

Intrinsic Spin and the Kaluza-Klein Fifth Dimension

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Abstract:

Kaluza-Klein Theories provide a compelling unification of classical gravitation with classical electrodynamics, but have long been plagued by the perceived absence of physical evidence of a compactified, hypercylindrical, spacelike fifth dimension. We examine the possibility that this fifth dimension may in fact exist physically, and be fundamentally responsible for the quantized “intrinsic” spins which, with the exception of the hypothesized Higgs boson, are exhibited by all of the known elementary particles in nature.

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1. Introduction

The possibility of employing a fifth spacetime dimension to unite classical gravitation and electrodynamics has intrigued physicists for almost a century. [1], [2] Indeed, the feature of Kaluza-Klein theories which most firmly commends their serious consideration, is their ability to seamlessly unite Einstein's gravitation and Maxwell's electrodynamics. This includes their ability to represent the Lorentz Force motion of charged particles in an electromagnetic field as geodesic motion, and to accommodate both of Maxwell's equations and the Maxwell Stress-Energy tensor (See [2], pp. 71-73.) But the main perceived shortcoming of these theories, quite simply, is that nobody to date has been able to point to physical evidence of a fifth dimension. [3], [4] Especially for theories in which the fifth dimension is taken to be a compactified hypercylindrical spatial dimension, see, e.g., [5], Figure 1, the clear benefits of this gravitational and electrodynamic union has often been outweighed by skepticism about a curled-up fifth dimension which appears to have no observed (or even observable) physical manifestation.

We demonstrate here, that the compactified fifth dimension may be quite real physically, and that it may well manifest itself directly in the so-called "intrinsic spin" exhibited by all of the known elementary particles, with the exception of the hypothesized scalar Higgs. In short, it is provisionally suggested herein, subject to further consideration, development, and scrutiny, that the compactified fifth dimension may well be the "intrinsic spin dimension."

2. Geodesic Motion in Five Dimensions

The foundation of Kaluza-Klein theory is a five-dimensional Riemannian geometry, often without any changes or enhancements, which merely extends the entire apparatus of gravitational theory into one more dimension. In five dimensions, $g_{MN} \equiv g_{NM}$ with uppercase Greek indexes $M, N = 0, 1, 2, 3, 5$ may be used to denote the metric tensor, so $g_{\mu\nu}$ with lowercase $\mu, \nu = 0, 1, 2, 3$ is the ordinary metric tensor in the spacetime subspace. Inverses are defined in the usual manner according to $g^{M\Sigma} g_{\Sigma N} = \delta^M_N$ and so $g^{M\Sigma}$ and $g_{\Sigma N}$ raise and lower indexes in the customary manner, but must be applied over all five dimensions to achieve proper five-covariance.

Kaluza-Klein theories typically maintain the usual interval in the 4-dimensional spacetime subspace, using $d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu$, and define a five-space interval as:

$$\begin{aligned}
d\mathbb{T}^2 &\equiv g_{MN} dx^M dx^N = g_{\mu\nu} dx^\mu dx^\nu + g_{5\nu} dx^5 dx^\nu + g_{\mu 5} dx^\mu dx^5 + g_{55} dx^5 dx^5 \\
&= d\tau^2 + 2g_{5\sigma} dx^5 dx^\sigma + g_{55} dx^5 dx^5 .
\end{aligned} \tag{2.1}$$

The above is independent of whether in the weak field linear approximation, $g_{55} \rightarrow \eta_{55} = \pm 1$, i.e., of whether the fifth dimension is timelike or spacelike, and is generally model-independent.

Like any metric equation, (2.1) can be algebraically-manipulated into:

$$1 = g_{MN} \frac{dx^M}{d\mathbb{T}} \frac{dx^N}{d\mathbb{T}}, \tag{2.2}$$

which is the first integral of the equation of motion.

In five dimensions, the Christoffel connections may also be specified in the usual manner as $\Gamma^M_{\Sigma T} = \frac{1}{2} g^{MA} (g_{A\Sigma, T} + g_{TA, \Sigma} - g_{\Sigma T, A})$, hence $\Gamma^M_{\Sigma T} = \Gamma^M_{T\Sigma}$. Similarly, $g_{MN, \Sigma} = 0$ as usual, with the usual first rank covariant derivative $A^M_{; \Sigma} = A^M_{, \Sigma} + \Gamma^M_{\Lambda \Sigma} A^\Lambda$. Thus, one may take the covariant derivative of each side of (2.2) above, and after the usual reductions employed in four dimensions, and multiplying the result through by $d\mathbb{T}^2 / d\tau^2$, may arrive at a five-dimensional geodesic equation:

$$\frac{d^2 x^M}{d\tau^2} + \Gamma^M_{\Sigma T} \frac{dx^\Sigma}{d\tau} \frac{dx^T}{d\tau} = 0, \tag{2.3}$$

which bears an exact resemblance to the four-dimensional gravitational equation.

The above contains five independent equations. The four equations for which $M = \mu$, which specify motion in ordinary spacetime, are given by:

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\Sigma T} \frac{dx^\Sigma}{d\tau} \frac{dx^T}{d\tau} = 0. \tag{2.4}$$

which expands, using the metric tensor symmetry $g_{MN} = g_{NM}$, to:

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\sigma\tau} \frac{dx^\sigma}{d\tau} \frac{dx^\tau}{d\tau} + 2\Gamma^\mu_{5\sigma} \frac{dx^5}{d\tau} \frac{dx^\sigma}{d\tau} + \Gamma^\mu_{55} \frac{dx^5}{d\tau} \frac{dx^5}{d\tau} = 0. \tag{2.5}$$

The $M = 5$ equation, specifies fifth-dimensional ‘‘acceleration,’’ $d^2 x^5 / d\tau^2$.

