Dear Friends:

I'd like to lay out a nifty little mathematical calculation which allows a "decomposition" of the intrinsic spin matrices $s^i = \frac{1}{2}\hbar\sigma^i$ to include the position and momentum operators x^i, p^i, $i = 1,2,3$. *In this draft, I add a new section 3, which may open a new avenue to understanding the anomalous (Schwinger) magnetic moments of the charged leptons and perhaps other elementary field quanta and the Heisenberg uncertainty principle.* To simplify matters, we will employ a Minkowski metric tensor with $\mathrm{diag}(\eta_{\mu\nu}) = (-1,+1,+1,+1)$ so that raising and lowering the space indexes $i = 1,2,3$ is simple and at will, and does not entail any sign reversal.

1. Triple Cross Products and the Intrinsic Spin Decomposition

We start with the general cross product for two three-vectors **A** and **B**. Written in covariant (index) notation:

$$(\mathbf{A}\times\mathbf{B})_i \equiv \varepsilon_{ijk}A^j B^k . \quad (1.1)$$

One can easily confirm this by taking, for example, $(\mathbf{A}\times\mathbf{B})_3 \equiv A^1B^2 - A^2B^1$. Now, let's take the triple cross product $(\mathbf{A}\times\mathbf{B})\times\mathbf{C}$. We can apply (1.1) to itself using $(\mathbf{A}\times\mathbf{B})^j \equiv \varepsilon^{jmn}A_m B_n$, to write:

$$[(\mathbf{A}\times\mathbf{B})\times\mathbf{C}]_i = \varepsilon_{ijk}(\mathbf{A}\times\mathbf{B})^j C^k = \varepsilon_{ijk}\varepsilon^{jmn}A_m B_n C^k . \quad (1.2)$$

The fact that the crossing of **A** and **B** takes precedence over crossing with **C** is retained in the fact that $A_m B_n$ sum with ε^{jmn}, while C^k alone sums into ε_{ijk}.

Let us now expand (1.2) for the component equation for which $i = 3$. The full calculation, shown for illustrative purposes, is as such:

$$\begin{aligned}
[(\mathbf{A}\times\mathbf{B})\times\mathbf{C}]_3 &= \varepsilon_{3jk}\varepsilon^{jmn}A_m B_n C^k \\
&= \varepsilon_{312}\varepsilon^{123}A_2B_3C^2 + \varepsilon_{312}\varepsilon^{132}A_3B_2C^2 + \varepsilon_{321}\varepsilon^{231}A_3B_1C^1 + \varepsilon_{321}\varepsilon^{213}A_1B_3C^1 \\
&= A_1B_3C^1 + A_2B_3C^2 - A_3B_1C^1 - A_3B_2C^2 \\
&= A_1B_3C^1 + A_2B_3C^2 + A_3B_3C^3 - A_3B_1C^1 - A_3B_2C^2 - A_3B_3C^3 \\
&= A_1B_3C^1 + A_2B_3C^2 + A_3B_3C^3 - A_3(\mathbf{B}\cdot\mathbf{C})
\end{aligned} \quad , \ (1.3)$$

where we have added $0 = A_3B_3C^3 - A_3B_3C^3$ to the fourth line. Now in the final line, we hit an impasse, because B_3 is sandwiched between the terms we would like to form into the other dot product $\mathbf{A}\cdot\mathbf{C}$. In order to complete the calculation, we must make an assumption that the A_i commute with B_3, i.e., that $[A_i, B_3]=0$. For now, let us make this assumption, which we will shortly revisit.

If we assume that that $[A_i, B_3]=0$, we may then out the commutation in (1.3), and conclude the calculation so as to write:

$$\begin{aligned}&[(\mathbf{A}\times\mathbf{B})\times\mathbf{C}]_3=\varepsilon_{3jk}\varepsilon^{jmn}A_mB_nC^k=A_1B_3C^1+A_2B_3C^2+A_3B_3C^3-A_3(\mathbf{B}\cdot\mathbf{C})\\&=B_3(\mathbf{A}\cdot\mathbf{C})-A_3(\mathbf{B}\cdot\mathbf{C})=B_3A_jC^j-A_3B_jC^j\end{aligned}. \quad (1.4)$$

Generalizing fully, we may now write (1.4) in two equivalent ways as:

$$\begin{cases}(\mathbf{A}\times\mathbf{B})\times\mathbf{C}=-\mathbf{A}(\mathbf{B}\cdot\mathbf{C})+\mathbf{B}(\mathbf{A}\cdot\mathbf{C})\\ \varepsilon_{ijk}\varepsilon^{jmn}A_mB_nC^k=-A_iB_jC^j+B_iA_jC^j\end{cases}. \quad (1.5)$$

The reader will observe the well-known formula for the cross product.

