Heisenberg Uncertainty and the Schwinger Anomaly

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1. Introduction

In a fundamental 1986 paper [1] which has to date not received nearly the recognition warranted, Ohanian demonstrates based on a 1939 analysis by Belinfante, [2] that the intrinsic spin of a Dirac spinor is not an "abstruse quantum property of the electron . . . not amenable to physical explanation." Rather, he demonstrates quite clearly how intrinsic spin "could be regarded as due to a circulating flow of energy, or a momentum density, in the electron wave field."

This demonstration implicitly originates with the canonical energy momentum tensor $T^{\mu}{}_{\nu} = \frac{\partial L}{\partial(\partial_\mu \psi)}\partial_\nu \psi - L\delta^{\mu}{}_{\nu}$ for $\overline{\psi}, \psi$ based on the Dirac Lagrangian $L = \overline{\psi}(i\gamma^\mu \partial_\mu - m)\psi$ and symmetrized, from which one may obtain the momentum density three-vector $\mathbf{G} \equiv T^{0k}$ employed beginning in Ohanian's equation (10). Then, in equation (16), Ohanian writes, essentially:

$$\mathbf{J} = \mathbf{L} + \mathbf{S} = \int (\mathbf{x} \times \mathbf{G}) d^3x = \frac{\hbar}{2i}\int \mathbf{x} \times [\psi^\dagger, \nabla]\psi d^3x + \frac{\hbar}{4}\int \mathbf{x} \times [\nabla \times (\psi^\dagger \mathbf{\Sigma} \psi)] d^3x \quad (1.1)$$

for the total angular momentum **J.** After expansion of the triple cross product and integration by parts in equation (18), he then demonstrates that:

$$\mathbf{S} = \frac{\hbar}{4}\int \mathbf{x} \times [\nabla \times (\psi^\dagger \mathbf{\Sigma} \psi)] d^3x = \frac{\hbar}{2}\int (\psi^\dagger \mathbf{\Sigma} \psi) d^3x,^* \quad (1.2)$$

which is the expectation value $(\hbar/2)\int (\psi^\dagger \mathbf{\Sigma} \psi) d^3x = (\hbar/2)\langle \mathbf{\Sigma} \rangle$ of the associated spin operator $\mathbf{S}_{\text{op}} = (\hbar/2)\mathbf{\Sigma}$.

This leads to a completely "physical picture of the spin as due to a circulating energy flow in the Dirac field." In particular, this picture of spin is based on the equation $\mathbf{J} = \mathbf{L} + \mathbf{S} = \int (\mathbf{x} \times \mathbf{G}) d^3x$ wherein one merely takes the classical cross product $\mathbf{x} \times \mathbf{G}$ of position **x** and momentum density **G** at each event over a three-dimensional volume (hypersurface) and

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* Here, we use $\mathbf{\Sigma}$ rather $\boldsymbol{\sigma}$ than to represent the spin (helicity)operator, to avoid confusion with the Pauli matrices to which they relate by $\text{diag}(\mathbf{\Sigma}) = (\boldsymbol{\sigma}, \boldsymbol{\sigma})$.

then integrates over the three-volume element d^3x. Because the total four momentum is given by $p^\mu = \int T^{0\mu} d^3x$ and so the three-momentum vector $\mathbf{p} = \int \mathbf{G}\, d^3x$, we have an entirely physical picture of the spin angular momentum as originating from the crossing of the physical position with the physical momentum at each event, integrated over the entire spatial expanse, just as how one would classically calculate the total angular momentum for any macroscopic body. The importance of this paper is thus that it demystifies intrinsic spin often thought to be a quintessential quantum mechanical phenomenon without classical basis, and instead constructs the spin out of crossing position and momentum operators in the usual, classical way.

Similarly, the magnetic moment is obtained based on a "circulating flow of charge" according to:

$$\mathbf{m} = -\frac{e\hbar}{2m}\int \psi^\dagger \gamma^0 \mathbf{\Sigma}\psi d^3x = -\mu_B \int \psi^\dagger \gamma^0 \mathbf{\Sigma}\psi d^3x, \quad (1.3)$$

which is the expectation value $-(e\hbar/2m)\int \psi^\dagger \gamma^0 \mathbf{\Sigma}\psi d^3x = -(e\hbar/2m)\langle \gamma^0 \mathbf{\Sigma}\rangle$ of the magnetic moment operator $\mathbf{m}_{\mathrm{op}} = -(e/m)\gamma^0 \mathbf{S}_{\mathrm{op}}$, where $\mu_B = e\hbar/2m$ is the Bohr magneton.

Because the spin derivation in Ohanian's approach does not proceed "via the Dirac equation by investigating the response of an electron to an external magnetic field" but rather is based on integrating the circulating flows of charge and energy without at any time introducing an external gauge field, this approach naturally leads to the consideration of the expectation values $\langle \mathbf{O}\rangle$ of various operators $\mathbf{O}$ appearing in the integrals $\langle \mathbf{O}\rangle = \int \psi^\dagger \mathbf{O}\psi d^3x$. As we shall see, this may lead to a direct connection with both the Heisenberg uncertainty relationship and the anomalous magnetic moment which is simply not as evident from the more customary derivation which, as Ohanian states, "fails to provide a physical picture of the mechanism underlying the magnetic moment.

2. A Recasting of the Uncertainty Relationship

Let us begin by considering a current density $j^\mu = \overline{\psi}\gamma^\mu \psi$. The $\mu = 0$ component of this current is of course the probability density $j^0 = \overline{\psi}\gamma^0\psi = \psi^\dagger\gamma^0\gamma^0\psi = \psi^\dagger\psi$, which, when integrated over an infinite volume, is then normalized such that $\int_{-\infty}^{+\infty} \psi^\dagger \psi d^3x = \langle 1\rangle = 1$. If the volume V is defined by $V \equiv \int_{-\infty}^{+\infty} d^3x$, then $\int_{-\infty}^{+\infty} \psi^\dagger \psi d^3x = \langle 1 \rangle = 1$ requires each wavefunction to include a covariant normalization $N = 1/\sqrt{V}$. This is also required to keep the total probability=1, dimensionless.

We start now with Ohanian's equation (25), which we write here in alternative forms as:

$$\begin{cases} \mathbf{m} = -\dfrac{e\hbar}{2m}\int \psi^\dagger \gamma^0 \mathbf{\Sigma} \psi d^3x = -g_D \dfrac{e\hbar}{4m}\int \psi^\dagger \gamma^0 \mathbf{\Sigma} \psi d^3x = -g_D \dfrac{e\hbar}{4m}\left\langle \gamma^0 \mathbf{\Sigma} \right\rangle \\ m_i = -\dfrac{e\hbar}{2m}\int \psi^\dagger \gamma^0 \Sigma_i \psi d^3x = -g_D \dfrac{e\hbar}{4m}\int \psi^\dagger \gamma^0 \Sigma_i \psi d^3x = -g_D \dfrac{e\hbar}{4m}\left\langle \gamma^0 \Sigma_i \right\rangle \end{cases} \quad (2.1)^*$$

where $g_D = 2$ is the Dirac gyromagnetic "g-factor" ratio, without any Schwinger-based correction to account for "anomaly." We know that $g_D = 2$ is the appropriate choice at this juncture, because Ohanian's (25) is based directly on Dirac's equation without any perturbative analysis. We also make use of the expected value relationship $\left\langle \gamma^0 \mathbf{\Sigma} \right\rangle = \int \psi^\dagger \gamma^0 \mathbf{\Sigma} \psi d^3x$. Our goal is to see if we can approach and understand the "anomalous" deviation of the g-factor upwards from $g_D = 2$, i.e., $g \geq g_D = 2$, based on the uncertainty principle $\Delta\mathbf{x}\Delta\mathbf{p} \geq \frac{1}{2}\hbar$.