Now, let us contrast (2.5) above to the gravitational geodesic equation which includes the Lorentz force law, namely, equation (20.41) of [6]:

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\sigma\tau} \frac{dx^\sigma}{d\tau} \frac{dx^\tau}{d\tau} - \frac{q}{m} F^\mu_{\sigma} \frac{dx^\sigma}{d\tau} = 0. \quad (2.6)$$

The first two terms in (2.5) and (2.6) are identical, and they specify geodesic motion in an ordinary gravitational field absent any electrodynamic fields or sources. The absence of any mass or charge in the first two terms captures the Galilean principle of equivalence, and further expresses Newtonian inertial motion in a gravitational field via the Christoffel connections $\Gamma^\mu_{\sigma\tau}$. The way in which the Lorentz force becomes geodesic motion in Kaluza-Klein theories, is by the effective equivalence between the third terms in (2.5) and (2.6), that is, by virtue of the relationship: e.g. [7], eqs. (13) versus (14), [8].

$$2\Gamma^\mu_{5\sigma} \frac{dx^5}{d\tau} \frac{dx^\sigma}{d\tau} = -\frac{q}{m} F^\mu_{\sigma} \frac{dx^\sigma}{d\tau}. \quad (2.7)$$

Whether one starts with the Lorentz force and derives other aspects of Kaluza-Klein, or starts elsewhere and derives the Lorentz force, is irrelevant. No matter where one starts, somewhere along the line, if the Lorentz force is to constitute geodesic motion, then Kaluza-Klein theories must have as one of their relationships, is the one given by (2.7) above.

3. The Spacetime Metric and Electromagnetic Field Strength Tensors

The equation in Klein's [2] between (6) and (7) establishes the relationship between the metric tensor and the five-dimensional electromagnetic field strength tensor, and may be represented in the notation employed here as $F_{MN} \propto g_{5M,N} - g_{5N,M}$.

Let us formalize the above relationship of Klein's a bit further. If we define:

$$\Gamma^M_{5\Sigma} \equiv -\frac{1}{4} b \bar{\kappa} F^M_{\Sigma}, \quad (3.1)$$

where b is a dimensionless numeric constant of proportionality and $\bar{\kappa} = \sqrt{16\pi G/c^4}$ is the constant employed, for example, in the expression $g_{MN} = \eta_{MN} + \bar{\kappa} h_{MN}$ from gravitational theory, then it is easy to show that (3.1) above is but an equivalent restatement of Klein's relationship set forth above.

To demonstrate this equivalence, we simply require the field strength tensor to be antisymmetric $F^{MN} \equiv -F^{NM}$ in the usual manner, including its fifth dimensional components.

Then, we can employ the Christoffels $\Gamma^M_{5T} = \frac{1}{2} g^{MA} (g_{A5,T} + g_{TA,5} - g_{5T,A})$ to write $F^{MN} = -F^{NM}$ completely in terms of the metric tensor g_{MN} and its first derivatives, as:

$$\frac{1}{4} b \bar{\kappa} F^{MN} = -\frac{1}{4} b \bar{\kappa} F^{NM} = -g^{MA} g^{\Sigma N} (g_{A5,\Sigma} + g_{\Sigma A,5} - g_{5\Sigma,A}) = g^{NA} g^{\Sigma M} (g_{A5,\Sigma} + g_{\Sigma A,5} - g_{5\Sigma,A}). \quad (3.2)$$

Renaming indexes, and using the symmetry of the metric tensor, this is readily reduced to:

$$g^{M\Sigma} g^{TN} g_{T\Sigma,5} = 0. \quad (3.3)$$

Then, using the inverse relationship $g^{TN} g_{T\Sigma} = \delta^N_{\Sigma}$, which we can differentiate to obtain

$$(g^{TN} g_{T\Sigma})_{,A} = g^{TN}{}_{,A} g_{T\Sigma} + g^{TN} g_{T\Sigma,A} = 0, \text{ i.e., } g^{TN} g_{T\Sigma,A} = -g^{TN}{}_{,A} g_{T\Sigma}, \text{ with } A = 5, \text{ we may reduce}$$

(3.3) to the very simple expressions, for both the covariant and contravariant metric tensor:

$$g^{MN}{}_{,5} = 0; g_{MN,5} = 0. \quad (3.4)$$

This is the second of Klein's "special assumptions" on page 68 of [2], in the notation presently employed, that "the quantities g_{MN} must not depend on the fifth coordinate x^5 ." In other words, Klein's second "special assumption" is, simply, the perfectly-reasonable $F^{MN} \equiv -F^{NM}$.

So, returning to the main point, this means that with (3.4), (3.1) expands to:

$$\Gamma^M_{5\Sigma} = -\frac{1}{4} b \bar{\kappa} F^M_{\Sigma} = -\frac{1}{2} g^{MA} (g_{A5,\Sigma} + g_{\Sigma A,5} - g_{5\Sigma,A}) = \frac{1}{2} g^{MA} (g_{5\Sigma,A} - g_{5A,\Sigma}), \quad (3.5)$$

which lowers to:

$$g_{TM} b \bar{\kappa} F^M_{\Sigma} = \frac{1}{4} b \bar{\kappa} F_{T\Sigma} = -\frac{1}{2} g_{TM} g^{MA} (g_{5A,\Sigma} - g_{5\Sigma,A}) = \frac{1}{2} (g_{5\Sigma,T} - g_{5T,\Sigma}). \quad (3.6)$$

So, via $F^{MN} \equiv -F^{NM}$, (3.1) is just another way of representing Klein's $F_{MN} \propto g_{5M,N} - g_{5N,M}$, with the addition of a constant of proportionality.