Now, let's take the cross product in which $\mathbf{A}=\mathbf{x}$, $\mathbf{B}=\mathbf{p}$ and $\mathbf{C}=\boldsymbol{\sigma}$, where **x** is the hermitian quantum mechanical position operator, **p** its conjugate hermitian momentum operator, and $\boldsymbol{\sigma}$ is the spin (helicity) operator $\sigma^1=-i\alpha^2\alpha^3$, $\sigma^2=-i\alpha^3\alpha^1$, $\sigma^3=-i\alpha^1\alpha^2$ of Dirac's equation when written in the Hamiltonian form $H\psi=(\boldsymbol{\alpha}\cdot\mathbf{p}+\beta m)\psi$. By the foregoing, this means that we must also take into account the Heisenberg canonical commutation relationship between the position and momentum operators, that is, $[x_\mu, p_\nu]=i\hbar\delta_{\mu\nu}$. This means that we will have to be careful at the juncture between equations (1.3) and (1.4), because the position and momentum operators, when coaligned, do not commute.

So, we return to the final line of (1.3) before the commutation, with $\mathbf{A}=\mathbf{x}$, $\mathbf{B}=\mathbf{p}$ and $\mathbf{C}=\boldsymbol{\sigma}$, to write:

$$[(\mathbf{x}\times\mathbf{p})\times\boldsymbol{\sigma}]_3=\varepsilon_{3jk}\varepsilon^{jmn}x_mp_n\sigma^k=x_1p_3\sigma^1+x_2p_3\sigma^2+x_3p_3\sigma^3-x_3(\mathbf{p}\cdot\boldsymbol{\sigma}). \quad (1.6)$$

To take the next step, we want to commute p_3 in front of the x_i. In so doing, we can commute p_3 with x_i for $i=1,2$. But, for $i=3$, we must employ $x_3p_3=p_3x_3+i\hbar$. Therefore, (1.6) now becomes:

$$\begin{aligned}&[(\mathbf{x}\times\mathbf{p})\times\boldsymbol{\sigma}]_3=\varepsilon_{3jk}\varepsilon^{jmn}x_mp_n\sigma^k=p_3x_1\sigma^1+p_3x_2\sigma^2+(p_3x_3+i\hbar)\sigma^3-x_3(\mathbf{p}\cdot\boldsymbol{\sigma})\\&=p_3(\mathbf{x}\cdot\boldsymbol{\sigma})-x_3(\mathbf{p}\cdot\boldsymbol{\sigma})+i\hbar\sigma_3=p_3x_j\sigma^j-x_3p_j\sigma^j+i\hbar\sigma_3\end{aligned}, \quad (1.7)$$

lowering the index on $i\hbar\sigma^3$ with $\mathrm{diag}(\eta_{ij}) = (+1,+1,+1)$. Now all of a sudden, $i\hbar\sigma^3$ has made an unexpected appearance. Generalizing (1.7), we may write:

$$\begin{cases} [(\mathbf{x}\times\mathbf{p})\times\boldsymbol{\sigma}] = -\mathbf{x}(\mathbf{p}\cdot\boldsymbol{\sigma}) + \mathbf{p}(\mathbf{x}\cdot\boldsymbol{\sigma}) + i\hbar\boldsymbol{\sigma} \\ \varepsilon_{ijk}\varepsilon^{jmn}x_m p_n \sigma^k = -x_i p_j \sigma^j + p_i x_j \sigma^j + i\hbar\sigma_i \end{cases}. \quad (1.8)$$

This is the also the well-known formula for the triple-cross product, but with an additional term $i\hbar\boldsymbol{\sigma}$ emerging from the canonical commutation relationship. In fact, moving terms, equation (1.8) gives us a way to decompose the intrinsic spin matrix so as to contain the position and momentum, and as we shall also see, orbital angular momentum operators.

First, we rewrite (1.8) as:

$$\begin{cases} i\hbar\mathbf{S} = [(\mathbf{x}\times\mathbf{p})\times\mathbf{S}] + \mathbf{x}(\mathbf{p}\cdot\mathbf{S}) - \mathbf{p}(\mathbf{x}\cdot\mathbf{S}) \\ i\hbar S_i = \varepsilon_{ijk}\varepsilon^{jmn}x_m p_n S^k + x_i p_j S^j - p_i x_j S^j \end{cases}, \quad (1.9)$$

where we have multiplied through by $\frac{1}{2}\hbar$ and then set $S_i \equiv \frac{1}{2}\hbar\sigma_i$. This decomposes the intrinsic spin matrix into an expression involving itself, as well as the position and momentum operators.

Now, using the definition (1.1) but with $\mathbf{A} = \mathbf{x}$ and $\mathbf{B} = \mathbf{p}$, let's introduce the orbital angular momentum operator :

$$\mathbf{L}^j \equiv (\mathbf{x}\times\mathbf{p})^j \equiv L^j \equiv \varepsilon^{jmn}x_m p_n \quad (1.10)$$

It is easy to see, for example, that $L^3 = x_1 p_2 - x_2 p_1$. This is possible, because the same **x** and **p** operators which appear in $\lfloor x_\mu, p_\nu \rfloor = i\hbar\delta_{\mu\nu}$ also are used to form **L** via the cross product in (1.10). Using (1.10), we now rewrite (1.9) as:

$$\boxed{\begin{cases} i\hbar\mathbf{S} = (\mathbf{L}\times\mathbf{S}) + \mathbf{x}(\mathbf{p}\cdot\mathbf{S}) - \mathbf{p}(\mathbf{x}\cdot\mathbf{S}) \\ i\hbar S_i = \varepsilon_{ijk}L^j S^k + x_i p_j S^j - p_i x_j S^j \end{cases}}, \quad (1.11)$$

We see that part of this decomposition includes the cross-product $\mathbf{L}\times\mathbf{S}$ of the orbital angular momentum operator with the intrinsic spin operator.

