

Heisenberg Uncertainty and the Schwinger Anomaly

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1. Introduction

In a fundamental 1986 paper [1] which has to date not received nearly the recognition warranted, Ohanian demonstrates based on a 1939 analysis by Belinfante, [2] that the intrinsic spin of a Dirac spinor is not an “abstruse quantum property of the electron . . . not amenable to physical explanation.” Rather, he demonstrates quite clearly how intrinsic spin “could be regarded as due to a circulating flow of energy, or a momentum density, in the electron wave field.”

This demonstration implicitly originates with the canonical energy momentum tensor $T^{\mu}_{\nu} = \frac{\partial L}{\partial(\partial_{\mu}\psi)}\partial_{\nu}\psi - L\delta^{\mu}_{\nu}$ for $\bar{\psi}, \psi$ based on the Dirac Lagrangian $L = \bar{\psi}(i\gamma^{\mu}\partial_{\mu} - m)\psi$ and symmetrized, from which one may obtain the momentum density three-vector $\mathbf{G} \equiv T^{0k}$ employed beginning in Ohanian’s equation (10). Then, in equation (16), Ohanian writes, essentially:

$$\mathbf{J} = \mathbf{L} + \mathbf{S} = \int (\mathbf{x} \times \mathbf{G}) d^3x = \frac{\hbar}{2i} \int \mathbf{x} \times [\psi^{\dagger}, \nabla] \psi d^3x + \frac{\hbar}{4} \int \mathbf{x} \times [\nabla \times (\psi^{\dagger} \boldsymbol{\Sigma} \psi)] d^3x \quad (1.1)$$

for the total angular momentum \mathbf{J} . After expansion of the triple cross product and integration by parts in equation (18), he then demonstrates that:

$$\mathbf{S} = \frac{\hbar}{4} \int \mathbf{x} \times [\nabla \times (\psi^{\dagger} \boldsymbol{\Sigma} \psi)] d^3x = \frac{\hbar}{2} \int (\psi^{\dagger} \boldsymbol{\Sigma} \psi) d^3x,^* \quad (1.2)$$

which is the expectation value $(\hbar/2) \int (\psi^{\dagger} \boldsymbol{\Sigma} \psi) d^3x = (\hbar/2) \langle \boldsymbol{\Sigma} \rangle$ of the associated spin operator $\mathbf{S}_{\text{op}} = (\hbar/2) \boldsymbol{\Sigma}$.

This leads to a completely “physical picture of the spin as due to a circulating energy flow in the Dirac field.” In particular, this picture of spin is based on the equation

$\mathbf{J} = \mathbf{L} + \mathbf{S} = \int (\mathbf{x} \times \mathbf{G}) d^3x$ wherein one merely takes the classical cross product $\mathbf{x} \times \mathbf{G}$ of position \mathbf{x} and momentum density \mathbf{G} at each event over a three-dimensional volume (hypersurface) and

* Here, we use $\boldsymbol{\Sigma}$ rather $\boldsymbol{\sigma}$ than to represent the spin (helicity) operator, to avoid confusion with the Pauli matrices to which they relate by $\text{diag}(\boldsymbol{\Sigma}) = (\boldsymbol{\sigma}, \boldsymbol{\sigma})$.

then integrates over the three-volume element d^3x . Because the total four momentum is given by $p^\mu = \int T^{0\mu} d^3x$ and so the three-momentum vector $\mathbf{p} = \int \mathbf{G} d^3x$, we have an entirely physical picture of the spin angular momentum as originating from the crossing of the physical position with the physical momentum at each event, integrated over the entire spatial expanse, just as how one would classically calculate the total angular momentum for any macroscopic body. The importance of this paper is thus that it demystifies intrinsic spin often thought to be a quintessential quantum mechanical phenomenon without classical basis, and instead constructs the spin out of crossing position and momentum operators in the usual, classical way.

Similarly, the magnetic moment is obtained based on a “circulating flow of charge” according to:

$$\mathbf{m} = -\frac{e\hbar}{2m} \int \psi^\dagger \boldsymbol{\gamma}^0 \boldsymbol{\Sigma} \psi d^3x, \quad (1.3)$$

which is the expectation value $-(e\hbar/2m) \int \psi^\dagger \boldsymbol{\gamma}^0 \boldsymbol{\Sigma} \psi d^3x = -(e\hbar/2m) \langle \boldsymbol{\gamma}^0 \boldsymbol{\Sigma} \rangle$ of the magnetic moment operator $\mathbf{m}_{\text{op}} = -(e/m) \boldsymbol{\gamma}^0 \mathbf{S}_{\text{op}}$.

Because the spin derivation in Ohanian’s approach does not proceed “via the Dirac equation by investigating the response of an electron to an external magnetic field” but rather is based on integrating the circulating flows of charge and energy without at any time introducing an external gauge field, this approach naturally leads to the consideration of the expectation values $\langle \mathbf{O} \rangle$ of various operators \mathbf{O} appearing in the integrals $\langle \mathbf{O} \rangle = \int \psi^\dagger \mathbf{O} \psi d^3x$. As we shall see, this may lead to a direct connection with both the Heisenberg uncertainty relationship and the anomalous magnetic moment which is simply not as evident from the more customary derivation which, as Ohanian states, “fails to provide a physical picture of the mechanism underlying the magnetic moment.