Now, let's engage in a simple exercise with Planck's constant $\hbar$. In particular, we can write $\hbar$ in several different ways, namely:

$$\begin{cases} \hbar = \langle \hbar \rangle = i\langle [\mathbf{x}, \mathbf{p}] \rangle \leq 2\Delta\mathbf{x}\Delta\mathbf{p} \\ \hbar\delta_{\mu\nu} = \langle \hbar\delta_{\mu\nu} \rangle = i\langle [x_\mu, p_\nu] \rangle \leq 2\Delta x_\mu \Delta p_\nu \end{cases}, \quad (2.2)$$

where, in the above, we have employed $[x_\mu, p_\nu] = i\hbar\delta_{\mu\nu}$, i.e., $[\mathbf{x}, \mathbf{p}] = i\hbar$ and the antisymmetric portion of the Robertson-Schrödinger relation in the form $\Delta\mathbf{x}\Delta\mathbf{p} \geq \frac{1}{2} i\langle [\mathbf{x}, \mathbf{p}] \rangle$ to obtain a direct expression of the uncertainty relationship $\Delta x_\mu \Delta p_\nu \geq \frac{1}{2}\hbar\delta_{\mu\nu}$, i.e., $\Delta\mathbf{x}\Delta\mathbf{p} \geq \frac{1}{2}\hbar$. Above, we can use the rule that the expectation value of a matrix is a matrix of the expected values, because $\hbar$ is a simple constant and is not a wavefunction operator. In contrast, this is not so for the term $\left\langle \gamma^0 \mathbf{\Sigma} \right\rangle = \int \psi^\dagger \gamma^0 \mathbf{\Sigma} \psi d^3x$ in (2.1), which must be a scalar number even though $\gamma^0 \mathbf{\Sigma}$ itself is a 4x4 matrix.

With (2.2) in mind, let's go back and write (2.1) as:

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[*] Unless otherwise stated, all integrals are presumed to be taken over the range $\int_{-\infty}^{\infty}$.

$$\begin{cases} |\mathbf{m}| = g_D \dfrac{e\hbar}{4m}\left\langle \gamma^0 \mathbf{\Sigma}\right\rangle = g_D \dfrac{e}{4m}\langle \hbar \rangle \left\langle \gamma^0 \mathbf{\Sigma}\right\rangle = i g_D \dfrac{e}{4m}\langle [\mathbf{x},\mathbf{p}]\rangle \left\langle \gamma^0 \mathbf{\Sigma}\right\rangle \le g_D \dfrac{e}{2m}\left\langle \gamma^0 \mathbf{\Sigma}\right\rangle \Delta\mathbf{x}\Delta\mathbf{p} \\ |m_i|\delta_{\mu\nu} = g_D \dfrac{e\hbar\delta_{\mu\nu}}{4m}\left\langle \gamma^0 \Sigma_i\right\rangle = g_D \dfrac{e}{4m}\langle \hbar\delta_{\mu\nu} \rangle \left\langle \gamma^0 \Sigma_i\right\rangle = i g_D \dfrac{e}{4m}\langle [x_\mu, p_\nu]\rangle \left\langle \gamma^0 \Sigma_i\right\rangle \le g_D \dfrac{e}{2m}\left\langle \gamma^0 \Sigma_i\right\rangle \Delta x_\mu \Delta p_\nu \end{cases} \quad (2.3)$$

The third term contains a product $\langle [x_\mu, p_\nu]\rangle \langle \gamma^0 \Sigma_i \rangle$ i.e., $\langle [\mathbf{x},\mathbf{p}]\rangle \langle \gamma^0 \mathbf{\Sigma}\rangle$ of two expectation values one of which is a 4x4 matrix $\langle [x_\mu, p_\nu]\rangle$ and the other of which $\langle \gamma^0 \mathbf{\Sigma}\rangle$ is a scalar number.

Now, let's go back and extract from the above:

$$\begin{cases} g_D \dfrac{e\hbar}{4m}\left\langle \gamma^0 \mathbf{\Sigma}\right\rangle \le g_D \dfrac{e}{2m}\left\langle \gamma^0 \mathbf{\Sigma}\right\rangle \Delta\mathbf{x}\Delta\mathbf{p} \\ g_D \dfrac{e\hbar\delta_{\mu\nu}}{4m}\left\langle \gamma^0 \Sigma_i\right\rangle \le g_D \dfrac{e}{2m}\left\langle \gamma^0 \Sigma_i\right\rangle \Delta x_\mu \Delta p_\nu \end{cases} \quad (2.4)$$

and then let us divide out most of the terms to reduce to:

$$\begin{cases} g_D \le g_D \dfrac{\Delta\mathbf{x}\Delta\mathbf{p}}{\hbar/2} \\ g_D \delta_{\mu\nu} \le g_D \dfrac{\Delta x_\mu \Delta p_\nu}{\hbar/2} \end{cases} . \quad (2.5)$$

Equation (2.5) above is obviously a truism, as it is just another way of writing the Heisenberg uncertainty relation $\Delta\mathbf{x}\Delta\mathbf{p} \ge \frac{1}{2}\hbar$, and in fact, we could have simply written (2.5) down on a sheet of paper without any derivation at all, because all we are doing is rearranging the uncertainty relationship and then multiplying through by g_D. Yet the way in which we have come across this relationship here in the context of considering the magnetic moment is suggestive of a way to supplement our understanding of the magnetic moment anomaly and the Heisenberg principle which is made more apparent based on Ohanian's approach than it is based on the more customary approach to the magnetic moment. Let's take this piece by piece.

3. A Hypothesis about Heisenberg Uncertainty and the Schwinger Anomaly

First, consider the circumstance where the wavefunction ψ under consideration is a perfect Gaussian, $\psi(x) = N \cdot u(\mathbf{p})\exp\left(-\frac{1}{2}Ax^2\right)$, where $u(\mathbf{p})$ is a dimensionless Dirac spinor which, as usual, is a function only of momentum and on-shell rest mass, $p^\sigma p_\sigma = m^2$, and $N = 1/\sqrt{V}$ as noted at the start of section 2. (For the this discussion, we consider *x* in $\psi(x)$ to represent a single spacetime dimension.) To maintain a dimensionless exponent in the wavefunction, it is apparent that the coefficient A must have mass dimension 2, because x^2 has mass dimension -2.

Here, for a Gaussian wavefunction, the equality $\Delta\mathbf{x}\Delta\mathbf{p} = \frac{1}{2}\hbar$ applies, and so (2.5) is:

$$g_D = 2 = 2\frac{\Delta\mathbf{x}\Delta\mathbf{p}}{\hbar/2} \quad (3.1)$$

Let us take this to be a statement that for a *perfect* Gaussian wavepacket, the general g-factor g is *exactly* equal to the Dirac value of 2. That is, for $\psi(x) = N\cdot u(\mathbf{p})\exp\left(-\frac{1}{2}Ax^2\right)$:

$$g = g_D = 2\frac{\Delta\mathbf{x}\Delta\mathbf{p}}{\hbar/2} = 2, \quad (3.2)$$

because $\Delta\mathbf{x}\Delta\mathbf{p} = \frac{1}{2}\hbar$. The foregoing also applies to the form $\psi(x) = N\cdot u(\mathbf{p})\exp\left(-\frac{1}{2}Ax^2 + Bx\right)$, because one can always turn this into the form $\psi(x) = N\cdot u(\mathbf{p})\exp\left(-\frac{1}{2}A'x^2\right)$ by a suitable replacement of variables.

Now, we know that for any wavefunction *other than a perfect Gaussian*, e.g., for a wavefunction of, e.g., the form $\psi(x) = N\cdot u(\mathbf{p})\exp\left(-\frac{1}{2}Ax^2 - Bx - V(x)\right)$, that $2\frac{\Delta\mathbf{x}\Delta\mathbf{p}}{\hbar/2} > 2$, i.e., the Heisenberg *inequality* now applies. Again, we are just restating the uncertainty principle. But, what of the "*intrinsic*" magnetic moment "g-factor" $g \cong 2$? Might it be, when a wavefunction is of a form other than Gaussian, that this concomitantly raises the (absolute value of the) magnetic moment g-factor above 2 as well, i.e., that $2\frac{\Delta\mathbf{x}\Delta\mathbf{p}}{\hbar/2} > 2$ <u>if and only if</u> $g > 2$?

To explore this further, let us in fact make *inductive hypothesis* that for any form of wavefunction *other than a perfect Gaussian*, e.g., for $\psi(x) = N\cdot u(\mathbf{p})\exp\left(-\frac{1}{2}Ax^2 - Bx - V(x)\right)$, the relationship (3.10) continues to hold, but now, in the more general form of an inequality:

$$\boxed{|g| = 2\frac{\Delta\mathbf{x}\Delta\mathbf{p}}{\hbar/2} \geq 2}, \quad (3.3)$$

where the equality in (3.2) applies only to the special case of a Gaussian wavefunction. In the above, we need to keep sight of the fact that there are an infinite number of wave functions which a given an electron could be in, which would also give an infinite number of uncertainty relations which, according to (3.3), would also give an infinite number of possible values of g. For one example: if the electron possesses orbital angular momentum, then its overall "Landé g-factor" will be a function of both the spin and the orbital angular momentum, see, e.g., [3]. For a second example, taking $V(x)$ in to be a generalized "externally-applied" potential, one can always manipulate $V(x)$ to yield a range of uncertainty relationships, and this would then yield a corresponding range of values of g in (3.3).