Equation (1.11) allows us to decompose the total spin **S** for a Dirac field ψ, as follows:

$$\begin{cases} \mathbf{S} = \int(\overline{\psi}\mathbf{s}\psi)d^3x = -\frac{i}{\hbar}\int(\overline{\psi}[(\mathbf{L}\times\mathbf{S}) + \mathbf{x}(\mathbf{p}\cdot\mathbf{S}) - \mathbf{p}(\mathbf{x}\cdot\mathbf{S})]\psi)d^3x \\ S_i = \int(\overline{\psi}s_i\psi)d^3x = -\frac{i}{\hbar}\int(\overline{\psi}[\varepsilon_{ijk}l^j S^k + x_i p_j S^j - p_i x_j S^j]\psi)d^3x \end{cases}, \quad (1.12)$$

See Ohanian, H., *What is Spin*, at http://jayryablon.wordpress.com/files/2008/04/ohanian-what-is-spin.pdf, equation (18). And, in general, we may use (1.11) to substitute for either of $\mathbf{S} \equiv \frac{1}{2}\hbar\boldsymbol{\sigma}$ or $\boldsymbol{\sigma}$ wherever they appear in a physics equation or operator, and, by such substitution, to effectively introduce the canonical commutation relationship $[x_\mu, p_\nu] = i\hbar\delta_{\mu\nu}$ because this relationship is implicit in (1.11). As we shall see in section 3, when this substitution is used in expressions and operators for the magnetic moment of the electron, the decomposition (1.11) may yield an alternative way of approaching the Schwinger magnetic anomaly.

2. Commutation Relationships Involving the Spin Operator

A key term in (1.11) is the cross product $\mathbf{L}\times\mathbf{S}$ between the orbital angular momentum and the intrinsic spin operators, which we shall new examine in further detail.

First, let's multiply the lower equation (1.11) through by ε^{mni} and then employ the commutation relationship $[S^m, S^n] = i\hbar\varepsilon^{mni}S_i$, to write:

$$[S^m, S^n] = i\hbar\varepsilon^{mni}S_i = L^m S^n + \varepsilon^{mni}x_i p_j S^j - \varepsilon^{mni}p_i x_j S^j . \quad (2.1)$$

Above, we also make use of $\varepsilon^{imn}\varepsilon_{ijk} = \delta^{imn}{}_{ijk}$. In the above, the $\mathbf{L}\times\mathbf{S}$ term yield the $L^m S^n$ term, which now motivates us to inquire specifically about the commutator $[L^m, S^n] = L^m S^n - S^n L^m$.

First, of course, we know that the intrinsic spin and the orbital angular momentum operators both have the same commutation relationships, that is:

$$\begin{matrix}[S^m, S^n] = i\hbar\varepsilon^{mni}S_i \\ [L^m, L^n] = i\hbar\varepsilon^{mni}L_i\end{matrix} . \quad (2.2)$$

While neither of these commutes with the Hamiltonian and so neither is separately conserved, the total angular momentum operator $J^m = L^m + S^m$ does so-commute, $[H, \mathbf{J}] = 0$, and so is conserved. Further, J^m follows the same commutation relationship as do S^m and L^m, i.e.,

$$[J^m, J^n] = i\hbar\varepsilon^{mni}J_i \quad (2.3)$$

If we then substitute $J^m = L^m + S^m$ into (2.3) and use (2.2), we obtain:

$$[J^m, J^n] = [L^m + S^m, L^n + S^n] = [L^m, L^n] + [S^m, S^n] + [L^m, S^n] + [S^m, L^n] = i\hbar\varepsilon^{mni}L_i + i\hbar\varepsilon^{mni}S_i , \quad (2.4)$$

and then, via (2.2) this reduces to $[L^m, S^n] + [S^m, L^n] = 0$ or, in a different form:

$$[L^m, S^n] = [L^n, S^m]. \quad (2.5)$$

That is, the commutator of $\left[L^m, S^n\right]$ is symmetric under exchange of the *m*, *n* space indexes. Thus, when it is employed to operate on wavefunctions in the form $\left\langle \overline{\psi} \middle| \left[L^m, S^n\right] \psi \right\rangle$, these objects transform as a symmetric (three-space dimensional) tensor.

If we can now turn (2.1) into a form where it contains not just $L^m S^n$, but the commutator $\left[L^m, S^n\right]$, then relationship (2.5) can be used to impose further constraints on (2.1), or at the very least, we can make sure that (2.1) is consistent with (2.2) and (2.3) and $J^m = L^m + S^m$, which it must be. That is, we wish to make use of (2.5) above, which has an independent basis, to see what further information can be obtained from (2.1), or to at least make sure that (2.1) does not in any way contradict these relationships.