2. A Recasting of the Uncertainty Relationship

Let us begin by considering a current density $j^\mu = \bar{\psi} \boldsymbol{\gamma}^\mu \psi$. The $\mu = 0$ component of this current is of course the probability density $j^0 = \bar{\psi} \boldsymbol{\gamma}^0 \psi = \psi^\dagger \boldsymbol{\gamma}^0 \boldsymbol{\gamma}^0 \psi = \psi^\dagger \psi$, which, when integrated over an infinite volume, is then normalized such that $\int_{-\infty}^{+\infty} \psi^\dagger \psi d^3x = \langle 1 \rangle = 1$. If the

volume V is defined by $V \equiv \int_{-\infty}^{+\infty} d^3x$, then $\int_{-\infty}^{+\infty} \psi^\dagger \psi d^3x = \langle 1 \rangle = 1$ requires each wavefunction to include a covariant normalization $N = 1/\sqrt{V}$. This is also required to keep the total probability=1, dimensionless.

We start now with Ohanian's equation (25), which we write here in alternative forms as:

$$\begin{cases} \mathbf{m} = -\frac{e\hbar}{2m} \int \psi^\dagger \boldsymbol{\gamma}^0 \boldsymbol{\Sigma} \psi d^3x = -g_D \frac{e\hbar}{4m} \int \psi^\dagger \boldsymbol{\gamma}^0 \boldsymbol{\Sigma} \psi d^3x = -g_D \frac{e\hbar}{4m} \langle \boldsymbol{\gamma}^0 \boldsymbol{\Sigma} \rangle \\ m_i = -\frac{e\hbar}{2m} \int \psi^\dagger \boldsymbol{\gamma}^0 \Sigma_i \psi d^3x = -g_D \frac{e\hbar}{4m} \int \psi^\dagger \boldsymbol{\gamma}^0 \Sigma_i \psi d^3x = -g_D \frac{e\hbar}{4m} \langle \boldsymbol{\gamma}^0 \Sigma_i \rangle \end{cases} \quad (2.1)^*$$

where $g_D = 2$ is the Dirac gyromagnetic “g-factor” ratio, without any Schwinger-based correction to account for “anomaly.” We know that $g_D = 2$ is the appropriate choice at this juncture, because Ohanian's (25) is based directly on Dirac's equation without any perturbative analysis. We also make use of the expected value relationship $\langle \boldsymbol{\gamma}^0 \boldsymbol{\Sigma} \rangle = \int \psi^\dagger \boldsymbol{\gamma}^0 \boldsymbol{\Sigma} \psi d^3x$. Our goal is to see if we can approach and understand the “anomalous” deviation of the g-factor upwards from $g_D = 2$, i.e., $g \geq g_D = 2$, based on the uncertainty principle $\Delta \mathbf{x} \Delta \mathbf{p} \geq \frac{1}{2} \hbar$.

Now, let's engage in a simple exercise with Planck's constant \hbar . In particular, we can write \hbar in several different ways, namely:

$$\begin{cases} \hbar = \langle \hbar \rangle = i \langle [\mathbf{x}, \mathbf{p}] \rangle \leq 2 \Delta \mathbf{x} \Delta \mathbf{p} \\ \hbar \delta_{\mu\nu} = \langle \hbar \delta_{\mu\nu} \rangle = i \langle [x_\mu, p_\nu] \rangle \leq 2 \Delta x_\mu \Delta p_\nu \end{cases}, \quad (2.2)$$

where, in the above, we have employed $[x_\mu, p_\nu] = i\hbar \delta_{\mu\nu}$, i.e., $[\mathbf{x}, \mathbf{p}] = i\hbar$ and the antisymmetric portion of the Robertson-Schrödinger relation in the form $\Delta \mathbf{x} \Delta \mathbf{p} \geq \frac{1}{2} i \langle [\mathbf{x}, \mathbf{p}] \rangle$ to obtain a direct expression of the uncertainty relationship $\Delta x_\mu \Delta p_\nu \geq \frac{1}{2} \hbar \delta_{\mu\nu}$, i.e., $\Delta \mathbf{x} \Delta \mathbf{p} \geq \frac{1}{2} \hbar$. Above, we can use the rule that the expectation value of a matrix is a matrix of the expected values, because \hbar is a simple constant and is not a wavefunction operator. In contrast, this is not so for the term $\langle \boldsymbol{\gamma}^0 \boldsymbol{\Sigma} \rangle = \int \psi^\dagger \boldsymbol{\gamma}^0 \boldsymbol{\Sigma} \psi d^3x$ in (2.1), which must be a scalar number even though $\boldsymbol{\gamma}^0 \boldsymbol{\Sigma}$ itself is a 4x4 matrix.