So it needs to be made clear that the g in (3.3) is to be regarded as the "*intrinsic* uncertainty" of an electron, defined to be that portion of the electron's uncertainty which is intrinsic and irreducible, in the baseline circumstance where the electron has only its intrinsic spin and charge, and no orbital angular momentum (which from [3] is seen to raise the overall g factor from $g_s \cong 2$), and in the circumstance where there is no *externally-applied*, "extrinsic" potential $V_{\text{ext}}(x)$. In other words, in $\psi(x) = N \cdot u(\mathbf{p}) \exp\left(-\frac{1}{2}Ax^2 - Bx - V(x)\right)$, we take $V(x) \equiv V(x)_{\text{int}}$ to be an intrinsic potential of the electron wavefunction, separate and distinct from any extrinsic potential, and we take (3.3) to apply to the electron's *intrinsic* g-factor, and not to any enhancements to this g-factor which might arise from orbital angular momentum or an extrinsic potential. We will, in the course of the forthcoming development, seek to give precise mathematical definition to this "intrinsic uncertainty."

So, let us now start to explore the downstream consequences of this hypothesis to see where it leads us and if it can be made consistent with other known physics, especially, the perturbative results which lead to Schwinger's explanation of the magnitude of the charged leptons' magnetic moments. In (3.3), we are simultaneously saying a number of things:

First, if the hypothesis embodied in (3.3) is true, then the greater than or equal to inequality of Heisenberg says, in this context, that the magnitude of the intrinsic g-factor of a charged wavefunction is always greater than or equal to 2. That is, the *inequality* $\Delta\mathbf{x}\Delta\mathbf{p} \geq \frac{1}{2}\hbar$ becomes another way of stating a parallel *inequality* $|g| \geq 2$. We know this to be true for the charged leptons, which have intrinsic $g_e/2 = 1.00115965218 59$, $g_\mu/2 = 1.0011659203$, and $g_\tau/2 = 1.0011773$ respectively. [4] By contrast, the leading terms in the Schwinger expansion with $\alpha = 1/137.036$ are given by $g/2 = 1 + a/2\pi = 1.00116140973$.

Secondly, given the experimental fact that the charged leptons have intrinsic g-factors only slightly above 2, hypothesis (3.3) suggests that a) these charged leptons, in their intrinsic state, differ from perfect Gaussian wavefunctions by only a very tiny amount, b) the intrinsic electron is slightly more Gaussian than the intrinsic muon, and the intrinsic muon slightly more-so than the intrinsic tauon. The three-quark proton, with $g_P/2 = 2.7928473565$, is definitively less-Gaussian than the charged leptons.

Third, (3.3) states that the magnetic moment *anomaly* via the g-factor is a *precise measure of the degree to which the intrinsic* $\Delta\mathbf{x}\Delta\mathbf{p}$ *exceeds* $\hbar/2$ *and the degree to which the intrinsic wavefunction differs from a perfect Gaussian*. This is best seen by writing (3.3) as:

$$\boxed{\Delta\mathbf{x}\Delta\mathbf{p} = \frac{|g|}{2}\frac{\hbar}{2} \geq \frac{\hbar}{2}}. \quad (3.4)$$

Thus, for the electron, the intrinsic $(\Delta\mathbf{x}\Delta\mathbf{p})_e = 1.0011596521\ 859 \cdot (\hbar/2)$, to give an exact numerical example. For a different example, for the proton, $(\Delta\mathbf{x}\Delta\mathbf{p})_p = 2.7928473565 \cdot (\hbar/2)$.

Fourth, as a philosophical and historical matter, one can achieve a new, deeper perspective about uncertainty. Classically, it was long thought that one can specify position and momentum simultaneously, with precision. To the initial consternation of many and the lasting consternation of some, it was found that even in principle, one could at best determine the standard deviations in position and momentum according to $\Delta\mathbf{x}\Delta\mathbf{p} \geq \frac{1}{2}\hbar$. There are two aspects of this consternation: First, that one can never have $\Delta\mathbf{x}\Delta\mathbf{p} = 0$ as in classical theory. Second, that this is merely an *inequality*, not an exact expression, so that even for a particle with $\Delta\mathbf{x}\Delta\mathbf{p} \geq \frac{1}{2}\hbar$, we do not know for sure what is its exact value of $\Delta\mathbf{x}\Delta\mathbf{p}$. This second issue is *not* an in-principle limitation on position and momentum measurements: there is nothing which says in principle, for a wavefunction with $\Delta\mathbf{x}\Delta\mathbf{p} \geq \frac{1}{2}\hbar$, that we cannot state exactly the degree to which $\Delta\mathbf{x}\Delta\mathbf{p}$ exceeds $\frac{1}{2}\hbar$, as, for example, in (3.4), or via a numeric factor employed similarly to g in (3.4). Our inability to do so is a limitation merely on the present state of human knowledge.

Now, while $\frac{1}{2}\hbar$ is a lower bound *in principle*, the question remains open to the present day, whether there is a way, for a given electron wavefunction, to specify the *precise* degree to which its $\Delta\mathbf{x}\Delta\mathbf{p}$ exceeds $\frac{1}{2}\hbar$, and how this would be measured. For example, one might ask, is there any particle in the real world which, in its intrinsic state, is a *perfect* Gaussian, and therefore can be located in spacetime and conjugate energy-momentum space, down to exactly $\frac{1}{2}\hbar$. Equation (3.4) above suggests that if such a particle exists, it must be a perfect Gaussian, and, *that we would know it was a perfect Gaussian, because its g-factor would be experimentally determined to be exactly equal to the Dirac value of 2*. Conversely, (3.4) tells us that *it is the* *intrinsic g-factor itself, which is the direct experimental indicator of the magnitude of the intrinsic* $\Delta\mathbf{x}\Delta\mathbf{p}$ for any given particle wavefunction. The classical precision of $\Delta\mathbf{x}\Delta\mathbf{p} = 0$ therefore comes full circle, and while it will never return, there would be the satisfaction of being able to replace this with the quantum mechanical precision of (3.4), $\Delta\mathbf{x}\Delta\mathbf{p} = |g|\hbar/4$, rather than the weaker inequality of $\Delta\mathbf{x}\Delta\mathbf{p} \ge \frac{1}{2}\hbar$.

Fifth, if (3.4) is a correct hypothesis, then since it is independently known from Schwinger that $\frac{g}{2} = 1 + \frac{a}{2\pi} + \dots$, this would mean that we would have to have:

$$\Delta\mathbf{x}\Delta\mathbf{p} = \frac{|g|}{2}\frac{\hbar}{2} = \left(1 + \frac{a}{2\pi} + \dots\right)\frac{\hbar}{2} \quad (3.5)$$

Thus, from the perturbative viewpoint, the degree to which the intrinsic $\Delta\mathbf{x}\Delta\mathbf{p}$ exceeds $\frac{1}{2}\hbar$ would have to be a function of the running coupling strength $\alpha = e^2/4\pi$ in Heaviside-Lorentz units. We note again, for $\alpha = 1/137.036$, that the first order terms $1 + a/2\pi = 1.00116140973$. We shall soon seek to exploit this connection between the Heisenberg principle and Schwinger's calculation of the magnetic anomaly.

Sixth, since the deviation of the g-factor upwards from 2 in (3.5) would have to arise from a *non-Gaussian* wavefunction, we shall consider wavefunctions including a generalized *intrinsic* potential, of the form $\psi(x) = N \cdot u(\mathbf{p})\exp\left(-\frac{1}{2}Ax^2 + Bx - V(x)\right)$. Because $\exp\left(-\frac{1}{2}Ax^2 + Bx\right)dx$ is itself a Gaussian which leads to $\Delta\mathbf{x}\Delta\mathbf{p} = \hbar/2$, the rise of the g-factor above 2 would have to stem from the $V(x)$ term in this non-Gaussian wavefunction. Please note, while we refer to $V(x)$ as an intrinsic "potential," the better, more "clinical" mathematical description which does not in any way presuppose its physical interpretation, is to simply refer to $V(x)$ as a generalized polynomial in *x*.