For this, we will first return to (1.1) and (1.2), to derive the triple cross product $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$, so this can be used to obtain $\boldsymbol{\sigma}\times(\mathbf{x}\times\mathbf{p}) = \boldsymbol{\sigma}\times\mathbf{L}$. This can then be turned into the term $S^n L^m$ needed together with $L^m S^n$ to form the commutator $\left[L^m, S^n\right]$. Then, finally, we will employ the independent symmetry relationship (2.5) as an independent condition.

First, analogous to (1.2), the desired cross product is:

$$\left[\mathbf{A}\times(\mathbf{B}\times\mathbf{C})\right]_i = \varepsilon_{ijk} A^j (\mathbf{B}\times\mathbf{C})^k = \varepsilon_{ijk}\varepsilon^{lmk} A^j B_l C_m \quad (2.6)$$

The $i=3$ equation analogous to (1.3), but omitting some intermediate steps, is:

$$\begin{aligned}\left[\mathbf{A}\times(\mathbf{B}\times\mathbf{C})\right]_3 &= \varepsilon_{3jk}\varepsilon^{lmk} A^j B_l C_m = A^1 B_3 C_1 + A^2 B_3 C_2 + A^3 B_3 C_3 - (\mathbf{A}\cdot\mathbf{B})C_3 \\ &= (\mathbf{A}\cdot\mathbf{C})B_3 - (\mathbf{A}\cdot\mathbf{B})C_3 = A^j C_j B_3 - A^j B_j C_3\end{aligned} \quad . \quad (2.7)$$

In the final line, we have employed $\left[B_3, C_i\right] = 0$, which is again an *assumption* about commutation that needs to be approached with care once position and momentum operators are under consideration. The generalization of (2.7), analogous to (1.5), is now:

$$\begin{cases} \left[\mathbf{A}\times(\mathbf{B}\times\mathbf{C})\right] = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C} \\ \varepsilon_{ijk}\varepsilon^{lmk} A^j B_l C_m = A^j C_j B_1 - A^j B_j C_i \end{cases} \quad (2.8)$$

This too, should be a familiar cross product to the reader.

Now, we substitute $\mathbf{A} = \boldsymbol{\sigma}$, $\mathbf{B} = \mathbf{X}$ and $\mathbf{C} = \mathbf{p}$ into (2.7), but make use of the canonical commutator $\left[x_\mu, p_\nu\right] = i\hbar\delta_{\mu\nu}$ to get to the final line, thus:

$$\left[\boldsymbol{\sigma}\times(\mathbf{x}\times\mathbf{p})\right]_3 = \varepsilon_{3jk}\varepsilon^{lmk}\sigma^j x_l p_m = (\boldsymbol{\sigma}\cdot\mathbf{p})x_3 - (\boldsymbol{\sigma}\cdot\mathbf{x})p_3 + i\hbar\sigma^3 = \sigma^j p_j x_3 - \sigma^j x_j p_3 + i\hbar\sigma_3 . \quad (2.9)$$

This is analogous to (1.7), and generalizes to the (1.8) analog:

$$\begin{cases} [\boldsymbol{\sigma}\times(\mathbf{x}\times\mathbf{p})] = (\boldsymbol{\sigma}\cdot\mathbf{p})\mathbf{x} - (\boldsymbol{\sigma}\cdot\mathbf{x})\mathbf{p} + i\hbar\boldsymbol{\sigma} \\ \varepsilon_{ijk}\varepsilon^{lmk}\sigma^j x_l p_m = \sigma^j p_j x_i - \sigma^j x_j p_i + i\hbar\sigma_i \end{cases} . \quad (2.10)$$

Isolating the spin matrix $S_i \equiv \frac{1}{2}\hbar\sigma_i$ as in (1.9) now yields:

$$\begin{cases} i\hbar\mathbf{S} = [\mathbf{S}\times(\mathbf{x}\times\mathbf{p})] - (\mathbf{S}\cdot\mathbf{p})\mathbf{x} + (\mathbf{S}\cdot\mathbf{x})\mathbf{p} \\ i\hbar S_i = \varepsilon_{ijk}\varepsilon^{lmk} S^j x_l p_m - S^j p_j x_i + S^j x_j p_i \end{cases} . \quad (2.11)$$

Then, introducing the orbital angular momentum $\mathbf{L} = (\mathbf{x}\times\mathbf{p})$ as in (1.11) yields:

$$\begin{cases} i\hbar\mathbf{S} = (\mathbf{S}\times\mathbf{L}) - (\mathbf{S}\cdot\mathbf{p})\mathbf{x} + (\mathbf{S}\cdot\mathbf{x})\mathbf{p} \\ i\hbar S_i = \varepsilon_{ijk} S^j L^k - S^j p_j x_i + S^j x_j p_i \end{cases} . \quad (2.12)$$

Finally, we use $[s^m, s^n] = i\hbar\varepsilon^{mni} s_i$ to convert over to:

$$[S^m, S^n] = S^m L^n - \varepsilon^{mni} S^j p_j x_i + \varepsilon^{mni} S^j x_j p_i \quad (2.13)$$