With (2.2) in mind, let's go back and write (2.1) as:

* Unless otherwise stated, all integrals are presumed to be taken over the range $\int_{-\infty}^{\infty}$.

$$\left\{ \begin{aligned} |\mathbf{m}| &= g_D \frac{e\hbar}{4m} \langle \gamma^0 \boldsymbol{\Sigma} \rangle = g_D \frac{e}{4m} \langle \hbar \rangle \langle \gamma^0 \boldsymbol{\Sigma} \rangle = i g_D \frac{e}{4m} \langle [\mathbf{x}, \mathbf{p}] \rangle \langle \gamma^0 \boldsymbol{\Sigma} \rangle \leq g_D \frac{e}{2m} \langle \gamma^0 \boldsymbol{\Sigma} \rangle \Delta \mathbf{x} \Delta \mathbf{p} \\ |m_i| \delta_{\mu\nu} &= g_D \frac{e\hbar \delta_{\mu\nu}}{4m} \langle \gamma^0 \Sigma_i \rangle = g_D \frac{e}{4m} \langle \hbar \delta_{\mu\nu} \rangle \langle \gamma^0 \Sigma_i \rangle = i g_D \frac{e}{4m} \langle [x_\mu, p_\nu] \rangle \langle \gamma^0 \Sigma_i \rangle \leq g_D \frac{e}{2m} \langle \gamma^0 \Sigma_i \rangle \Delta x_\mu \Delta p_\nu. \end{aligned} \right. \quad (2.3)$$

The third term contains a product $\langle [x_\mu, p_\nu] \rangle \langle \gamma^0 \Sigma_i \rangle$ i.e., $\langle [\mathbf{x}, \mathbf{p}] \rangle \langle \gamma^0 \boldsymbol{\Sigma} \rangle$ of two expectation values one of which is a 4x4 matrix $\langle [x_\mu, p_\nu] \rangle$ and the other of which $\langle \gamma^0 \boldsymbol{\Sigma} \rangle$ is a scalar number.

Now, let's go back and extract from the above:

$$\left\{ \begin{aligned} g_D \frac{e\hbar}{4m} \langle \gamma^0 \boldsymbol{\Sigma} \rangle &\leq g_D \frac{e}{2m} \langle \gamma^0 \boldsymbol{\Sigma} \rangle \Delta \mathbf{x} \Delta \mathbf{p} \\ g_D \frac{e\hbar \delta_{\mu\nu}}{4m} \langle \gamma^0 \Sigma_i \rangle &\leq g_D \frac{e}{2m} \langle \gamma^0 \Sigma_i \rangle \Delta x_\mu \Delta p_\nu \end{aligned} \right. \quad (2.4)$$

and then let us divide out most of the terms to reduce to:

$$\left\{ \begin{aligned} g_D &\leq g_D \frac{\Delta \mathbf{x} \Delta \mathbf{p}}{\hbar/2} \\ g_D \delta_{\mu\nu} &\leq g_D \frac{\Delta x_\mu \Delta p_\nu}{\hbar/2} \end{aligned} \right. \quad (2.5)$$

Equation (2.5) above is obviously a truism, as it is just another way of writing the Heisenberg uncertainty relation $\Delta \mathbf{x} \Delta \mathbf{p} \geq \frac{1}{2} \hbar$, and in fact, we could have simply written (2.5) down on a sheet of paper without any derivation at all. Yet the way in which we have come across this relationship here in the context of considering the magnetic moment is suggestive of a way to supplement our understanding of the magnetic moment anomaly and the Heisenberg principle which is made more apparent based on Ohanian's approach than it is based on the more customary approach to the magnetic moment. Let's take this piece by piece.

3. A Hypothesis about Heisenberg Uncertainty and the Schwinger Anomaly

First, consider the circumstance where the wavefunction ψ under consideration is a perfect Gaussian, $\psi(x) = N \cdot u(\mathbf{p}) \exp(-\frac{1}{2} A x^2)$, where $u(\mathbf{p})$ is a dimensionless Dirac spinor which, as usual, is a function only of momentum and on-shell rest mass, $p^\sigma p_\sigma = m^2$, and $N = 1/\sqrt{V}$ as noted at the start of section 2. (For the moment, to simplify upcoming Gaussian calculations, we consider x in $\psi(x)$ to represent a single spacetime dimension; this will later be covariantly-generalized.) To maintain a dimensionless exponent in the wavefunction, it is apparent that the coefficient A must have mass dimension 2, because x^2 has mass dimension -2.

Here, for a Gaussian wavefunction, the equality $\Delta\mathbf{x}\Delta\mathbf{p} = \frac{1}{2}\hbar$ applies, and so (2.5) is:

$$g_D = 2 = 2 \frac{\Delta\mathbf{x}\Delta\mathbf{p}}{\hbar/2} \quad (3.1)$$

Let us take this to be a statement that for a *perfect* Gaussian wavepacket, the general g-factor g is *exactly* equal to the Dirac value of 2. That is, for $\psi(x) = N \cdot u(\mathbf{p}) \exp(-\frac{1}{2}Ax^2)$:

$$g = g_D = 2 \frac{\Delta\mathbf{x}\Delta\mathbf{p}}{\hbar/2} = 2, \quad (3.2)$$

because $\Delta\mathbf{x}\Delta\mathbf{p} = \frac{1}{2}\hbar$.