We pause briefly before proceeding, to examine the term $\Gamma^{\mu}_{55} \frac{dx^5}{d\tau} \frac{dx^5}{d\tau}$ in the geodesic equation (2.5). Using $g_{MN,5} = 0$ from (3.4), a.k.a. $F^{MN} \equiv -F^{NM}$, to reduce, this connection is $\Gamma^{\mu}_{55} = \frac{1}{2} g^{\mu A} (g_{A5,5} + g_{5A,5} - g_{55,A}) = -\frac{1}{2} g_{55}{}^{,\mu}$. Whether Γ^{μ}_{55} is zero or non-zero, thereby depends upon the constancy or not, of g_{55} , i.e., on the oft-debated question of whether or not the theory is "restricted." As Klein notes in [2] on page 68, after examining five-dimensional coordinate

transformations, “the assumption $g_{55} = \text{constant}$ is . . . allowed,” and Leibowitz and Rosen in 1973 overcame earlier perceived problems with such a restriction. [9] We take no position here on this question, as it is not relevant to the analysis to follow.

4. Introduction of a Compactified, Spacelike Fifth Dimension

Now, we return to (2.7), and substitute (3.1) as $\Gamma^{\mu}_{5\sigma} \equiv -\frac{1}{4} \overline{b\kappa} F^{\mu}_{\sigma}$, to obtain:

$$\frac{1}{2} \overline{b\kappa} F^{\mu}_{\sigma} \frac{dx^5}{d\tau} \frac{dx^{\sigma}}{d\tau} = \frac{q}{m} F^{\mu}_{\sigma} \frac{dx^{\sigma}}{d\tau}. \quad (4.1)$$

Reducing leaves us with:

$$\frac{dx^5}{d\tau} = \frac{2}{\overline{b\kappa}} \frac{q}{m} = \frac{2c^2}{b\sqrt{16\pi}G} \frac{q}{m} \quad (4.2)$$

Irrespective of whether the fifth dimension is timelike or spacelike, we take dx^5 to be given in dimensions of time, so $dx^5/d\tau$ is a dimensionless ratio. When taking the fifth dimension to be spacelike, one need merely divide through by c . What (4.2) clearly states, is that the charge-to-mass ratio is a measure of motion through the x^5 dimension. If x^5 is “curled up,” and spacelike, then motion through the x^5 dimension should manifest as angular motion. We will now explore the possible connection between such angular motion, and intrinsic spin.

First, we ask: can we apply (4.2) to individual particles such as the charged leptons? After all, the term $F^{\mu}_{\sigma} (dx^{\sigma} / d\tau)$ factored out from (4.1) is related to the Lorentz force equation of motion which is classical, and does not extend to individual charges other than to describe their “expected” motion. However, what is left behind in (4.2) is a q/m term which can be sensibly-considered *even for individual particles* such as the charged leptons, since these do have a definite, certain charge and a definite, certain mass. In effect, in going from (4.1) to (4.2), we have factored out that part of the Lorentz force which can only be regarded classically, and left behind that part which has precise meaning even for individual charged particles, namely, the q/m ratio.

Let us therefore, consider a single charged lepton with a single unit of Coulomb charge, such as the electron, or its mu or tau partners (or their antiparticles). In rationalized Heaviside-Lorentz units which we shall use here, with fundamental constants restored, the electric charge

strength q for such a unit charge is related to the dimensionless (running) electromagnetic coupling by $\alpha = q^2/4\pi\hbar c$, which approaches $\alpha \rightarrow 1/137.036$ at low energy. The value of α is the same in all systems of units but the numerical value of q is different. Making use of the inverse $q = \sqrt{4\pi\hbar c \alpha}$, we thereby extend (4.2) above to:

$$\frac{dx^5}{d\tau} = \frac{2}{b\kappa} \frac{q}{m} = \frac{2c^2}{b\sqrt{16\pi G}} \frac{q}{m} = \frac{c^2}{b} \frac{\sqrt{\hbar c \alpha}}{\sqrt{Gm}}. \quad (4.3)$$

In the above, $dx^5/d\tau$ is dimensionless, as is $\sqrt{\hbar c \alpha} / \sqrt{Gm}$. Therefore, to restore complete dimensional balance, with all fundamental constants properly restored, we divide the final three terms in the above through by c^2 , to obtain:

$$\frac{dx^5}{d\tau} = \frac{2}{b\kappa c^2} \frac{q}{m} = \frac{2}{b\sqrt{16\pi G}} \frac{q}{m} = \frac{1}{b} \frac{\sqrt{\hbar c \alpha}}{\sqrt{Gm}}. \quad (4.4)$$

Now, following many who study Kaluza-Klein, let's take the fifth dimension to be spacelike, and compactified into a hypercylinder $x^5 \equiv R\phi$ (see [5], Figure 1). We further, as is customary, take R to be a constant radius, so that $dx^5 \equiv Rd\phi$. We then substitute this into the first term of (4.4), so as to write:

$$\frac{Rd\phi}{cd\tau} = \frac{2}{b\kappa c^2} \frac{q}{m} = \frac{2}{b\sqrt{16\pi G}} \frac{q}{m} = \frac{1}{b} \frac{\sqrt{\hbar c \alpha}}{\sqrt{Gm}}. \quad (4.5)$$

Note that we have also put an extra c into the denominator of the first term to maintain all terms as completely dimensionless, because $x^5 \equiv R\phi$ is defined to be spacelike and, as written, has spatial dimension. Now, let's study the dimensionless equation (4.5).