Now we have a desired term $S^m L^n$ with **S** to the left of **L**, but we need to reverse indexes in order form the anticommutator term $S^n L^m$ to be used with (2.1). This can be done with some renaming and simple rearrangement of (2.13), thus:

$$[S^m, S^n] = -S^n L^m - \varepsilon^{mni} S^j p_j x_i + \varepsilon^{mni} S^j x_j p_i \quad (2.14)$$

Then, we may add together (2.1) with (2.14), and also use $2[S^m, S^n] = 2i\hbar\varepsilon^{mni} S_i$, to obtain:

$$i\hbar\varepsilon^{mni} S_i = [S^m, S^n] = \tfrac{1}{2}[L^m, S^n] + \tfrac{1}{2}\varepsilon^{mni}[x_i, p_j S^j] - \tfrac{1}{2}\varepsilon^{mni}[p_i, x_j S^j] \quad (2.15)$$

This contains the desired commutator $[L^m, S^n]$. To arrive at (2.15) by adding (2.1) with (2.14), we note that the S^j can be commuted with **p** and **x**, and so employ $[S^j, p_j] = [S^j, x_j] = 0$. We can then multiply this through by ε_{mnk} to obtain, with $\varepsilon^{imn}\varepsilon_{ijk} = \delta^{imn}{}_{ijk}$, and subsequent to some index renaming:

$$i\hbar S_i = \tfrac{1}{4}\varepsilon_{ijk}[L^j, S^k] + \tfrac{1}{2}[x_i, p_j S^j] - \tfrac{1}{2}[p_i, x_j S^j] \quad (2.16)$$

Isolating $[L^m, S^n]$ from (2.15) also yields:

$$[L^m, S^n] = 2[S^m, S^n] - \varepsilon^{mni}[x_i, p_j S^j] + \varepsilon^{mni}[p_i, x_j S^j] \quad (2.17)$$

This is our desired commutator between **L** and **S**.

Now, it is time to make use of the independently-obtained $[L^m, S^n] = [L^n, S^m]$ from (2.5), which is based on $J^m = L^m + S^m$ together with the angular momentum commutation relationships (2.2) and (2.3). Using (2.5) in (2.17) allows us to eliminate $[L^m, S^n] = [L^n, S^m]$, leaving behind, following some rearrangement:

$$i\hbar\varepsilon^{mni}S_i = [S^m, S^n] = \tfrac{1}{2}\varepsilon^{mni}[x_i, p_j S^j] - \tfrac{1}{2}\varepsilon^{mni}[p_i, x_j S^j], \quad (2.18)$$

where we have also employed $[S^m, S^n] = i\hbar\varepsilon^{mni}S_i$. Comparing and combining (2.18) with (2.15) then yields the independent result:

$$\boxed{[L^m, S^n] = 0} \quad (2.19)$$

That is, the net result of combining $[L^m, S^n] = [L^n, S^m]$ with (2.17) yields (2.19) which tells us that the orbital and spin angular momentum operators are mutually-commuting, which is another way of saying that L^m and S^m are distinct, independent operators, which is another way of saying that the orbital and spin angular momentum are two distinct, mutually-independent degrees of freedom. This, of course, is already known, but it is of interest that this known fact can be deduced via the spin decomposition (1.11) and $J^m = L^m + S^m$ and the commutation relationships (2.2), (2.3).

With (2.19), (2.16) further reduces to:

$$i\hbar S_i = \tfrac{1}{2}[x_i, p_j S^j] - \tfrac{1}{2}[p_i, x_j S^j], \quad (2.20)$$

which is yet another way of decomposing S_i to include position and momentum operators. This, however, is just the canonical commutation relationship in disguise, because S^j can be removed from and placed to the right of the commutators. Then, since $[x_\mu, p_\nu] = i\hbar\delta_{\mu\nu} = [x_\nu, p_\mu]$ is symmetric under transposition of indexes μ, ν, we may rewrite (2.20) identically, as:

$$i\hbar S_i = \tfrac{1}{2}\left[[x_i, p_j] - [p_i, x_j]\right]S^j = \tfrac{1}{2}\left[i\hbar\delta_{ij} + i\hbar\delta_{ij}\right]S^j = i\hbar S_i, \quad (2.21)$$

This serves as a consistency check, and assures us that the results derived thus far are fully consistent with all of the fundamental commutation relationships $[x_\mu, p_\nu] = i\hbar\delta_{\mu\nu}$, $[S^m, S^n] = i\hbar\varepsilon^{mni}S_i$, $[L^m, L^n] = i\hbar\varepsilon^{mni}L_i$ and $[J^m, J^n] = i\hbar\varepsilon^{mni}J_i$, as well as $J^m = L^m + S^m$.

3. The Schwinger Magnetic Moment Anomaly, and Heisenberg Uncertainty

Of equal importance to Ohanian's derivation of intrinsic spin as a circulating flow of energy in the electron wave field, is his related derivation of the electron magnetic moment as an oppositely-circulating flow of electric charge. The decomposition of Ohanian's equation (25) for the magnetic moment, may yield some new insight into the *anomalous* part of the observed magnetic moment, as will now be developed below.