Now, we know that for any wavefunction *other than a perfect Gaussian*, e.g., for a wavefunction of the form $\psi(x) = N \cdot u(\mathbf{p}) \exp(-\frac{1}{2}Ax^2 - Bx - V(x))$, that $2 \frac{\Delta\mathbf{x}\Delta\mathbf{p}}{\hbar/2} > 2$, i.e., the Heisenberg *inequality* now applies. Again, we are just restating the uncertainty principle. But, what of the magnetic moment? Might it be, when a wavefunction is of a form other than Gaussian, that this concomitantly raises the (absolute value of the) magnetic moment g-factor above 2 as well, i.e., that $2 \frac{\Delta\mathbf{x}\Delta\mathbf{p}}{\hbar/2} > 2$ if and only if $g > 2$?

To pursue this further, let us in fact make *inductive hypothesis* that for any form of wavefunction *other than a perfect Gaussian*, e.g., for $\psi(x) = N \cdot u(\mathbf{p}) \exp(-\frac{1}{2}Ax^2 - Bx - V(x))$, the relationship (3.10) continues to hold, but now, in the more general form of an inequality:

$$\boxed{|g| = 2 \frac{\Delta\mathbf{x}\Delta\mathbf{p}}{\hbar/2} \geq 2}, \quad (3.3)$$

where the equality in (3.2) applies only to the special case of a Gaussian wavefunction. Then, let us explore the downstream consequences of this hypothesis to see where it leads us and if it can be made consistent with other known physics, especially, the perturbative results which lead to Schwinger's explanation of the magnitude of the charged leptons' magnetic moments. In (3.3), we are simultaneously saying a number of things:

First, if the hypothesis embodied in (3.3) is true, then the greater than or equal to inequality of Heisenberg says, in this context, that the magnitude of the intrinsic g-factor of a charged wavefunction is always greater than or equal to 2. That is, the *inequality* $\Delta\mathbf{x}\Delta\mathbf{p} \geq \frac{1}{2}\hbar$ becomes another way of stating a parallel *inequality* $|g| \geq 2$. We know this to be true for the

charged leptons, which have $g_e/2 = 1.0011596521859$, $g_\mu/2 = 1.0011659203$, and $g_\tau/2 = 1.0011773$ respectively. [3] By contrast, the leading terms in the Schwinger expansion with $\alpha = 1/137.036$ are given by $g/2 = 1 + a/2\pi = 1.00116140973$.

Secondly, given the experimental fact that the charged leptons have g-factors only slightly above 2, hypothesis (3.3) suggests that a) these charged leptons differ from perfect Gaussian wavefunctions by only a very tiny amount, b) the electron is slightly more Gaussian than the muon, and the muon slightly more-so than the tauon. The three-quark proton, with $g_p/2 = 2.7928473565$, is definitively less-Gaussian than the charged leptons.

Third, (3.3) states that the magnetic moment *anomaly* via the g-factor is a *precise measure of the degree to which $\Delta\mathbf{x}\Delta\mathbf{p}$ exceeds $\hbar/2$ and the degree to which a wavefunction differs from a perfect Gaussian*. This is best seen by writing (3.3) as:

$$\boxed{\Delta\mathbf{x}\Delta\mathbf{p} = \frac{|g|}{2} \frac{\hbar}{2} \geq \frac{\hbar}{2}}. \quad (3.4)$$

Thus, for the electron, $(\Delta\mathbf{x}\Delta\mathbf{p})_e = 1.0011596521859 \cdot (\hbar/2)$, to give an exact numerical example. For a different example, for the proton, $(\Delta\mathbf{x}\Delta\mathbf{p})_p = 2.7928473565 \cdot (\hbar/2)$.

Fourth, as a philosophical and historical matter, one can achieve a new, deeper perspective about uncertainty. Classically, it was long thought that one can specify position and momentum simultaneously, with precision. To the initial consternation of many and the lasting consternation of some, it was found that even in principle, one could at best determine the standard deviations in position and momentum according to $\Delta\mathbf{x}\Delta\mathbf{p} \geq \frac{1}{2}\hbar$. There are two aspects of this consternation: First, that one can never have $\Delta\mathbf{x}\Delta\mathbf{p} = 0$ as in classical theory. Second, that this is merely an *inequality*, not an exact expression, so that even for a particle with $\Delta\mathbf{x}\Delta\mathbf{p} \geq \frac{1}{2}\hbar$, we do not know for sure what is its exact value of $\Delta\mathbf{x}\Delta\mathbf{p}$. This second issue is *not* an in-principle limitation on position and momentum measurements: there is nothing which says in principle, for a wavefunction with $\Delta\mathbf{x}\Delta\mathbf{p} \geq \frac{1}{2}\hbar$, that we cannot state exactly the degree to which $\Delta\mathbf{x}\Delta\mathbf{p}$ exceeds $\frac{1}{2}\hbar$, as, for example, in (3.4), or via a numeric factor employed similarly to g in (3.4). Our inability to do so is a limitation merely on the present state of human knowledge.