Thus, the goal from here is to calculate precisely, the form of the uncertainty principle for a non-Gaussian wavefunction. To frame the problem precisely: Consider a non-Gaussian wavefunction given by the general form $\psi(x) = \exp\left(-\frac{1}{2}Ax^2 + Bx - V(x)\right)$, where $V(x)$ is any polynomial in *x*. For the moment, we take *x* to b along a single t, x, y, z dimension of spacetime. We wish to deduce the product $\Delta x \Delta p$ of the root mean square deviations Δx and Δp_x as a function of A, B, V. Once we have this, we will see what constraints need to be placed on

A, B, V to render this wave function consistent with Schwinger-type perturbation theory, and we will be able to more-precisely distinguish the *intrinsic* g-factor and uncertainty, from *extrinsic* variations in the g-factor and the uncertainty.

4. Selection of a Generalized Wavefunction, and Derivation of the Associated Integral Identity

In this section, as just noted, we start with a non-Gaussian wavefunction of the form:

$$\psi(x) = Ne^{-\frac{1}{2}A'x^2+B'x-V'(x)}, \quad (4.1)$$

with a normalization N which we shall generally not show and rather leave implicit. For the moment, to remain perfectly general, we make no suppositions as to whether each of A', B', V' are real or imaginary. Therefore, from (4.1), we may define the probability density:

$$\rho = |\psi(x)|^2 = \psi(x)^* \psi(x) = e^{-\frac{1}{2}Ax^2+Bx-V(x)} \quad (4.2)$$

where we have defined $A \equiv A' + A'^*$, $B \equiv B' + B'^*$, and $V \equiv V' + V'^*$. We also take the position operator x to be self-adjoint, i.e., Hermitian, as usual. We use the "primes" in (4.1) simply so that we can use unprimed symbols in (4.2).

First, let us obtain the Gaussian integral which we will need to use to carry out the critical steps of the calculations to follow. We start with the well-known Gaussian integral:

$$\int e^{-\frac{1}{2}Ax^2+Bx}dx = \sqrt{\frac{2\pi}{A}}e^{\frac{B^2}{2A}}. \quad (4.3)$$

Now, let us seek a closed expression for the integral $\int \rho dx = 1$ over the probability density in (4.2), that is, we seek:

$$1 = \int e^{-\frac{1}{2}Ax^2-V(x)+Bx}dx = \int e^{-V(x)}e^{-\frac{1}{2}Ax^2+Bx}dx = \int \left(1 - V(x) + \frac{1}{2!}V(x)^2 - ...\right)e^{-\frac{1}{2}Ax^2+Bx}dx, \quad (4.4)$$

where in the final expression we show the first two terms in the series expansion for $e^{-V(x)}$.

The "intrinsic potential" $V(x)$, we now take be given by the perfectly-general polynomial:

$$V(x) \equiv \sum_{n=0}^{\infty} C^{(n)}x^n, \quad (4.5)$$

where the $C^{(n)}$ represent an infinite set of coefficients. As noted, while we refer to this as an "intrinsic potential," it is perhaps better to think about this simply as an unspecified, completely-general polynomial in *x*.

Substituting (4.5) into (4.4) then allows us to write:

$$\int e^{-\frac{1}{2}Ax^2 - V(x) + Bx}dx = \int\left(1 - \sum_{n=0}^{\infty} C^{(n)}x^n + \frac{1}{2!}\left(\sum_{n=0}^{\infty} C^{(n)}x^n\right)^2 - \ldots\right)e^{-\frac{1}{2}Ax^2 + Bx}dx$$
$$= \int\left(1 - \sum_{n=0}^{\infty} C^{(n)}\left(\frac{d}{dB}\right)^n + \frac{1}{2!}\left(\sum_{n=0}^{\infty} C^{(n)}\left(\frac{d}{dB}\right)^n\right)^2 - \ldots\right)e^{-\frac{1}{2}Ax^2 + Bx}dx \qquad . \quad (4.6)$$
$$= \int e^{-\sum_{n=0}^{\infty} C^{(n)}\left(\frac{d}{dB}\right)^n} e^{-\frac{1}{2}Ax^2 + Bx}dx = e^{-\sum_{n=0}^{\infty} C^{(n)}\left(\frac{d}{dB}\right)^n}\int e^{-\frac{1}{2}Ax^2 + Bx}dx$$

Between the first two lines, the polynomial in *x* becomes a polynomial in the operator d/dB. In the final line, we are able to move $e^{-\sum_{n=0}^{\infty} C^{(n)}\left(\frac{d}{dB}\right)^n}$ outside the integral, because it is no longer a direct function of *x*. Now, we define the "potential" polynomial as a function of d/dB:

$$V\left(\frac{d}{dB}\right) \equiv \sum_{n=0}^{\infty} C^{(n)}\left(\frac{d}{dB}\right)^n, \quad (4.7)$$

so that (4.6), using the Gaussian integral (4.3), may finally be rewritten as:

$$\boxed{1 = \int e^{-\frac{1}{2}Ax^2 - V(x) + Bx}dx = e^{-V\left(\frac{d}{dB}\right)}\int e^{-\frac{1}{2}Ax^2 + Bx}dx = e^{-V\left(\frac{d}{dB}\right)}\sqrt{\frac{2\pi}{A}}e^{\frac{B^2}{2A}} = \sqrt{\frac{2\pi}{A}}e^{-V\left(\frac{d}{dB}\right) + \frac{B^2}{2A}}}. \quad (4.8)$$

This is the integral expression which underlies what Zee [5] at 460 refers to as the "Central Identity of Quantum Field Theory."

One other integral which we shall need is (4.8) above, but in the circumstance where $B = 0$. One can set $B = 0$ above to arrive at the right-hand side $\exp[-V(d/dB)]\sqrt{2\pi/A}$ directly, but it is prudent to be certain by calculating this integral explicitly. If we start with $\int e^{-\frac{1}{2}Ax^2 - V(x)}dx$, then by the substitution of variables $x \rightarrow x - B/A$, and moving terms out from the integral which are not functions of *x*, we may write out:

$$\int e^{-\frac{1}{2}Ax^2-V(x)}dx=\int e^{-V(x)}e^{-\frac{1}{2}Ax^2}dx=\int e^{-V(x)}e^{-\frac{1}{2}A\left(x-\frac{B}{A}\right)^2}dx=\int e^{-V(x)}e^{-\frac{1}{2}Ax^2+Bx-\frac{1}{2}\frac{B^2}{A}}dx$$
$$=e^{-\frac{1}{2}\frac{B^2}{A}}\int e^{-V(x)}e^{-\frac{1}{2}Ax^2+Bx}dx=e^{-\frac{1}{2}\frac{B^2}{A}}\int e^{-V\left(\frac{d}{dB}\right)}e^{-\frac{1}{2}Ax^2+Bx}dx=e^{-\frac{1}{2}\frac{B^2}{A}}e^{-V\left(\frac{d}{dB}\right)}\int e^{-\frac{1}{2}Ax^2+Bx}dx \quad . \quad (4.9)$$
$$=e^{-\frac{1}{2}\frac{B^2}{A}}e^{-V\left(\frac{d}{dB}\right)}\sqrt{\frac{2\pi}{A}}e^{\frac{1}{2}\frac{B^2}{A}}=e^{-V\left(\frac{d}{dB}\right)}\sqrt{\frac{2\pi}{A}}$$

where we also employ (4.3) in the final step. So we see that indeed, when $B=0$, we may simply substitute $B=0$ into (4.8), and still retain the operator term $\exp[-V(d/dB)]$ on the right hand side.

Before moving forward, a brief comment regarding the mass-dimensionality of A,B,V is in order. The exponent $-(1/2)Ax^2-V(x)+Bx$ in (4.8) must itself be dimensionless, otherwise one could not form a proper series from $\exp(-(1/2)Ax^2-V(x)+Bx)$. Because *x* has a mass dimension of -1, this means that *A* must have a mass-dimension of 2, and *B* a mass dimension of 1. The polynomial $V(x)$ must be dimensionless as well, so from (4.5), or from (4.7) together with the fact that *B* has a mass dimension of 1, each of the polynomial coefficients $C^{(n)}$ must have mass dimension *n*. With an overall factor $\propto 1/\sqrt{A}$ on the right hand side which thereby has mass dimension of -1, this then matches up in units of $\hbar=c=1$ with the left hand side for which the dimensionality originates in dx which also has mass dimension of -1.