5. A Possible Connection Between the Fifth Dimension and Intrinsic Spin

Taking R to be constant, it is noteworthy that the physics ratio q/m measures an "angular frequency" $d\phi/d\tau$ of fifth-dimensional rotation. Of special interest, given constant R , and holding q fixed, *this frequency runs inversely to the mass*. By classical analysis, if one doubles the mass, one halves the tangential velocity, and given a constant radius and charge, the angular momentum will also be constant. In other words, by classical principles, this means that

the angular momentum remains constant, no matter what the mass. If the charged lepton is, e.g., a muon rather than an electron, then given that their charges are identical, $d\phi/d\tau \propto 1/m$ will vary inversely to the mass, i.e., the frequency of fifth-dimensional rotation will be reduced by the ratio of the muon mass to the electron mass. But the angular momentum will remain the same. One might therefore suspect that this constant angular momentum is, by virtue of its constancy independently of mass, related to intrinsic spin. In fact, following this line of reasoning, one can arrive at an exact expression for the compactification radius R , via the following “semi-classical” analysis, which turns out to be on the order of the Planck length, just as many who work with compactified spatial dimensions expect:

In (4.5), move the c away from the first term and move the m over to the first term. Then, multiply all terms through by another R . Everything is now dimensioned as an angular momentum $L = m \cdot v \cdot R$ with $v = Rd\phi/d\tau$, which L we have just ascertained is constant irrespective of mass because $v \propto 1/m$. Because this angular momentum is a constant, independent of the mass, let us *define* this angular momentum, *by hypothesis*, as the *intrinsic spin* of the subject charged lepton.

In particular, let's us first consider *magnitude*, then orientation. The magnitude of the angular momentum arising from an intrinsic spin $S = \frac{1}{2}$ is given by $L = \hbar\sqrt{s(s+1)} = \frac{\sqrt{3}}{2}\hbar$.

Therefore, by hypothesis, we define:

$$m \frac{Rd\phi}{d\tau} R = \frac{2}{b\kappa c} qR = \frac{2c}{b\sqrt{16\pi G}} qR = \frac{1}{b} \frac{\sqrt{\hbar c^3 \alpha}}{\sqrt{G}} R \equiv \frac{\sqrt{3}}{2} \hbar. \quad (5.1)$$

Now, keeping in mind that the final term with α is independent of our choice of units but that the q/m terms are not, we may take the final two terms and solve for R , which yields:

$$R = \frac{b\sqrt{3}}{2\sqrt{\alpha}} \sqrt{\frac{G\hbar}{c^3}} = \frac{b\sqrt{3}}{2\sqrt{\alpha}} L_p, \quad (5.2)$$

where $L_p = \sqrt{G\hbar/c^3}$ is the Planck length.

*This intrinsic spin hypothesis in (5.1) thereby gives a definitive size for the compactification radius, and it is very close to the Planck length.** Because we have based the foregoing semi-classical analysis on a unit charge with $S = \frac{1}{2}$, (5.2) would appear to characterize the constant compactification radius R for all of the charged leptons, and to provide a *classical geometric foundation* for the ostensibly *quantum-mechanical*, “*non-classical two-valuedness*” of the intrinsic spin of the charged leptons. [10] For $\alpha = 1$ or on the order of unity, the compactification radius of the fifth dimension for the charged leptons appears to be very close to the Planck length of Wheeler’s geometrodynamics vacuum “foam” and the Schwarzschild radius of the vacuum. [6] at §43.4, [11]** And even relative to the low energy $\alpha \rightarrow 1/137.036$, this is still very close to the Planck scale.

Now, with magnitude in hand, let’s consider the intrinsic spin *orientation*. It is helpful to start by envisioning the classical example, of a material body rotating about an origin in the x-y plane, with an angular momentum of given magnitude L . The orientation of the angular momentum vector, in this instance, is definitively along the z-axis, orthogonal to the x-y plane of rotation. But what about a single electron, moving through a curled-up, fourth, hyper-cylindrical spatial dimension x^5 ?

The magnitude of the spin angular momentum for this movement through the fourth space dimension is $(\sqrt{3}/2)\hbar$, as postulated in (5.1). And the orientation, following suit with the classical example above, will be orthogonal to the “plane” of the curled-up x^5 . But what is orthogonal to x^5 , is very much different from the single z axis which is orthogonal to the x-y plane in classical physics. Rather, *all three dimensions of ordinary space* are orthogonal to the fourth space dimension. That is, the three-dimensional volume of ordinary space is what is orthogonal to x^5 , and so the spin angular momentum is oriented, *non-classically, into all three of the space dimensions*. From the foundation of pure Riemannian geometry, we extract a

* In a complete Kaluza-Klein analysis, which we shall not do here, it turns out that when one considers the way in which the Maxwell’s stress energy tensor embeds into Kaluza-Klein, the fifth dimension must in fact be spacelike, and $b^2 = 8$. As a result, (5.2) becomes $R = L_p \sqrt{6/\alpha}$.

** By way of review, the Planck mass, defined from the term atop Newton’s law as a mass for which $GM_p^2 \equiv \hbar c$, is thus $M_p = \sqrt{\hbar c/G}$. In the geometrodynamics vacuum, the negative gravitational energy between Planck masses separated by the Planck length $L_p = \sqrt{G\hbar/c^3}$ precisely counterbalances and cancels the positive energy of the Planck masses themselves. The Schwarzschild radius of a Planck mass $R_s = 2GM_p/c^2 = 2\sqrt{G\hbar/c^3} = 2L_p$.

decidedly non-classical result, of a spin orientation which cannot be directed along a single axis of three-dimensional space. Let's follow this a step further.