Now, while $\frac{1}{2}\hbar$ is a lower bound *in principle*, the question remains open to the present day, whether there is a way, for a given particle, to specify the *precise* degree to which its $\Delta\mathbf{x}\Delta\mathbf{p}$ exceeds $\frac{1}{2}\hbar$, and how this would be measured. For example, one might ask, is there any particle in the real world that is a *perfect* Gaussian, and therefore can be located in spacetime and conjugate energy-momentum space, down to exactly $\frac{1}{2}\hbar$. Equation (3.4) above suggests that if such a particle exists, it must be a perfect Gaussian, and, *that we would know it was a perfect Gaussian, because its g-factor would be experimentally determined to be exactly equal to the Dirac value of 2*. Conversely, (3.4) tells us that *it is the g-factor itself, which is the direct experimental indicator of the magnitude of $\Delta\mathbf{x}\Delta\mathbf{p}$* for any given particle wavefunction. The classical precision of $\Delta\mathbf{x}\Delta\mathbf{p} = 0$ therefore comes full circle, and while it will never return, there would be the satisfaction of being able to replace this with the quantum mechanical precision of (3.4), $\Delta\mathbf{x}\Delta\mathbf{p} = |g|\hbar/4$, rather than the weaker inequality of $\Delta\mathbf{x}\Delta\mathbf{p} \geq \frac{1}{2}\hbar$.

Fifth, if (3.4) is a correct hypothesis, then since it is independently known from Schwinger that $\frac{g}{2} = 1 + \frac{a}{2\pi} + \dots$, this would mean that we would have to have:

$$\Delta\mathbf{x}\Delta\mathbf{p} = \frac{|g|}{2} \frac{\hbar}{2} = \left(1 + \frac{a}{2\pi} + \dots\right) \frac{\hbar}{2} \quad (3.13)$$

Thus, from the perturbative viewpoint, the degree to which $\Delta\mathbf{x}\Delta\mathbf{p}$ exceeds $\frac{1}{2}\hbar$ would have to be a function of the running coupling strength $\alpha = e^2/4\pi$ in Heaviside-Lorentz units. We note again, for $\alpha = 1/137.036$, that the first order terms $1 + a/2\pi = 1.00116140973$. We shall soon seek to exploit this connection between the Heisenberg principle and Schwinger's calculation of the magnetic anomaly.

Sixth, since the deviation of the g-factor upwards from 2 in (3.13) would have to arise from a *non-Gaussian* wavefunction, we shall consider wavefunctions including a generalized potential, of the form $\psi(x) = N \cdot u(\mathbf{p}) \exp(-\frac{1}{2}Ax^2 + Bx - V(x))$. Because $\exp(-\frac{1}{2}Ax^2 + Bx)dx$ is itself a Gaussian which leads to $\Delta\mathbf{x}\Delta\mathbf{p} = \hbar/2$, the rise of the g-factor above 2 would have to stem from the $V(x)$ term in this non-Gaussian wavefunction.

Thus, the goal from here is to calculate precisely, the form of the uncertainty principle for a non-Gaussian wavefunction. To frame the problem precisely: Consider a non-Gaussian

wavefunction given by the general form $\psi(x) = \exp\left(-\frac{1}{2}Ax^2 + Bx - V(x)\right)$. For the moment, we take x to be along a single t, x, y, z dimension of spacetime. We wish to deduce the product $\Delta x \Delta p$ of the root mean square deviations Δx and Δp_x as a function of A, B, V . Once we have this, we will see what constraints need to be placed on A, B, V to render this wave function consistent with Schwinger-type perturbation theory.

4. Selection of a Generalized Wavefunction, and Derivation of the Associated Integral Identity

In this section, as just noted, we start with a non-Gaussian wavefunction of the form:

$$\psi(x) = Ne^{-\frac{1}{2}A'x^2 + B'x - V'(x)}, \quad (4.1)$$

with a normalization N which we shall generally not show and rather leave implicit. For the moment, to remain perfectly general, we make no suppositions as to whether each of A', B', V' are real or imaginary. Therefore, from (4.1), we may define the probability density:

$$\rho = |\psi(x)|^2 = \psi(x) * \psi(x) = e^{-\frac{1}{2}Ax^2 + Bx - V(x)} \quad (4.2)$$

where we have defined $A \equiv A' + A'^*$, $B \equiv B' + B'^*$, and $V \equiv V' + V'^*$. We also take the position operator x to be self-adjoint, i.e., Hermitian, as usual.

First, let us obtain the Gaussian integral which we will need to use to carry out the critical steps of the calculations to follow. We start with the well-known Gaussian integral:

$$\int e^{-\frac{1}{2}Ax^2 + Bx} dx = \sqrt{\frac{2\pi}{A}} e^{\frac{B^2}{2A}}. \quad (4.3)$$

Now, let us seek a closed expression for the integral:

$$\int e^{-\frac{1}{2}Ax^2 - V(x) + Bx} dx = \int e^{-V(x)} e^{-\frac{1}{2}Ax^2 + Bx} dx = \int \left(1 - V(x) + \frac{1}{2!}V(x)^2 + \dots\right) e^{-\frac{1}{2}Ax^2 + Bx} dx, \quad (4.4)$$

where in the final expression we show the first two terms in the series expansion for $e^{-V(x)}$.