5. Calculation of the Position Variance, for a Non-Gaussian Wavefunction

Now, we shall engage in three distinct calculations. First, for the wavefunction $\psi(x)=Ne^{-\frac{1}{2}A'x^2+B'x-V'(x)}$ of (4.1), we obtain the variance $(\Delta x)^2=\langle x^2\rangle-\langle x\rangle^2$. Second, we obtain the Fourier transform of the wavefunction $\psi(p)$. Finally, we then obtain the variance $(\Delta p)^2=\langle p^2\rangle-\langle p\rangle^2$. That will give us the ingredients necessary to calculate the uncertainty $\Delta x\Delta p$ which will be greater than $\hbar/2$ if the potential $V(d/dB)$ is non-zero. We can then compare our results with our hypothesis relationship (3.4), and develop a more precise formulation of this hypothesis as a mathematically-based theorem.

First, using (4.2) and (4.8) and $A=A'+A'^*$, $B=B'+B'^*$, we may write:

$$\left\langle x^2\right\rangle=\frac{\int\rho x^2 dx}{\int\rho dx}=\frac{\int x^2 e^{-\frac{1}{2}Ax^2+Bx-V(x)}dx}{\int e^{-\frac{1}{2}Ax^2+Bx-V(x)}dx}=-2\frac{d}{dA}\ln\int e^{-\frac{1}{2}Ax^2+Bx-V(x)}dx=-2\frac{d}{dA}\ln\left(\sqrt{\frac{2\pi}{A}}e^{-V\left(\frac{d}{dB}\right)}e^{\frac{B^2}{2A}}\right) \quad (5.1)$$
$$=\frac{1}{A}+\frac{B^2}{A^2}=\frac{1}{A'+A'^*}+\frac{(B'+B'^*)^2}{(A'+A'^*)^2}$$

Next, we may similarly write:

$$\left\langle x\right\rangle=\frac{\int\rho x dx}{\int\rho dx}=\frac{\int x e^{-\frac{1}{2}Ax^2+Bx-V(x)}dx}{\int e^{-\frac{1}{2}Ax^2+Bx-V(x)}dx}=\frac{d}{dB}\ln\int e^{-\frac{1}{2}Ax^2+Bx-V(x)}dx=\frac{d}{dB}\ln\left(e^{-V\left(\frac{d}{dB}\right)}\sqrt{\frac{2\pi}{A}}e^{\frac{B^2}{2A}}\right) \quad (5.2)$$
$$-\frac{dV}{dB}+\frac{B}{A}=-\frac{dV}{d\left(B'+B'^*\right)}+\frac{\left(B'+B'^*\right)}{\left(A'+A'^*\right)}$$

where, from (4.7):

$$\frac{dV}{dB}=\frac{d}{dB}V\left(\frac{d}{dB}\right)=\sum_{n=0}^{\infty}C^{(n)}\left(\frac{d}{dB}\right)^{n+1} . \quad (5.3)$$

Combining (5.1) and (5.2) then yields the x variance:

$$\boxed{(\Delta x)^2=\left\langle x^2\right\rangle-\left\langle x\right\rangle^2=\frac{1}{A}-\left(\frac{dV}{dB}\right)^2+2\frac{dV}{dB}\frac{B}{A}=\frac{1}{A}-2V\frac{d^2V}{dB^2}+2\frac{V}{A}}, \quad (5.4)$$

where we have also employed (4.7) to obtain:

$$\frac{dV}{dB}\frac{B}{A}=\sum_{n=0}^{\infty}C^{(n)}\left(\frac{d}{dB}\right)^{n+1}\frac{B}{A}=\sum_{n=0}^{\infty}C^{(n)}\left(\frac{d}{dB}\right)^{n}\left(\frac{d}{dB}\right)\frac{B}{A}=\sum_{n=0}^{\infty}C^{(n)}\left(\frac{d}{dB}\right)^{n}\frac{1}{A}=\frac{V}{A}, \quad (5.5)$$

$$V^2=\sum_{n=0}^{\infty}C^{(n)2}\left(\frac{d}{dB}\right)^{2n}, \quad (5.6)$$

and

$$\left(\frac{dV}{dB}\right)^2=\sum_{n=0}^{\infty}C^{(n)2}\left(\frac{d}{dB}\right)^{2n+2}=\frac{d^2}{dB^2}\sum_{n=0}^{\infty}C^{(n)2}\left(\frac{d}{dB}\right)^{2n}=\frac{d^2}{dB^2}V^2=2V\frac{d^2V}{dB^2}. \quad (5.7)$$

In the circumstance where $V=0$ and $A'=A'^*$ is real hence $A=A'+A'^*=2A'$, (5.4) reduces to the usual result for a Gaussian wavefunction, $(\Delta x)^2=1/2A'$. Were A' to be imaginary, then $A=A'+A'^*=0$, and (5.4) would become infinite. Because of this, we will henceforth select $A'=A'^*$ to be real, so that $A=2A'$.

While (5.4) clearly applies where B' is real, hence $B=B'+B'^*=2B'$, it is also necessary to consider the situation where B' is imaginary, because in that circumstance,

$B = B' + B'^* = 0$. This is this same circumstance which caused us to explicitly evaluate (4.9). In this event, using (4.9) in an intermediate step, and $A = 2A'$, (5.1) now becomes, for B' imaginary:

$$\left\langle x^2 \right\rangle = \frac{\int x^2 e^{-\frac{1}{2}Ax^2 - V(x)} dx}{\int e^{-\frac{1}{2}Ax^2 - V(x)} dx} = -2\frac{d}{dA}\ln\int e^{-\frac{1}{2}Ax^2 - V(x)} dx = -2\frac{d}{dA}\ln\left(\sqrt{\frac{2\pi}{A}} e^{-V\left(\frac{d}{dB}\right)}\right) = \frac{1}{A} = \frac{1}{2A'} , \quad (5.8)$$

Similarly (5.2) now becomes, for B' imaginary,

$$\left\langle x \right\rangle = \frac{\int x e^{-\frac{1}{2}Ax^2 - V(x)} dx}{\int e^{-\frac{1}{2}Ax^2 - V(x)} dx} = \frac{d}{dB}\ln\int e^{-\frac{1}{2}Ax^2 - V(x)} dx = \frac{d}{dB}\ln\left(\sqrt{\frac{2\pi}{A}} e^{-V\left(\frac{d}{dB}\right)}\right) = -\frac{dV}{dB} \quad (5.9)$$

Therefore, for B' imaginary, combining (5.7), (5.8) and (5.9):

$$\boxed{(\Delta x)^2 = \left\langle x^2 \right\rangle - \left\langle x \right\rangle^2 = \frac{1}{A} - \left(\frac{dV}{dB}\right)^2 = \frac{1}{A} - 2V\frac{d^2V}{dB^2}}. \quad (5.10)$$

This is the same as (5.4), absent the $2\frac{dV}{dB}\frac{B}{A} = 2\frac{V}{A}$ term expanded in (5.5).

6. Calculation of the Fourier Transform

Now we return to (4.1) to calculate the Fourier transform wavefunction $\psi(p)$. This may be obtained according to:

$$\boxed{\begin{aligned}\psi(p) &= \int e^{ipx} e^{-\frac{1}{2}A'x^2 + B'x - V'(x)} dx = \int e^{-\frac{1}{2}A'\left(x - \frac{(B'+ip)}{A'}\right)^2 + \frac{1}{2}\frac{(B'+ip)^2}{A'} - V'(x)} dx = \int e^{-\frac{1}{2}A'x^2 + \frac{1}{2}\frac{(B'+ip)^2}{A'} - V'(x)} dx \\ &= e^{\frac{1}{2}\frac{(B'+ip)^2}{A'}} \int e^{-\frac{1}{2}A'x^2 - V'(x)} dx = \sqrt{\frac{2\pi}{A'}} e^{\frac{1}{2}\frac{(B'+ip)^2}{A'}} e^{-V'\left(\frac{d}{dB}\right)} = \sqrt{\frac{2\pi}{A'}} e^{-\frac{1}{2A'}p^2 + i\frac{B'}{A'}p + \frac{B'^2}{2A'} - V'\left(\frac{d}{dB}\right)}\end{aligned}}, \quad (6.1)$$

where we have completed the square in the usual way via $x - (B' + ip)/A' \to x$ and also used (4.9). The probability density ρ_p in momentum space, akin to (4.2), is then formed from (6.1) according to:

$$\rho_p = \left|\psi(p)\right|^2 = \psi(p)^* \psi(p) = \frac{2\pi}{A'} e^{-\frac{1}{A'}p^2 + i\left(\frac{B' - B'^*}{A'}\right)p + \frac{B'^{*2} + B'^2}{2A'} - V} . \quad (6.2)$$

Above, we employ the earlier selection $A' = A'^*$ real, see just after (5.7), and $V = V' + V'^*$.

