion unless acted upon by a "force." [17] With the exception of the tidal forces later found in the General Theory of Relativity, this *gedanken* remains an accurate guide to the present day.

But electrical masses have long presented a dilemma, because the electrical mass of a material body, say, an electron, is *not* equal to its inertial mass, and this inequivalence is the mainspring of the forces we feel which clearly, as a physical sensation, differentiate the acceleration of Newton's a=F/m from that of the gravitational a=9.8 meters/sec^2 near the surface of the earth. The General Theory of Relativity, in the end, captured inertial motion and its close cousin of free-fall motion in a gravitational field, in the most elegant way, as simple geodesic motion in a curved Riemannian geometry along geodesic paths which coincide precisely with the paths one observes for bodies moving under gravitational influences. But the electrical motion of the Lorentz force has long been the "odd man out," because it was something distinct from gravitation: it did not appear to follow a geodesic path, and it did not "feel" like inertial or gravitational free fall motions because it created the sensation of a force which we can measure when we place a scale between ourselves and the ground on which we stand or the elevator which accelerates us upward, because of the collective electrical repulsion between billions of electrons in our bodies and billions more in the surface against which we are pressing.

The key to unlocking this mystery, and ultimately, to placing gravitation and electromagnetism within the same framework, is to understand the motion of electrical masses – as governed by the experimentally-grounded Lorentz force law – as geodesic motion no less that that of gravitation, but in a spacetime that is extended to contain a single additional fifth dimension: the dimension first proposed long ago by Kaluza and Klein. By placing electrical masses onto their own geodesics in a five-dimensional Riemannian geometry which embeds the spacetime of our daily experience into its seamless fabric, we find that the long-standing quest to unite gravitation and electrodynamics may finally arrive at a safe haven on a firm foundation.

Because the remaining interactions of nature are but carbon copies of electrodynamics based on a group theory pioneered by Yang and Mills, the five-dimensional unity of electro-gravitational phenomenon exposited here, may presage the development of a non-Abelian Kaluza-Klein spacetime geometry which could make further strides toward uncovering nature's underlying unity.

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