We start now with Ohanian's equation (25), which we rewrite here as:

$$|\mathbf{m}| = \left| -\frac{e\hbar}{2m}\psi^{\dagger}\gamma^{0}\boldsymbol{\sigma}\psi \right| = \left| -g_{D}\frac{e\hbar}{4m}\psi^{\dagger}\gamma^{0}\boldsymbol{\sigma}\psi \right| \quad (3.1)$$

where $g_D = 2$ is the Dirac gyromagnetic "g-factor" ratio, without any Schwinger-based correction to account for "anomaly." We know that $g_D = 2$ is the appropriate choice at this juncture, because Ohanian's (25) is based directly on Dirac's equation without any perturbative analysis. We also employ the absolute value $|\ |$ of all of the foregoing.

Now, let's make use of the decomposition (1.11), expressed in terms of $\boldsymbol{\sigma}$, to write:

$$|\mathbf{m}| = \left| -g_{D}\frac{e\hbar}{4m}\int\psi^{\dagger}\gamma^{0}\boldsymbol{\sigma}\psi d^{3}x \right| = \left| ig_{D}\frac{e}{4m}\int\psi^{\dagger}\gamma^{0}\left[(\mathbf{L}\times\boldsymbol{\sigma}) + \mathbf{x}(\mathbf{p}\cdot\boldsymbol{\sigma}) - \mathbf{p}(\mathbf{x}\cdot\boldsymbol{\sigma})\right]\psi d^{3}x \right|. \quad (3.2)$$

Interestingly, the above introduces the cross product $\mathbf{L}\times\boldsymbol{\sigma}$ as one of the terms which determines the magnetic moment, though we will not consider this term right now.

To keep things in the simplest case, let's consider an electron for which the orbital angular momentum is zero, $\mathbf{L} = 0$, that is, an electron with only intrinsic spin and no orbital angular momentum. Then, (3.2) becomes:

$$|\mathbf{m}| = \left| ig_{D}\frac{e}{4m}\int\psi^{\dagger}\gamma^{0}\left[\mathbf{x}(\mathbf{p}\cdot\boldsymbol{\sigma}) - \mathbf{p}(\mathbf{x}\cdot\boldsymbol{\sigma})\right]\psi d^{3}x \right|. \quad (3.3)$$

The terms sandwiched between $\psi^{\dagger}, \psi$ can now be converted over into an expectation value for the magnetic moment operator, via $\langle x \rangle = \psi^{\dagger} x \psi$. Thus, we now write (3.3) as:

$$|\mathbf{m}_{op}| = \left| ig_{D}\frac{e}{4m}\left\langle \gamma^{0}\left[\mathbf{x}(\mathbf{p}\cdot\boldsymbol{\sigma}) - \mathbf{p}(\mathbf{x}\cdot\boldsymbol{\sigma})\right]\right\rangle \right|, \quad (3.4)$$

where $\mathbf{m}_{op}$ is now the operator representing the magnetic moment. However, a detailed component-by-component calculation shows that $[\mathbf{x}(\mathbf{p}\cdot\boldsymbol{\sigma}) - \mathbf{p}(\mathbf{x}\cdot\boldsymbol{\sigma})] = [\mathbf{x},\mathbf{p}]\boldsymbol{\sigma}$, so that (3.4) may be further simplified to:

$$|\mathbf{m}_{op}| = \left| ig_{D}\frac{e}{4m}\left\langle \gamma^{0}[\mathbf{x},\mathbf{p}]\boldsymbol{\sigma}\right\rangle \right|, \quad (3.5)$$

Now, we take $\gamma^0, \boldsymbol{\sigma}$ to be constant operators (which they certainly are if defined using the Dirac anticommutator $\frac{1}{2}\{\gamma^\mu, \gamma^\nu\} \equiv \eta^{\mu\nu}$), and since $\langle Kx \rangle = K\langle x \rangle$ for any constant K, we take the constant factors outside the expectation brackets, and move most terms outside of the absolute value operator, to write:

$$|\mathbf{m}_{op}| = \left| ig_D \frac{e}{4m} \gamma^0 \langle [\mathbf{x}, \mathbf{p}] \rangle \boldsymbol{\sigma} \right| = \left| -g_D \frac{e\hbar}{4m} \gamma^0 \boldsymbol{\sigma} \right| = g_D \frac{e\hbar}{4m} \gamma^0 \boldsymbol{\sigma} = -ig_D \frac{e}{4m} \gamma^0 \left| \langle [\mathbf{x}, \mathbf{p}] \rangle \right| \boldsymbol{\sigma}, \quad (3.6)$$

also using $\langle [\mathbf{x}, \mathbf{p}] \rangle = \langle i\hbar \rangle = i\hbar$. This is just another way of stating (and arriving at) Ohanian's equation (27), and is a check on our calculation to this point. But, the presence of $\left| \langle [\mathbf{x}, \mathbf{p}] \rangle \right|$ in the above now gives us the connection we need to the Heisenberg uncertainty inequality. For, it is well known by the theorem of Robertson, that $i\Delta\mathbf{x}\Delta\mathbf{p} \geq \frac{1}{2}\left| \langle [\mathbf{x}, \mathbf{p}] \rangle \right| = \frac{1}{2} i\hbar$. As written in this form, this contains both $\Delta\mathbf{x}\Delta\mathbf{p} \geq \frac{1}{2}\hbar$ and $\frac{1}{2}\left| \langle [\mathbf{x}, \mathbf{p}] \rangle \right| = \frac{1}{2} i\hbar$. For the Hermitian **x**, **p**, the equality, rather than the inequality, applies to the special situation where the electron wavefunction is a precise, perfect Gaussian $\psi(x) = N \exp\left(-\frac{1}{2} A x^2\right)$. For any other form of wavefunction, for example, $\psi(x) = N \exp\left(-\frac{1}{2} A x^2 + Bx\right)$, it is the inequality which applies.