The potential $V(x)$, which is not known but something we would certainly like to know, we now take to be given by the perfectly-general polynomial:

$$V(x) \equiv \sum_{n=0}^{\infty} C^{(n)} x^n, \quad (4.5)$$

where the $C^{(n)}$ represent an infinite set of coefficients. Substituting (4.5) into (4.4) then allows us to write:

$$\begin{aligned}
\int e^{-\frac{1}{2}Ax^2 - V(x) + Bx} dx &= \int \left(1 - \sum_{n=0}^{\infty} C^{(n)} x^n + \frac{1}{2!} \left(\sum_{n=0}^{\infty} C^{(n)} x^n \right)^2 + \dots \right) e^{-\frac{1}{2}Ax^2 + Bx} dx \\
&= \int \left(1 - \sum_{n=0}^{\infty} C^{(n)} \left(\frac{d}{dB} \right)^n + \frac{1}{2!} \left(\sum_{n=0}^{\infty} C^{(n)} \left(\frac{d}{dB} \right)^n \right)^2 + \dots \right) e^{-\frac{1}{2}Ax^2 + Bx} dx \quad . \quad (4.6) \\
&= \int e^{-\sum_{n=0}^{\infty} C^{(n)} \left(\frac{d}{dB} \right)^n} e^{-\frac{1}{2}Ax^2 + Bx} dx = e^{-\sum_{n=0}^{\infty} C^{(n)} \left(\frac{d}{dB} \right)^n} \int e^{-\frac{1}{2}Ax^2 + Bx} dx
\end{aligned}$$

In the final step, we are able to move $e^{-\sum_{n=0}^{\infty} C^{(n)} \left(\frac{d}{dB} \right)^n}$ outside the integral, because it is no longer a direct function of x . Now, we proceed to define the function:

$$V\left(\frac{d}{dB}\right) \equiv \sum_{n=0}^{\infty} C^{(n)} \left(\frac{d}{dB}\right)^n, \quad (4.7)$$

so that (4.6), using the Gaussian integral (4.3), may finally be rewritten as:

$$\boxed{\int e^{-\frac{1}{2}Ax^2 - V(x) + Bx} dx = e^{-V\left(\frac{d}{dB}\right)} \int e^{-\frac{1}{2}Ax^2 + Bx} dx = e^{-V\left(\frac{d}{dB}\right)} \sqrt{\frac{2\pi}{A}} e^{\frac{B^2}{2A}} = \sqrt{\frac{2\pi}{A}} e^{-V\left(\frac{d}{dB}\right) + \frac{B^2}{2A}}.} \quad (4.8)$$

This is the integral expression which underlies what Zee [4] at 460 refers to as the ‘‘Central Identity of Quantum Field Theory.’’

5. Calculation of the Position Variance, for a Non-Gaussian Wavefunction

Now, we shall engage in three distinct calculations. First, for the wavefunction

$\psi(x) = Ne^{-\frac{1}{2}Ax^2 + Bx - V(x)}$ of (4.1), we obtain the variance $(\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2$. Second, we obtain

the Fourier transform of the wavefunction $\psi(p)$. Finally, we then obtain the variance

$(\Delta p)^2 = \langle p^2 \rangle - \langle p \rangle^2$. That will give us the ingredients necessary to calculation the uncertainty $\Delta x \Delta p$ which will be greater than $\hbar/2$ if the potential $V(d/dB)$ is non-zero, which we can then compare with our hypothesis relationship (3.4) of uncertainty to magnetic anomaly g-factor,

namely, $\Delta \mathbf{x} \Delta \mathbf{p} = \frac{|g|}{2} \frac{\hbar}{2} \geq \frac{\hbar}{2}$.

First, using (4.2) and (4.8) and $A = A' + A'^*$, $B = B' + B'^*$, we may write:

$$\begin{aligned}\langle x^2 \rangle &= \frac{\int \rho x^2 dx}{\int \rho dx} = \frac{\int x^2 e^{-\frac{1}{2}Ax^2+Bx-V(x)} dx}{\int e^{-\frac{1}{2}Ax^2+Bx-V(x)} dx} = -2 \frac{d}{dA} \ln \int e^{-\frac{1}{2}Ax^2+Bx-V(x)} dx = -2 \frac{d}{dA} \ln \left(\sqrt{\frac{2\pi}{A}} e^{-V\left(\frac{d}{dB}\right)} e^{\frac{B^2}{2A}} \right) \\ &= \frac{1}{A} + \frac{B^2}{A^2} = \frac{1}{A' + A'^*} + \frac{(B' + B'^*)^2}{(A' + A'^*)^2}\end{aligned}\quad (5.1)$$

Next, we may write:

$$\begin{aligned}\langle x \rangle &= \frac{\int \rho x dx}{\int \rho dx} = \frac{\int x e^{-\frac{1}{2}Ax^2+Bx-V(x)} dx}{\int e^{-\frac{1}{2}Ax^2+Bx-V(x)} dx} = \frac{d}{dB} \ln \int e^{-\frac{1}{2}Ax^2+Bx-V(x)} dx = \frac{d}{dB} \ln \left(e^{-V\left(\frac{d}{dB}\right)} \sqrt{\frac{2\pi}{A}} e^{\frac{B^2}{2A}} \right) \\ &= -\frac{dV}{dB} + \frac{B}{A} = -\frac{dV}{d(B' + B'^*)} + \frac{(B' + B'^*)}{(A' + A'^*)}\end{aligned}\quad (5.2)$$

where, from (4.7):

$$\frac{dV}{dB} = \frac{d}{dB} V\left(\frac{d}{dB}\right) = \sum_{n=0}^{\infty} C^{(n)} \left(\frac{d}{dB}\right)^{n+1}. \quad (5.3)$$