7. Calculation of the Momentum Variance

Let us now calculate the momentum variance $(\Delta p)^2 = \langle p^2 \rangle - \langle p \rangle^2$. This is specified by:

$$\langle p^2 \rangle = \frac{\int p^2 \rho_p dp}{\int \rho_p dp} = \frac{\int p^2 e^{-\frac{1}{A'}p^2 + i\left(\frac{B'-B'^*}{A'}\right)p - V} dp}{\int e^{-\frac{1}{A'}p^2 + i\left(\frac{B'-B'^*}{A'}\right)p - V} dp}$$

$$= -\frac{d}{dA'^{-1}} \ln \int e^{-\frac{1}{A'}p^2 + i\left(\frac{B'-B'^*}{A'}\right)p - V} dp + i(B' - B'^*) \frac{\int p e^{-\frac{1}{A'}p^2 + i\left(\frac{B'-B'^*}{A'}\right)p - V} dp}{\int e^{-\frac{1}{A'}p^2 + i\left(\frac{B'-B'^*}{A'}\right)p - V} dp} \quad , \quad (7.1)$$

$$= -\frac{d}{dA'^{-1}} \ln \int e^{-\frac{1}{A'}p^2 + i\left(\frac{B'-B'^*}{A'}\right)p - V} dp + i(B' - B'^*)\langle p \rangle$$

where $2\pi / A'$ and $e^{\frac{B'^{*2}+B'^2}{2A'}}$ from (6.2) may be removed from the integral and then divided out from the ratio $\int \rho_p p^2 dp / \int \rho_p dp$ since they are not functions of *p*. The above expression is more complicated than the analogous expression (5.1) for $\langle x^2 \rangle$, because following the Fourier transform, $1/A'$ appears in the coefficient for both p^2 and *p*, leading to the additional term $i(B' - B'^*)\langle p \rangle$. Here, if B' is real so that $B' - B'^* = 0$ and $B = B' + B'^* = 2B'$, then the final term of above will drop out. However, if $B' \equiv iB''$ is imaginary, then $B' - B'^* = 2iB'' = 2B'$, and the term $i(B' - B'^*)\langle p \rangle = 2iB'$ will *not* drop out.

We have already established that $A' = A'^*$ must be real, see just after (5.7). However, we do not want to rule out the possibility that B' is imaginary, because $B' \to iB'$ together with the $V = 0$ portion of wavefunction (4.1) leads to $\psi(x, V' = 0) = N \exp\left[-\tfrac{1}{2}A'x^2 + iB'x\right]$, the real portion $\mathrm{Re}(\psi(x, V' = 0)) = N \cos(B'x) \exp\left[-\tfrac{1}{2}A'x^2\right]$ of which is a non-dispersive wavepacket such as that illustrated at [6], and this is a desirable wavepacket to be able to consider. So, in the discussion to follow, we shall calculate the variance for each situation: where B' is real, *and* where B' is imaginary.

If B' is *real*, $B' - B'^* = 0$, the second $i(B' - B'^*)\langle p \rangle$ term of the above is eliminated, and the coefficient of *p* is also eliminated from the exponent. In that event, employing

$\int e^{-\frac{1}{A'}p^2-V}dp = e^{-V}\sqrt{\pi A'}$ based on (4.8), and $dA'^{-1} = -A'^{-2}dA'$, while keeping in mind that $V = V(d/dB)$, we can now complete the calculation in (7.1) to obtain:

$$\langle p^2\rangle = -\frac{d}{dA'^{-1}}\ln\int e^{-\frac{1}{A'}p^2-V}dp = -\frac{d}{dA'^{-1}}\ln\left(e^{-V}\sqrt{\pi A'}\right) = A'^2\frac{d}{dA'}\ln\left(e^{-V}\sqrt{\pi A'}\right) = \frac{1}{2}A' \quad (7.2)$$

Similarly, we may calculate:

$$\langle p\rangle = \frac{\int \rho_p p dp}{\int \rho_p dp} = \frac{\int pe^{-\frac{1}{A'}p^2-V}dp}{\int e^{-\frac{1}{A'}p^2-V}dp} = \frac{\int pe^{-\frac{1}{A'}p^2}dp}{\int e^{-\frac{1}{A'}p^2}dp} = 0 \quad (7.3)$$

where we can factor $e^{-V(d/dB)}$ outside of the integral since it is not a function of *p*. What remains is clearly equal to zero by the symmetry of the clearly-Gaussian term $\exp\left[-(1/A')p^2\right]$. Thus, for B' real:

$$\boxed{(\Delta p)^2 = \langle p^2\rangle - \langle p\rangle^2 = \frac{1}{2}A'} \quad (7.4)$$

For B' imaginary, $B' - B'^* = 2B'$, to complete the calculation (7.1), we need to directly specify $\langle p\rangle$, which we may do, again factoring out $e^{-V(d/dB)}$, as follows:

$$\langle p\rangle = \frac{\int p\rho_p dp}{\int \rho_p dp} = \frac{\int pe^{-\frac{1}{A'}p^2+2i\frac{B'}{A'}p-V}dp}{\int e^{-\frac{1}{A'}p^2+2i\frac{B'}{A'}p-V}dp} = \frac{\int pe^{-\frac{1}{A'}p^2+2i\frac{B'}{A'}p}dp}{\int e^{-\frac{1}{A'}p^2+2i\frac{B'}{A'}p}dp} = -\frac{1}{2}iA'\frac{d}{dB'}\ln\int e^{-\frac{1}{A'}p^2+2i\frac{B'}{A'}p}dp \quad (7.5)$$

Now, we return to write (7.1), using (7.5), and again factoring out $e^{-V(d/dB)}$, to find that:

$$\begin{aligned}\langle p^2\rangle &= \frac{\int p^2 e^{-\frac{1}{A'}p^2+2i\frac{B'}{A'}p}dp}{\int e^{-\frac{1}{A'}p^2+2i\frac{B'}{A'}p}dp} = -\frac{d}{dA'^{-1}}\ln\int e^{-\frac{1}{A'}p^2+2i\frac{B'}{A}p}dp + A'B'\frac{d}{dB'}\ln\int e^{-\frac{1}{A'}p^2+2i\frac{B'}{A'}p}dp \\ &= -\frac{d}{dA'^{-1}}\ln\left(\sqrt{\pi A'}e^{-\frac{B'^2}{A'}}\right) + A'B'\frac{d}{dB'}\ln\left(\sqrt{\pi A'}e^{-\frac{B'^2}{A'}}\right) = \frac{1}{2}A' - B'^2\end{aligned}\quad , \quad (7.6)$$

also using $\int e^{-\frac{1}{A'}p^2+2i\frac{B'}{A}p}dp = \sqrt{\pi A'}e^{-\frac{B'^2}{A'}}$ based on (4.3), and $dA'^{-1} = -A'^{-2}dA'$. Similarly, we may use (7.5) directly, to obtain:

$$\langle p\rangle = -\frac{1}{2}iA'\frac{d}{dB'}\ln\int e^{-\frac{1}{A'}p^2+2i\frac{B'}{A'}p}dp = -\frac{1}{2}iA'\frac{d}{dB'}\ln\sqrt{\pi A'}e^{-\frac{B'^2}{A'}} = iB'. \quad (7.7)$$

This means that $\langle p\rangle^2 = -B'^2$, so putting this together with (7.6), we obtain:

$$\boxed{(\Delta p)^2 = \langle p^2\rangle - \langle p\rangle^2 = \frac{1}{2}A'}\,, \quad (7.8)$$

which is identical to the result (7.4) for real B'.

8. The Uncertainty Relationship

Now, we have all of the ingredients we need to calculate a precise value for the uncertainty of the wavefunction (4.1). We have learned just after (5.7) that A' *must* be real for $(\Delta x)^2$ obtained in (5.4) to contain $1/A = 1/2A'$ rather than $1/A = 1/0 = \infty$, and we have shown how to obtain non-divergent expressions whether B' is real or imaginary, so that we may describe, among other possibilities, the uncertainty associated with a non-dispersive wavepacket such as that illustrated in [6].