Applying Poincaré symmetry, there is, of course, no “preferred” frame of rotation, that is, we take the three dimensions of ordinary space to be completely isotropic. Thus, if the spin angular momentum is to project orthogonally out of x^5 into x^1, x^2, x^3 , one might envision not a single vector, but rather infinite number of vectors emanating outwardly from a single point, forming a sort of spherical porcupine, because that is how one would represent a "vector" orthogonal to the fourth-compactified dimension given that there is no preferred direction in the three-dimensional space. And, of course, one can then draw three mutually-orthogonal space coordinate axes against this spherical porcupine of vectors, and can then ask about the component of this angular momentum which projects along each of these three axes x^1, x^2, x^3 .

So, let's now try to formalize this. If we designate $\sigma^1, \sigma^2, \sigma^3$ to denote that orthogonal projection of the spin angular momentum out of x^5 and onto each of the x^1, x^2, x^3 axes, then we know, first, from (5.1), that:

$$\sigma^{1^2} + \sigma^{2^2} + \sigma^{3^2} = \frac{3}{4} \hbar^2 = s(s+1)\hbar^2, \quad (5.3)$$

This ensures that the magnitude of the spin angular momentum is properly projected. Next, because the spin angular momentum must project isotropically into x^1, x^2, x^3 because otherwise there would be a preferred direction in ordinary space, we may, from (5.3), also obtain:

$$\sigma^{1^2} = \sigma^{2^2} = \sigma^{3^2} = \frac{1}{4} \hbar^2, \quad (5.4)$$

Clearly, though based strictly on Riemannian geometry, this is already non-classical, because classically, angular momentum has a definite orientation whereas in (5.4) the angular momentum – squared – is oriented equally into all three spatial dimensions.

The next question arises: how does one take the square root of (5.4) to determine the σ^i , $i = 1,2,3$? And, closely-related, what sort of “numbers” do we use for the σ^i ? If the σ^i are ordinary commuting numbers, then one might write $\sigma^1 = \sigma^2 = \sigma^3 = \frac{1}{2} \hbar$, in which case there would be one-half a unit of angular momentum oriented along each axis, projected out from x^5 .

But, this is not what we observe physically. So, one might suppose that the σ^i in (5.4) are non-commuting *operators*. In this case, we might instead employ the Pauli spin matrices to write:

$$\sigma^1 = \frac{1}{2}\hbar \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \frac{1}{2}\hbar \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \frac{1}{2}\hbar \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (5.5)$$

which, as it happens, also perfectly satisfy (5.3) and (5.4). And, of course, the above matrices do lay the foundation for describing the quantum mechanical, non-classical two-valued phenomenon of intrinsic spin, as it is observed in nature. That is, (5.5) describes, physically, how the intrinsic spin generated out of the fourth space dimension, projects into the three ordinary dimensions of space.

Consequently, from a strictly geometrically-based, classical motion through a compactified fourth space dimension x^5 , we orthogonally project the spin angular momentum into x^1, x^2, x^3 , and find this projection to be described by the definitively non-classical (5.4), $\sigma^{1^2} = \sigma^{2^2} = \sigma^{3^2} = \frac{1}{4}\hbar^2$. Then, if we take the σ^i to be non-commuting operators rather than commuting numbers, we find that the solution to (5.4), is given by the Pauli matrices (5.5).

It is interesting to note that orbital angular momentum – like spin angular momentum – also can only be projected definitively (as a “good” quantum number) onto a single spatial axis of quantization. Because this is non-classical in the same way as spin, it appears as if the orbital as well as the spin quantum numbers of an electron emanate by an orthogonal projection of angular momentum out of out of x^5 , and onto each of the x^1, x^2, x^3 .

6. Derivation of the Heisenberg Commutators for Position and Momentum

Having the Pauli matrices in hand, we are but a step away from the Heisenberg principle, because the fact that spin angular momentum can only be projected with certainty onto a single axis of quantization, customarily taken to be the z-axis of the diagonalized operator σ^3 .

Specifically, the Pauli matrices (5.5) obey the well-known commutation relationship:

$$[\sigma^i, \sigma^j] = i\hbar \epsilon^{ijk} \sigma^k, \quad (6.1)$$

Then, we assume that “components of . . . angular momentum in quantum mechanics can be defined in an analogous manner to the corresponding components of classical angular momentum.” That is, we assume that the classical equations

$$L^i = \epsilon^{ijk} x^j p^k, \quad (6.2)$$

which specify the angular momentum L^i in terms of the position x^i and linear momentum p^i , also specify the quantum mechanical “angular momentum *operators* in terms of the position and linear momentum *operators*.” [12]

Therefore, taking (6.2) to an equation among quantum mechanical operators as well as among classical numbers, and treating (6.1) as an angular momentum, we may set:

$$\sigma^i = L^i. \quad (6.3)$$

This leads directly to the commutation relationship of the Heisenberg principle in the following manner.

Using (6.1), (6.2) and (6.3), we may write:

$$[L^i, L^j] = [\sigma^i, \sigma^j] = i\hbar \epsilon^{ijk} L^k = i\hbar \epsilon^{ijk} \sigma^k. \quad (6.4)$$

Equation (6.4) will be true, if and only if, the quantum mechanical operators for position x^i and linear momentum p^i are given by:

$$[x^i, x^j] = 0. \quad (6.5)$$

$$[p^i, p^j] = 0. \quad (6.6)$$

$$[x^i, p^j] = i\hbar \delta^{ij}. \quad (6.6)$$

Equations (6.5) and (6.6) specify that position commutes with position and momentum with momentum, while (6.7) is the Heisenberg canonical commutation relationship between position with momentum, from which the so-called “uncertainty principle” can be derived in a well know manner.