Therefore, first, let us use $i\Delta\mathbf{x}\Delta\mathbf{p} \geq \frac{1}{2}\left| \langle [\mathbf{x}, \mathbf{p}] \rangle \right| = \frac{1}{2} i\hbar$ to write (3.6) as:

$$|\mathbf{m}_{op}| = -ig_D \frac{e}{4m} \gamma^0 \left| \langle [\mathbf{x}, \mathbf{p}] \rangle \right| \boldsymbol{\sigma} = g_D \frac{e\hbar}{4m} \gamma^0 \boldsymbol{\sigma} \leq g_D \frac{e}{2m} \gamma^0 \boldsymbol{\sigma} (\Delta\mathbf{x}\Delta\mathbf{p}) \quad (3.7)$$

Now, let us focus on the Dirac gyromagnetic g-factor, $g_D = 2$. Specifically, let us set $g_D = |g_D| = 2$ in the $(\Delta\mathbf{x}\Delta\mathbf{p})$ term, but leave it as $|g_D|$ elsewhere. Then, let's extract out:

$$|g_D| = 2 \leq 2 \frac{(\Delta\mathbf{x}\Delta\mathbf{p})}{\hbar / 2} \quad (3.8)$$

This is obviously a truism, as it is just another way of writing the Heisenberg relation uncertainty relation $\Delta\mathbf{x}\Delta\mathbf{p} \geq \frac{1}{2}\hbar$. Yet the way in which we have come across this relationship here is suggestive of a way to supplement our understanding of the magnetic moment anomaly and the Heisenberg principle. Let's take this piece by piece.

First, consider the circumstance where the wave packet under consideration is a perfect Gaussian, $\psi(x) = N \exp\left(-\frac{1}{2} A x^2\right)$. Here, the equality $\Delta\mathbf{x}\Delta\mathbf{p} = \frac{1}{2}\hbar$ applies, and so (3.8) is:

$$|g_D| = 2 = 2\frac{(\Delta\mathbf{x}\Delta\mathbf{p})}{\hbar/2} \quad (3.9)$$

Let us take this to be a statement that for a perfect Gaussian wavepacket, the g-factor g is *exactly* equal to the Dirac value of 2. That is, for $\psi(x) = N\exp\left(-\tfrac{1}{2}Ax^2\right)$:

$$|g| = |g_D| = 2\frac{(\Delta\mathbf{x}\Delta\mathbf{p})}{\hbar/2} = 2\,, \quad (3.10)$$

because $\Delta\mathbf{x}\Delta\mathbf{p} = \tfrac{1}{2}\hbar$. Now, let us make *inductive hypothesis* that for anything other than a perfect Gaussian, e.g., $\psi(x) = N\exp\left(-\tfrac{1}{2}Ax^2 + Bx\right)$, the relationship (3.10) continues to hold, but now, in the more general form of an inequality:

$$\boxed{|g| = 2\frac{(\Delta\mathbf{x}\Delta\mathbf{p})}{\hbar/2} \geq 2}, \quad (3.11)$$

where the equality in (3.11) applies only to the special case of a Gaussian wavefunction. In (3.11), we are simultaneously saying a number of things:

First, if the hypothesis embodied in (3.11) is true, then the *greater than or equal to inequality* of Heisenberg says, in this context, that the magnitude of the *intrinsic* g-factor of a charged wavefunction is always *greater than or equal to* 2. That is, the *inequality* $\Delta\mathbf{x}\Delta\mathbf{p} \geq \tfrac{1}{2}\hbar$ becomes another way of stating a parallel inequality $|g| \geq 2$. We know this to be true for the charged leptons, which have $g_e/2 = 1.0011596521859$, $g_\mu/2 = 1.0011659203$, and $g_\tau/2 = 1.0011773$ respectively. [The foregoing data is extracted from *W.-M. Yao et al.,* J. Phys. G 33, 1 (2006)]

Secondly, the fact that the charged leptons have g-factors only slightly above 2, suggests that these a) differ from perfect Gaussian wavefunctions by only a very tiny amount, b) the electron is slightly more Gaussian than the muon, and the muon slightly more-so than the tauon. The three-quark proton, with $g_P/2 = 2.7928473565$, is definitively less-Gaussian the charged leptons. But, it is intriguing that the g-factor is now seen as a precise measure of the degree to which a wavefunction differs form a perfect Gaussian.