Combining (5.1) and (5.2) then yields the x variance:

$$\boxed{(\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2 = \frac{1}{A} + 2 \frac{dV}{dB} \frac{B}{A} - \left(\frac{dV}{dB}\right)^2 = \frac{1}{A} + 2 \frac{V}{A} - 2V \frac{d^2V}{dB^2}}, \quad (5.4)$$

where we have also employed (4.7) to obtain:

$$\frac{dV}{dB} \frac{B}{A} = \sum_{n=0}^{\infty} C^{(n)} \left(\frac{d}{dB}\right)^{n+1} \frac{B}{A} = \sum_{n=0}^{\infty} C^{(n)} \left(\frac{d}{dB}\right)^n \left(\frac{d}{dB}\right) \frac{B}{A} = \sum_{n=0}^{\infty} C^{(n)} \left(\frac{d}{dB}\right)^n \frac{1}{A} = \frac{V}{A}, \quad (5.5)$$

$$V^2 = \sum_{n=0}^{\infty} C^{(n)2} \left(\frac{d}{dB}\right)^{2n}, \quad (5.6)$$

and

$$\left(\frac{dV}{dB}\right)^2 = \sum_{n=0}^{\infty} C^{(n)2} \left(\frac{d}{dB}\right)^{2n+2} = \frac{d^2}{dB^2} \sum_{n=0}^{\infty} C^{(n)2} \left(\frac{d}{dB}\right)^{2n} = \frac{d^2}{dB^2} V^2 = 2V \frac{d^2V}{dB^2}. \quad (5.7)$$

In the circumstance where $V = 0$ and $A' = A'^*$ is real hence $A = 2A'$, (5.4) reduces to the usual result for a Gaussian wavefunction, $(\Delta x)^2 = 1/2A'$. Were A' to be imaginary, then $A = A' + A'^* = 0$, and (5.4) becomes infinite. Because of, we will henceforth select $A' = A'^*$ to be real.

6. Calculation of the Fourier Transform

Now we return to (4.1) to calculate the Fourier transform wavefunction $\psi(p)$. This may be obtained according to:

$$\begin{aligned}\psi(p) &= \int e^{ipx} e^{-\frac{1}{2}A'x^2 + B'x - V'(x)} dx = \int e^{-\frac{1}{2}A'\left(x - \frac{(B'+ip)}{A'}\right)^2 + \frac{1}{2}\frac{(B'+ip)^2}{A'} - V'(x)} dx = \int e^{-\frac{1}{2}A'x^2 + \frac{1}{2}\frac{(B'+ip)^2}{A'} - V'(x)} dx \\ &= e^{\frac{1}{2}\frac{(B'+ip)^2}{A'}} \int e^{-\frac{1}{2}A'x^2 - V'(x)} dx = \sqrt{\frac{2\pi}{A'}} e^{\frac{1}{2}\frac{(B'+ip)^2}{A'}} e^{-V'\left(\frac{d}{dB}\right)} = \sqrt{\frac{2\pi}{A'}} e^{-\frac{1}{2A'}p^2 + i\frac{B'}{A'}p + \frac{B'^2}{2A'} - V'\left(\frac{d}{dB}\right)},\end{aligned}\quad (6.1)$$

where we have completed the square in the usual way via $x - (B' + ip)/A' \rightarrow x$ and also used (4.8) with $B = 0$. The probability density ρ_p in momentum space, akin to (4.2), is then formed according to:

$$\rho_p = |\psi(p)|^2 = \psi(p) * \psi(p) = \frac{2\pi}{A'} e^{-\frac{1}{A'}p^2 + i\left(\frac{B'-B'^*}{A'}\right)p + \frac{B'^2 + B'^2}{2A'} - V}. \quad (6.2)$$

Above, we employ $V \equiv V' + V'^*$ and $A' = A'^*$ real.

7. Calculation of the Momentum Variance

Let us now calculate the momentum variance $(\Delta p)^2 = \langle p^2 \rangle - \langle p \rangle^2$. This is specified by:

$$\begin{aligned}\langle p^2 \rangle &= \frac{\int \rho_p p^2 dp}{\int \rho_p dp} = \frac{\int p^2 e^{-\frac{1}{A'}p^2 + i\left(\frac{B'-B'^*}{A'}\right)p - V} dp}{\int e^{-\frac{1}{A'}p^2 + i\left(\frac{B'-B'^*}{A'}\right)p - V} dp} \\ &= -\frac{d}{dA'^{-1}} \ln \int e^{-\frac{1}{A'}p^2 + i\left(\frac{B'-B'^*}{A'}\right)p - V} dp + i(B' - B'^*) \frac{\int p e^{-\frac{1}{A'}p^2 + i\left(\frac{B'-B'^*}{A'}\right)p - V} dp}{\int e^{-\frac{1}{A'}p^2 + i\left(\frac{B'-B'^*}{A'}\right)p - V} dp}\end{aligned}\quad (7.1)$$

where $2\pi/A'$ and $e^{\frac{B'^2 + B'^2}{2A'}}$ may be factored out since they are not functions of p . Here, if B' is real so that $B' - B'^* = 0$ and $B = B' + B'^* = 2B'$, then the final term of the above will drop out.