To see that (6.5)-(6.7) must be a consequence of (6.4), explicitly write out one of the three equations in (5), say, the $i = 1, j = 2$ equation. Employing (6.1) and (6.2), this expands to:

$$[L^1, L^2] = [x^2 p^3 - x^3 p^2, x^3 p^1 - x^1 p^3] = [\sigma^1, \sigma^2] = i\hbar \sigma^3 = i\hbar L^3 = i\hbar (x^1 p^2 - x^2 p^1). \quad (6.7)$$

This relationship may be solved if we define the bridge term $x^2 [p^3, x^3] p^1 + x^1 p^2 [x^3, p^3]$ by:

$$[x^2 p^3 - x^3 p^2, x^3 p^1 - x^1 p^3] \equiv x^2 [p^3, x^3] p^1 + x^1 p^2 [x^3, p^3] \equiv i\hbar (x^1 p^2 - x^2 p^1). \quad (6.8)$$

The full expansion and re-consolidation of these terms demonstrates that (10) is true if and only if (6.5)-(6.7) are true.

Thus, we have demonstrated how two foundations of quantum theory – the Pauli spin matrices and the Heisenberg canonical commutation relationship – can both be canonically generated from an even deeper level, out of the compactified fifth space dimension of a Kaluza-Klein theory. Quantum theory thereby acquires a foundation in Riemannian geometry, and the compactified fifth-dimension, often regarded as the Achilles heel of Kaluza-Klein theory because it is not directly observed, now becomes the geometrodynamical foundation standing at the root of quantum theory.

7. Fifth-Dimensional Super- or Sub-Luminosity?

Equation (5.1), which is the foundation of the connection drawn between the compactified fifth dimension and intrinsic spin and Heisenberg uncertainty, prompts consideration of a further question as to whether an electron, when moving through the compactified fifth dimension to generate its intrinsic spin and its q/m ratio, is traveling at super- or sub-luminal velocity. This, in turn, compels us to a direct examination of physics at the Planck scale, and to the development of a rudimentary Planck-scale “model” of the electron.

The problem now to be considered is seen most easily by returning to semi-classical analysis, and dividing (5.1) through by $m \cdot R$ to leave behind the linear velocity $v = R d\phi / d\tau$ through the fifth dimension, with $d\tau$ representing the electron’s own proper time, as such:

$$v = \frac{R d\phi}{d\tau} = \frac{2}{bkc} \frac{q}{m} = \frac{2c}{b\sqrt{16\pi G}} \frac{q}{m} = \frac{1}{b} \frac{\sqrt{\hbar c \alpha}}{\sqrt{Gm}} c \equiv \frac{\sqrt{3}}{2} \hbar \frac{1}{m \cdot R}. \quad (7.1)$$

Focusing on the sub-relation $\frac{v}{c} = \frac{1}{b} \frac{\sqrt{\hbar c \alpha}}{\sqrt{Gm}}$, we observe that at low probe energy, with

$\alpha \rightarrow 1/137.036$ and $b^2 = 8$ (a result that requires a detailed analysis beyond this paper), and with $m_e = .511 \text{ MeV}$ and $M_p = 1.22 \times 10^{22} \text{ MeV}$, one can derive that for the electron, the fifth-dimensional $v/c = 7.211 \times 10^{20}$. This is extremely superluminal, for which there are but three explanations: First, this result, and all that leads to it, is just plain wrong. Second, superluminosity is permitted in the fifth-dimension, even for a massive particle like the electron.

Third – and the path we shall now explore – is the prospect that the only reason (7.1) leads to $v/c = 7.211 \times 10^{20}$, is because the value we are using for the electron rest mass, $m_e = .511 \text{ MeV}$, is the *screened* electron mass, as observed from probe energies twenty orders of magnitude removed from the Planck scale. We know that probe energy is a key parameter in particle physics, and that how deeply we probe does affect what we observe. In particular, we shall consider the prospect that when observed unscreened from a Planck-scale probe energy, the electron rest mass is very much larger than $m_e = .511 \text{ MeV}$, and is, indeed, on the order of the Planck mass itself. This puts the ratio $\sqrt{\hbar c \alpha} / \sqrt{G m_e}$ on the order of unity (1), renders the true velocity of the electron into a sub-luminal range, and explains that $v/c = 7.211 \times 10^{20}$ result as an illusory outgrowth of the twenty orders of magnitude worth of charge and mass screening which sits between the perch of our experimental observations, and the actual physics which occurs tucked right in, riding an electron at the Planck scale. Recognizing that at Planck scale, the gravitational curvatures are so great as to generate a sea of virtual black holes, this is not unlike observing a black hole from an intergalactic distance rather than from right at its boundary. In effect, these twenty orders of magnitude of screening act as a powerful “lens,” distorting greatly, our view of what actually happens at the Planck scale. The principle of “relativity” extends in this way to probe energy, such that one must always specify the probe energy “relative to” which the phenomena in question are being observed.

The touchstone for this approach is Wheeler’s article [11], which lays the foundation for the geometrodynamics approach to physics, and makes quite clear in the very last paragraph that “the question of the origin of spin is decisive for the assessment of quantum geometrodynamics.”

In this seminal article, Wheeler admits a multiply-connected spacetime topology in lieu of one that is simply-connected, and, at 604, regards “classical charge . . . as the flux of lines of force trapped in a multiply connected metric.” Here, we make a course correction, and by the relationship (5.1), regard the electron charge to be emanating from the motion of the electron through the fifth compactified spacetime dimension, which simultaneously accounts for the electron’s intrinsic spin and via orthogonal projection into the three ordinary dimensions of space, for the non-classical two-valuedness of this spin when observed in ordinary spacetime, and for the Heisenberg commutation relationships, as earlier demonstrated.