Third, (3.11) states that the magnetic moment *anomaly* via the g-factor is a *precise* measure of the degree to which $\Delta\mathbf{x}\Delta\mathbf{p}$ exceeds $\hbar/2$. This is best seen by writing (3.11) as:

$$\boxed{\Delta\mathbf{x}\Delta\mathbf{p} = \frac{|g|}{2}\frac{\hbar}{2}}. \quad (3.12)$$

Thus, for the electron, $(\Delta\mathbf{x}\Delta\mathbf{p})_e = 1.0011596521859 \cdot (\hbar/2)$, to give an exact numerical example. For a different example, for the proton, $(\Delta\mathbf{x}\Delta\mathbf{p})_P = 2.7928473565 \cdot (\hbar/2)$.

Fourth, as a philosophical and historical matter, one can achieve a new, deeper perspective abour the uncertainty. Classically, it was long thought that one can specify position and momentum simultaneously, with precision. To the initial consternation of many and the lasting consternation of some, it was found that even in principle, one could at best determine the standard deviations in position and momentum according to $\Delta\mathbf{x}\Delta\mathbf{p} \ge \frac{1}{2}\hbar$. There are two aspects of this consternation: First, one can never have $\Delta\mathbf{x}\Delta\mathbf{p} = 0$ as in classical theory. Second, that this is merely an *inequality*, not an exact expression, so that even for a particle with $\Delta\mathbf{x}\Delta\mathbf{p} \ge \frac{1}{2}\hbar$, we do not know for sure what is its exact value of $\Delta\mathbf{x}\Delta\mathbf{p}$. This latter issue is *not* an in-principle limitation on position and momentum measurements; it is a limitation on the present state of human knowledge.

Now, while $\frac{1}{2}\hbar$ is a lower bound *in principle*, the question remains open to the present day, whether there is a way, for a given particle, to specify the *precise* degree to which its $\Delta\mathbf{x}\Delta\mathbf{p}$ exceeds $\frac{1}{2}\hbar$, and how this would be measured. For example, one might ask, is there any particle in the real world that is a *perfect* Gaussian, and therefore can be located in spacetime and conjugate momentum space, down to exactly $\frac{1}{2}\hbar$. Equation (3.12) above suggests that if such a particle exists, it must be a perfect Gaussian, and, *that we would know it was a perfect Gaussian, if its g-factor was experimentally determined to be exactly equal to the Dirac value of 2*. Conversely, (3.12) tells us that it is the g-factor itself, which is the direct experimental indicator of the magnitude of $\Delta\mathbf{x}\Delta\mathbf{p}$ for any given particle wavefunction. The classical precision of $\Delta\mathbf{x}\Delta\mathbf{p} = 0$ comes full circle, and while it will never return, there is the satisfaction of being able to replace this with the quantum mechanical precision of (3.12), $\Delta\mathbf{x}\Delta\mathbf{p} = |g|\hbar/4$, rather than the weaker inequality of $\Delta\mathbf{x}\Delta\mathbf{p} \ge \frac{1}{2}\hbar$.

Fifth, if (3.12) is a correct hypothesis, then since it is independently known from Schwinger that $\frac{g}{2} = 1 + \frac{a}{2\pi} + \ldots$, this would mean that we would have to have:

$$\Delta\mathbf{x}\Delta\mathbf{p} = \frac{|g|}{2}\frac{\hbar}{2} = \left(1 + \frac{a}{2\pi} + \ldots\right)\frac{\hbar}{2} \quad (3.13)$$

Thus, from the perturbative viewpoint, the degree to which $\Delta\mathbf{x}\Delta\mathbf{p}$ exceeds $\frac{1}{2}\hbar$ would have to be a function of the running coupling strength $\alpha = e^2/4\pi$ in Heaviside-Lorentz units.

Sixth, since deviation of the g-factor above 2 would arise from a non-Gaussian wavefunction such as $\psi(x) = N\exp\left(-\frac{1}{2}Ax^2 + Bx\right)$, the rise of the g-factor above 2 would have to stem from the Bx term in this non-Gaussian wavefuction. In this regard, we note to start, that $N\int \exp\left(-\frac{1}{2}Ax^2 + Bx\right)dx = \sqrt{2\pi/A}\exp\left(B^2/2A\right)$, for a non-Gaussian wavefunction, versus $N\int \exp\left(-\frac{1}{2}Ax^2\right)dx = \sqrt{2\pi/A}$ for a perfect Gaussian.

Finally, to calculate this all out precisely, one would need to employ a calculation similar to that shown at http://en.wikipedia.org/wiki/Uncertainty_principle#Wave_mechanics, but for the non-Gaussian $N\int \exp\left(-\frac{1}{2}Ax^2 + Bx\right)dx = \sqrt{2\pi/A}\exp\left(B^2/2A\right)$ rather than the Gaussian $N\int \exp\left(-\frac{1}{2}Ax^2\right)dx = \sqrt{2\pi/A}$, to arrive at the modified bottom line equation of this Wiki section. That is the next calculation I plan, but this is enough, I believe, to post at this time.

Looking forward to comments.

Jay R. Yablon

Jay R. Yablon, April 24, 2008