Now, let's collect all the ingredients. For B' real, $B = 2B'$, we use (5.4), (7.4) and $A = 2A'$ to obtain:

$$(\Delta x)^2(\Delta p)^2 = \frac{1}{4} - \frac{1}{8}A'\left(\frac{dV}{dB'}\right)^2 + \frac{1}{2}B'\frac{dV}{dB'} = \frac{1}{4} - \frac{1}{4}A'V\frac{d^2V}{dB'^2} + \frac{1}{2}V \ , \quad (8.1)$$

For B' imaginary, $B = B' + {}^*B' = 0$, we use (5.10) and (7.8) (=(7.4)) to write:

$$(\Delta x)^2(\Delta p)^2 = \frac{1}{4} - \frac{1}{2}A'\left(\frac{dV}{dB}\right)^2 = \frac{1}{4} - A'V\frac{d^2V}{dB^2} . \quad (8.2)$$

which is (8.1) without the final term $\frac{1}{2}B'\frac{dV}{dB'} = \frac{1}{2}V$, and where we have left $B = B' + {}^*B' = 0$ in the denominator terms $dB = d(B' + {}^*B') = d0$. We shall consider this more closely shortly.

Now, we come to the polynomial $V \equiv V' + V'^*$ in the above. If V' is real, $V \equiv 2V'$. If V' is imaginary, then $V = 0$, and all of the above reduce to the expression $\Delta x \Delta p = \hbar/2$ for a Gaussian wavefunction. In other words, an imaginary polynomial V maintains the wavefunction to have a Gaussian character, insofar as the uncertainty relationship is maintained to be equal to $\hbar/2$. Therefore, we will now take V' to be real, hence $V \equiv 2V'$, and so we recompute (8.1) and (8.2) with $V \equiv 2V'$ to obtain, for real B':

$$(\Delta x)^2(\Delta p)^2 = \frac{1}{4} - \frac{1}{2}A'\left(\frac{dV'}{dB'}\right)^2 + B'\frac{dV'}{dB'} = \frac{1}{4} - A'V'\frac{d^2V'}{dB'^2} + V' \ , \quad (8.3)$$

and for B' imaginary:

$$(\Delta x)^2(\Delta p)^2 = \frac{1}{4} - 2A'\left(\frac{dV'}{dB}\right)^2 = \frac{1}{4} - 4A'V'\frac{d^2V'}{dB^2}. \quad (8.4)$$

Now, finally, we take the square root of the foregoing, and restoring dimensionality using $\hbar$, (8.3) and (8.4) for real and imaginary B', respectively, reduce to:

$$\boxed{\Delta x \Delta p = \frac{\hbar}{2}\sqrt{1 - 2A'\left(\frac{dV'}{dB'}\right)^2 + 4B'\frac{dV'}{dB'}} = \frac{\hbar}{2}\sqrt{1 - 4A'V'\frac{d^2V'}{dB'^2} + 4V'}}, \quad (8.5)$$

$$\Delta x \Delta p = \frac{\hbar}{2}\sqrt{1 - 8A'\left(\frac{dV'}{dB}\right)^2} = \frac{\hbar}{2}\sqrt{1 - 16A'V'\frac{d^2V'}{dB^2}}. \quad (8.6)$$

These are the exact uncertainty relationships, for a wavefunction (4.1), $\psi(x) = Ne^{-\frac{1}{2}A'x^2 + B'x - V'(x)}$, and its momentum space counterpart (6.1), $\psi(p) = \sqrt{\frac{2\pi}{A'}}e^{-\frac{1}{2A'}p^2 + i\frac{B'}{A'}p + \frac{B'^2}{2A'} - V'\left(\frac{d}{dB}\right)}$, where $B = B' + B'^*$. For the "potential" polynomial $V' = 0$, the uncertainty $\Delta x \Delta p = \hbar/2$.

Now, we consider that in (8.6), we have maintained $dB = d(B' + {}^*B') = d0$ in the denominator. The question then arises, whether there is some way to tame these terms and remove dB from the denominator, to obtain a finite expression, using, for example, the definitions (4.5) and (4.7) for the polynomial *V*, similarly to equations (5.5) to (5.7). For example, starting out from (4.7), and playing with the indexes over which the sum is taken, one may take note of the fact that:

$$\frac{dV}{dB} = \sum_{n=0}^{\infty} C^{(n)}\left(\frac{d}{dB}\right)^{n+1} = \sum_{n=1}^{\infty} C^{(n-1)}\left(\frac{d}{dB}\right)^n = \sum_{n=1}^{\infty} \frac{C^{(n-1)}}{C^{(n)}} C^{(n)}\left(\frac{d}{dB}\right)^n \quad (8.7)$$

and:

$$\sum_{n=1}^{\infty} C^{(n)}\left(\frac{d}{dB}\right)^n = V - C^{(0)} \quad (8.8)$$

This looks like it might yield an expression in $V - C^{(0)}$ which eliminates the unruly derivatives $dB = d0$ in the denominators, however, $\sum_{n=1}^{\infty} \frac{C^{(n-1)}}{C^{(n)}} C^{(n)}\left(\frac{d}{dB}\right)^n \neq \sum_{n=1}^{\infty} \frac{C^{(n-1)}}{C^{(n)}} \sum_{n=1}^{\infty} C^{(n)}\left(\frac{d}{dB}\right)^n$ that is, the sum of the products is not equal to the product of the sums, and so, at least by this approach, it appears that (8.6) cannot be tamed. Similarly, there does not appear to be any simple route to re-formulate (8.6) so as to eliminate all of the $dB = d0$ terms from the denominators. Therefore, we make the final choice, that B' is to be real.

Thus, for a wavefunction $\psi(x) = Ne^{-\frac{1}{2}A'x^2+B'x-V'(x)}$ where all of A', B', and V' are now real, the uncertainty is described, in a well-behaved fashion, by the boxed uncertainty relationship (8.5). This completes our exploration of the uncertainty relationship on its own terms, and is what we shall now use in the ensuing discussion of the Schwinger anomaly.

9. Development of the Possible Connection between the Schwinger Anomaly and Heisenberg Uncertainty

As we return to the earlier hypothesis (3.3), (3.4), it is important to point out that all of the development in sections 4 through 8 is completely independent of this hypothesis, and stands alone and apart. In these sections 4 through 8, we simply started with the wavefunction $\psi(x) = Ne^{-\frac{1}{2}A'x^2+B'x-V'(x)}$ in (4.1), and asked what its *precise* uncertainty would be, rather than settling for the usual inequality $\Delta x \Delta p \ge \hbar/2$. We found in the process, to obtain a well-behaved uncertainty expression, that A' and B' need to be real, and that if the polynomial V' is not real, then the uncertainty remains fixed at exactly $\Delta x \Delta p = \hbar/2$, and so chose V' to be real as well. The result of all these five sections worth of calculation, is equation (8.5).

Now, we return to the hypothesis (3.4) regarding the magnetic moment anomaly which, if true, must also satisfy (3.5). Taking (3.4) and (3.5) together with (8.5), this means that for this hypothesis to be true, we must have:

$$\frac{\Delta x \Delta p}{\hbar/2} = \sqrt{1 + 4B'\frac{dV'}{dB'} - 2A'\left(\frac{dV'}{dB'}\right)^2} = \sqrt{1 + 4V' - 4A'V'\frac{d^2V'}{dB'^2}} = \frac{|g|}{2} = 1 + \frac{a}{2\pi} + \dots \ , \quad (9.1)$$

where α is the running electromagnetic coupling for which $\alpha(\mu) \to 1/137.036$ at low probe energy μ.