Continuing, Wheeler shows how positive Planck mass fluctuations which are typical of the Planck vacuum are typically separated by the Planck length and so render a negative gravitational energy which precisely counterbalances the positive Planck masses. And so, at 610, Wheeler concludes, “the possibility is open to have a vacuum state with zero net energy density.” We adopt the same view here, but add to it the understanding gleaned in recent decades that Higgs-type fluctuations in the vacuum are spinless (scalar) fluctuations, and apply this to model Wheelers vacuum as a vacuum filled with *scalar* fluctuations, with an expected mass equal to the Planck mass, with an expected separation equal to the Planck length, and with an expected negative gravitational energy equal to the negative of the Planck energy. Thus, the net vacuum expectation energy over any region substantially larger than the Planck length, is precisely equal to zero. And, there is an expected Schwarzschild black hole radius equal to twice the Planck-length enveloping this vacuum, so by Hawking’s celebrated analysis from 1974, [13] it is not implausible to suppose that the statistical distribution which leads to the foregoing expectation values is a blackbody spectrum, and perhaps, that when observed in a screened state, this is related to the cosmic blackbody radiation background. So, how do we now get real quantum electrons with intrinsic spin and non-zero masses?

By the forgoing result derived here, the electron is moving through a compactified fifth spacetime dimension with a $dx^5 \equiv Rd\phi \neq 0$ which by (5.1) is made to account with precision for the intrinsic spin not only of the electron, but of the muon and tauon. On the other hand, the spinless scalar Planck-scale masses which inhabit Wheeler’s vacuum foam must, by definition as spinless, be stationary along the fifth dimension, with $dx^5 = 0$. Thus, we posit here, a rudimentary model of the electron as a Planck mass which distinguishes itself from all the other Planck masses in the vacuum, by having a non-zero dx^5 specified by (5.1). Now, by virtue of this motion through dx^5 , there is an additional amount of kinetic energy that arises, and this provides the energy needed to perturb the vacuum ever-so-slightly above its natural energy density of zero. The electron, from this view, is a Planck mass which has gotten caught in the fifth dimension (as opposed to a wormhole which has “trapped” electric flux lines), moves through the fifth dimension according to (5.1), and thereby acquires its intrinsic spin and its electric charge and its “undressed” rest mass all at once, and all because of this $dx^5 \equiv Rd\phi \neq 0$ motion.

Then, we come to the question of how the electron acquires its experimental mass. Continuing at 610, Wheeler observed that “in electron theory one distinguishes between the mass and charge of the “undressed electron” and the mass and charge of the experimental electron.” As regards the charge strength, as is well-known in particle theory, this of course scales with the running coupling $\alpha \rightarrow 1/137/036$ at low energy, which is hypothesized, at the Planck scale, to merge in magnitude with all other interaction couplings (and thus exhibit one aspect of “unification”). Regarding the electron mass, Wheeler at 611 considers the view that “the electron is nothing but a collective state of excitation of the foam-like medium [which] is suggested in Fig. 2 by the slightly closer spacing of the wormholes within the dashed circle. The fractional increase in the concentration of electromagnetic mass-energy within the electron . . . is fantastically small compared to the concentration of . . . energy already present in the vacuum.”

Here, we take a slightly-modified view, that the electron mass arises from the differential energy provided by its $dx^5 \equiv Rd\phi \neq 0$ motion through the fifth dimension which also distinguishes the electron in a clear and stable way from the unstable scalar masses of the foam which characterize the vacuum. This additional dx^5 -derived kinetic energy is still “fantastically small compared to the concentration of . . . energy already present in the vacuum,” such that when it is viewed from the same perch at which $\alpha \rightarrow 1/137/036$, also leads one to observe that $m_e = .511 \text{ MeV}$. We leave unanswered for now, whether the muon and tauon are distinguished by the electron because their fifth-dimensional velocity, at Planck scale, is slower than that of the electron, or whether this distinction arises because, at low energies, these three charged leptons are differently-screened. That is, one must either consider whether the difference in mass between the electron and the other two charge lepton masses arises by virtue of differences already rooted at the Planck scale, or by virtue of screening differences down to low energies for each of these three leptons.

Wheeler’s goal, stated at 612, is “to find out, not whether pure quantum geometrodynamics can account for elementary particle physics, but whether there is some way to prove that it *cannot*” (original emphasis). Here, we have demonstrated that with slight changes, the results developed within, whereby the electron spin and charge and mass all derive from fifth dimensional motion, remain fully consistent with this analysis by Wheeler, with one bonus of inestimable value:

The final question posited by Wheeler is this: “How can a classical theory endowed with fields of integral spin possibly give on quantization a spin $\frac{1}{2}$ such as is required to account for the neutrino, the electron, and other particles? From the beginning Pauli referred to spin as ‘nonclassical two valuedness.’ Is there anything . . . that forces the introduction of any such non-classical two-valuedness?*

Unless there is, *pure quantum geometrodynamics must be judged deficient as a basis for elementary particle physics.* Therefore, the question of the origin of spin is decisive for the assessment of quantum geometrodynamics.”

Understanding intrinsic spin on the basis of motion through the compactified fifth dimension, with its concomitant production of the electron’s charge and mass, and of the Pauli spin operators and the Heisenberg canonical commutation relationship, suggests that quantum geometrodynamics remains alive and well as a foundation for the description of nature, rooted in real physical phenomena occurring at the Planck scale.