Let us in fact now select B' to be real, so that the second term of the above is eliminated. In that event, (4.2) further reduces to:

$$\rho_p = |\psi(p)|^2 = \psi(p) * \psi(p) = \frac{2\pi}{A'} e^{-\frac{1}{A'}p^2 + \frac{B'^2}{A'} - V}. \quad (7.2)$$

Then, using $\int e^{-\frac{1}{A'}p^2 - V} dp = e^{-V} \sqrt{\pi A'}$ based on (4.8), and $dA'^{-1} = -A'^{-2} dA'$, while keeping in mind that $V = V(d/dB)$, we can now complete the calculation in (7.1) to obtain:

$$\langle p^2 \rangle = -\frac{d}{dA'^{-1}} \ln \int e^{-\frac{1}{A'}p^2 - V} dp = -\frac{d}{dA'^{-1}} \ln(e^{-V} \sqrt{\pi A'}) = A'^2 \frac{d}{dA'} \ln(e^{-V} \sqrt{\pi A'}) = \frac{1}{2} A' \quad (7.3)$$

Finally, we calculate:

$$\langle p \rangle = \frac{\int \rho_p p dp}{\int \rho_p dp} = \frac{\int p e^{-\frac{1}{A'}p^2 - V} dp}{\int e^{-\frac{1}{A'}p^2 - V} dp} = \frac{\int p e^{-\frac{1}{A'}p^2} dp}{\int e^{-\frac{1}{A'}p^2} dp} = 0 \quad (7.4)$$

where we can factor $e^{-V(\frac{d}{dB})}$ outside of the integral since it is not a function of p . What remains is clearly equal to zero by symmetry. Thus:

$$\boxed{(\Delta p)^2 = \langle p^2 \rangle - \langle p \rangle^2 = \frac{1}{2} A'} \quad (7.5)$$

Now we return to (5.4) and employ $A = 2A'$ and $B = 2B'$ and also now regard the potential V' as real so that $V = V' + V'^* = 2V'$. Thus, we write:

$$(\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2 = \frac{1}{2A'} + 2 \frac{dV'}{dB'} \frac{B'}{A'} - \left(\frac{dV'}{dB'} \right)^2 = \frac{1}{2A'} + 2 \frac{V'}{A'} - 2V' \frac{d^2V'}{dB'^2} \quad (7.6)$$

The relationships (5.5) to (5.7) carry over in form for $A = 2A'$, $B = 2B'$, and $V = 2V'$.

Finally, we remove the “primes” so that we are working with the parameters and the potential which appear directly in the wavefunction (4.1), thereby rewriting this wavefunction as

$\psi(x) = N e^{-\frac{1}{2}Ax^2 + Bx - V(x)}$, we combine (7.5) and (7.6), we take the square root, and we restore the dimensional factor \hbar to obtain the uncertainty relationship:

$$\boxed{\Delta x \Delta p = \hbar \sqrt{\frac{1}{4} + V \left(1 - A \cdot \frac{d^2V}{dB^2} \right)} = \frac{\hbar}{2} \sqrt{1 + 4V \left(1 - A \cdot \frac{d^2V}{dB^2} \right)}} \quad (7.7)$$

This yields the Heisenberg “equality” $\Delta x \Delta p = \hbar/2$ whenever $V = 0$, and otherwise, so long as $A \cdot \frac{d^2V}{dB^2} < 1$, this will yield the Heisenberg inequality $\Delta x \Delta p > \hbar/2$. However, this is an

exact equation for $\Delta x \Delta p$, as a function of V and $A \cdot \frac{d^2 V}{dB^2}$. Now, finally, we can return to our

hypothesis (3.4), that $\Delta x \Delta p = \frac{|g| \hbar}{2}$. For, if this hypothesis is to be true, it is necessary that:

$$\boxed{\frac{|g|}{2} = \sqrt{1 + 4V \left(1 - A \cdot \frac{d^2 V}{dB^2} \right)} = 1 + \frac{a}{2\pi} + \dots}, \quad (7.8)$$

where we now also include the leading term of the Schwinger expansion $\frac{|g|}{2} = \left(1 + \frac{a}{2\pi} + \dots \right)$.

8. Development of the Connection between the Schwinger Anomaly and Heisenberg Uncertainty

Now, we use the series expansion

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \dots \quad (8.1)$$

with $x = 4V \left(1 - A \cdot \frac{d^2 V}{dB^2} \right)$ to rewrite (7.8) as:

$$\frac{|g|}{2} = \sqrt{1 + 4V \left(1 - A \cdot \frac{d^2 V}{dB^2} \right)} = 1 + 2V - 2V \cdot A \cdot \frac{d^2 V}{dB^2} - V^2 \left(1 - A \cdot \frac{d^2 V}{dB^2} \right)^2 + \dots = 1 + \frac{a}{2\pi} + \dots \quad (8.2)$$

WORK IN PROGRESS, TO BE CONTINUED.

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