8. Limitations, Discussion, and Conclusion

While the above was derived by considering the charged leptons, there are questions which arise when other particles / field quanta are considered. For example, from (5.1), it appears at first sight that for a neutral body, $q = 0$, such as the neutrino, we have $d\phi/d\tau = 0$, and so there is no fifth-dimensional rotation. One might take this to suggest that the neutrino has no fifth-dimensional motion and thus no intrinsic spin, the latter of which, of course, is contradicted by empirical knowledge. Strictly speaking, however, in the context of a Kaluza-Klein theory based on $U(1)_{em}$, one really ought not try to discuss any particles other than charged leptons and photons. But to provide at least some guidance, we know that in electroweak theory, $Q = Y/2 + I^3$, so there are in fact other charge generators implicit in (5.1). Therefore, it is plausible that this question about the neutrino may be resolved if one considers Kaluza-Klein in a non-Abelian (Yang-Mills) $SU(2)_W \times U(1)_Y$ rather than the present abelian $U(1)_{em}$ context. Here, the neutrino will have a non-zero weak isospin $I^3 = +\frac{1}{2}$ coupled via a weak running coupling α_W related to the electromagnetic running coupling by $\alpha = \alpha_W \sin^2 \theta_W$ via the weak mixing angle θ_W . Perhaps, this can lay a foundation for the intrinsic spin of the

* We defer the question of the neutrino for the following section, sticking for the moment with just the electrons.

neutrinos similar to that of (5.1) and (5.2) for the charged leptons. It should be observed also, that the intrinsic spin interpretation (5.1) suggests that any elementary scalar particle which has no intrinsic spin, must be electrically neutral. This is, in fact, true of the hypothesized Higgs boson. One must finally point out, that (5.1) does not shed any direct light on the intrinsic spin of *massless* particles, such as the photon. However, we also note that in electroweak theory, the absence of mass for the photon emerges from a spontaneous breaking of symmetry which does leave three other bosons, the $W^{\pm\mu}$ and the Z^μ , with a non-zero mass. We also note that for the photon, the fundamental starting point is (2.1), with $d\tau^2 = 0$.

So, with the clear caution that particles other than the charged leptons do raise further questions, *might it nevertheless be possible that the compactified fifth dimension of Kaluza-Klein, which has long begged for a physical foundation, is in fact the Riemannian-geometric foundation of intrinsic spin?* If one is scrupulous about the use of language, and if one is so quixotic as to still be looking for a geometric foundation even for an ostensibly-quantum phenomenon such as intrinsic spin, one may discern that using the term “intrinsic” to describe an “inherent” quantized angular momentum of elementary particles, linguistically papers over a deep ignorance of what “intrinsic spin” really means, *geometrically*. [10] Why?

For a material body to have an angular momentum, there must implicitly be a radius R with which that body circles about a rotational origin. Even the smallest bodies, if they have an angular momentum, must be rotating or spinning – at some *finite spatial radius* – about an origin. At the same time, nobody believes that intrinsic spin represents an angular momentum about a radius R in the three usual spatial dimensions. By associating intrinsic spin with motion through a fourth, compactified, hyper-cylindrical spatial dimension, one simultaneously makes sense of intrinsic spin and of a compact fourth spatial dimension. The material body now has a spatial radius R of rotation through a fourth spatial dimension other than the usual three spatial dimensions to give geometric plausibility to its “intrinsic” spin. At the same time, the compactified fourth spatial dimension now takes on real, physical meaning as something which is physically observed, via the phenomenon of intrinsic spin, and not merely a fictional mathematical device which by its perceived contrivance, gives people pause about Kaluza-Klein theories specifically, and dimensional compactification in general. Especially, considering that such angular momentum must be projected orthogonally to the fifth dimension, and therefore will reside non-classically along all three ordinary space dimensions, we are led directly to the

Pauli spin operators (5.5), and from there, to the Heisenberg canonical commutation relationship (6.6).

In sum, this provisional, tentative, cautious, hypothesized understanding of intrinsic spin as cyclical motion through a fourth dimension of space which is curled up into a radius on the order of the Planck length, if it can be developed further and sustained for particles other than the charged leptons, may be useful to overcome one of the most nagging objections about Kaluza-Klein theories. It would do so by underscoring a clearly-observed, physical manifestation of the fourth space dimension, namely, the intrinsic spins which pervade particle physics and which have, up to the present time, defied explanation on any sort of classical geometric foundation. This in turn, would remove a primary cause which has led many to abandon entirely, not only the attractive possibility of explaining nature on a solely geometrodynamical footing, but also the compelling gravitational and electrodynamic union otherwise provided by Kaluza-Klein theories. Finally, this would also remove the need for string theorists generally, and Kaluza-Klein theorists specifically, to reply with some disingenuity to skeptics, that the extra space dimension(s) is/are “so small that nobody will ever see it/them anyway.”

Going forward, it is also worth keeping in mind that at highly-relativistic energies, the non-classical concept of chirality, which employs the Dirac γ^5 emerging from representing the “square root” of the metric tensor via a Clifford algebra, becomes identified with the classical notion of helicity. While subject also to further investigation, perhaps this means that chirality is associated with rotational motion through the fourth spatial dimension, helicity with rotation manifest in the ordinary three-dimensions of space, and that relativistic motion is the arena in which these two forms of geometric rotation come to coincide.

Thus, with all of the above-noted caveats, and recognizing that the mathematical result in (5.2) is based strictly and solely on particles with non-zero mass which carry a single unit of electrostatic charge, i.e., an electron, muon, or tauon or their antiparticles, we conclude by cautiously affirming the provisional, tentative hypothesis, to be studied in other contexts and for other elementary particles, that the fourth spatial dimension in Kaluza-Klein theories, may best be thought of as the “intrinsic spin dimension” of a real, physical, five-dimensional spacetime in which classical gravitation and classical electrodynamics stand united under one roof.

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