If we use the first two terms of the series expansion:

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \dots \quad (9.2)$$

with, from (9.1):

$$x = 4B'\frac{dV'}{dB'} - 2A'\left(\frac{dV'}{dB'}\right)^2 = 4V' - 4A'V'\frac{d^2V'}{dB'^2} \quad (9.3)$$

we may rewrite (9.1) in expanded form as:

$$1 + 2B'\frac{dV'}{dB'} - A'\left(\frac{dV'}{dB'}\right)^2 - \ldots = 1 + 2V' - 2A'V'\frac{d^2V'}{dB'^2} - \ldots = 1 + \frac{a}{2\pi} + \ldots \ . \quad (9.4)$$

Now, there is no association which is compelled, but the simplest association in (9.4) is of $2V'$ to $\alpha/2\pi$. Let us in fact make this association, and then see what else it implies. That is, looking at the leading terms in the expansion (9.1) and in the Schwinger series, we set the dimensionless polynomial:

$$V' \equiv \alpha/4\pi \quad (9.5)$$

This means that the wavefunction with which we started way back in (4.1) now becomes:

$$\boxed{\psi(x) = Ne^{-\frac{1}{2}A'x^2 + B'x - \frac{\alpha}{4\pi}}} \quad (9.6)$$

and that the uncertainty relationship (9.1) now becomes:

$$\boxed{\frac{\Delta x \Delta p}{\hbar/2} = \sqrt{1 + \frac{1}{\pi}B'\frac{d\alpha}{dB'} - \frac{1}{8\pi^2}A'\left(\frac{d\alpha}{dB'}\right)^2} = \sqrt{1 + \frac{\alpha}{\pi} - A'\frac{\alpha}{4\pi^2}\frac{d^2\alpha}{dB'^2}} = \frac{|g|}{2} = 1 + \frac{a}{2\pi} + \ldots}, \quad (9.7)$$

Directly to the point: an electron with the wavefunction (9.6), with A' and B' real, *will* have the uncertainty relationship (9.7), period. For $\alpha = 1/137.036$, the leading uncertainty term $\sqrt{1 + \frac{a}{\pi}} = 1.00116073607$, while the leading anomaly term $1 + \frac{a}{2\pi} = 1.00116140973$. These two terms differ by just under 7 parts in 10^{-7}. Therefore, we can state the following:

Theorem: *For a wavefunction* $\psi(x) = Ne^{-\frac{1}{2}A'x^2 + B'x - \frac{\alpha}{4\pi}}$*, the uncertainty ratio* $\frac{\Delta x \Delta p}{\hbar/2}$*, to leading order in* α*, differs from the intrinsic Schwinger g-factor* $g/2$ *by less than 7 parts in* 10^{-7}.

We have stated this as a theorem, because this is a simple statement of fact, and involves no interpretation or hypothesis whatsoever. However, in order to sustain the broader hypothesis (3.4), we do need to engage in some interpretation.

First, we shall now define (9.6) as the *intrinsic wavefunction* of a ground state electron with no orbital angular momentum and no applied external potential. And, we define (9.7) as the *intrinsic uncertainty* of this intrinsic wavefunction. Not every electron will have this wavefunction or this uncertainty or this g-factor, but this wavefunction becomes the baseline electron wavefunction from which any variation is due to *extrinsic* factors, such as possessing orbital angular momentum or being placed into an external potential, for example, that of a proton. Thus, our hypothesis (3.4) is a hypothesis about the intrinsic uncertainty associated with the intrinsic wavefunction, and it says that:

Reformulated Hypothesis: *The intrinsic uncertainty associated with the intrinsic electron wavefunction is identical with the intrinsic g-factor of the anomalous magnetic moment.*

10. Introduction of an Extrinsic Potential and Extrinsic g-Factor

Now, let us return to (9.6), and reintroduce a term $V_{\text{ext}}(x)$, which is dimensionless, and which we shall call the *extrinsic potential*. In this vein, $V_{\text{int}} \equiv V' = \alpha / 4\pi$ of (9.5) is now to be referred to as the *intrinsic potential*. This means that the running coupling $\alpha = 4\pi V_{\text{int}}$, which measures the strength with which the electron charge interacts, may alternatively be considered as a measure of an intrinsic potential through which the ground state electron interacts with the vacuum, absent the application of *any* extrinsic potential. In different terms, this may be thought of as an intrinsic *self-potential* of the electron. This $V_{\text{ext}}(x)$ may readily be added to V_{int} to obtain a total potential $V_{\text{total}} = V_{\text{int}} + V_{\text{ext}}$. Thus, we now extend (9.6) to read:

$$\psi(x) = Ne^{-\frac{1}{2}A'x^2 + B'x - V_{\text{int}}(x) - V_{\text{ext}}(x)} = Ne^{-\frac{1}{2}A'x^2 + B'x - \frac{\alpha}{4\pi} - V_{\text{ext}}(x)} \qquad (10.1)$$

Then, without re-doing all of the calculation in sections 4 through 8, we can simply make the substitution $(4\pi V_{\text{int}} = \alpha) \rightarrow (4\pi V_{\text{int}} + 4\pi V_{\text{ext}} = \alpha + 4\pi V_{\text{ext}})$ in (9.7) to write:

$$\begin{aligned}\frac{\Delta x \Delta p}{\hbar / 2} &= \sqrt{1 + \frac{1}{\pi} B' \frac{d\alpha + 4\pi dV_{\text{ext}}}{dB'} - \frac{1}{8\pi^2} A' \left(\frac{d\alpha + 4\pi dV_{\text{ext}}}{dB'} \right)^2} \\ &= \sqrt{1 + \frac{\alpha + 4\pi V_{\text{ext}}}{\pi} - A' \frac{\alpha + 4\pi V_{\text{ext}}}{4\pi^2} \frac{d^2\alpha + 4\pi d^2 V_{\text{ext}}}{dB'^2}} \equiv \frac{|g_{\text{int}}|}{2} + \frac{|g_{\text{ext}}|}{2}\end{aligned} \qquad (10.2)$$

where $|g_{\text{ext}}|$ is an *extrinsic* contribution to the g-factor. Now, with the application of a V_{ext}, the uncertainty rises above the baseline intrinsic uncertainty. If the hypothesis (3.4) is to hold, that the g-factor is synonymous with the uncertainty, then the g-factor will increase as well, which we capture with the final term $|g_{\text{int}}|/2 + |g_{\text{ext}}|/2$.

Now the question becomes, to what degree, exactly, does the uncertainty increase upon the application of a given V_{ext}? The measure of this, is simply $|g_{\text{ext}}|/2$. So, if we now take $|g| = |g_{\text{int}}|$ in (9.7), and combine (9.7) with (10.2), we may calculate this directly as:

$$\boxed{\begin{aligned}\frac{|g_{\text{ext}}|}{2} &= \sqrt{1+\frac{1}{\pi}B'\frac{d\alpha+4\pi dV_{\text{ext}}}{dB'}-\frac{1}{8\pi^2}A'\left(\frac{d\alpha+4\pi dV_{\text{ext}}}{dB'}\right)^2}-\sqrt{1+\frac{1}{\pi}B'\frac{d\alpha}{dB'}-\frac{1}{8\pi^2}A'\left(\frac{d\alpha}{dB'}\right)^2} \\ &= \sqrt{1+\frac{\alpha+4\pi V_{\text{ext}}}{\pi}-A'\frac{\alpha+4\pi V_{\text{ext}}}{4\pi^2}\frac{d^2\alpha+4\pi d^2V_{\text{ext}}}{dB'^2}}-\sqrt{1+\frac{\alpha}{\pi}-A'\frac{\alpha}{4\pi^2}\frac{d^2\alpha}{dB'^2}}\end{aligned}} \quad (10.3)$$

Now, the question becomes, does $|g_{\text{ext}}|$, which is driven by the extrinsic potential V_{ext}, show up anywhere in observed physics?

To start, consider that the orbital angular momentum serves to shift the overall g-factor according to the Landé relationship: [3]

$$g_J = g_L\frac{J(J+1)-S(S+1)+L(L+1)}{2J(J+1)}+g_S\frac{J(J+1)+S(S+1)-L(L+1)}{2J(J+1)} \quad (10.4)$$

where $g_L = 1$ and $g_S \cong 2$. More generally, the magnetic moment:

$$\vec{\mu} = -\mu_B\left(g_L\vec{l}+g_S\vec{s}\right) \quad (10.5)$$

where $\mu_B = e\hbar/2m$ is the Bohr magneton, see our starting point (1.3). Whereas Zeeman splitting takes place in a weak magnetic field, consider instead, a strong field, which produces the Paschen-Back effect. For a state $|A\rangle$, this is specified by: [7]

$$\left\langle A\left|\left(H_0+\frac{B_z\mu_B}{\hbar}(L_Z+g_SS_Z)\right)\right|A\right\rangle = E_0+B_z\mu_B(m_L+g_Sm_S). \quad (10.6)$$

Referring to (9.7), we now associate $g_s = g = g_{\text{int}}$, and retain $g_L = 1$. We now use dimensional arguments, noting that E_0 is an energy with mass dimension of 1, that the magnetic field B_z has mass dimension 2, and that μ_B has mass dimension -1. Therefore, if we want g_{ext} with mass dimension to fit with g_L and g_{int} in (10.5) and (10.6), and we want the effect of g_{ext} to be independent of the magnetic field, then we need to increase the mass dimension of g_{ext} by 1. One way to do this, taking an educated guess, is to use the Bohr magneton as a divisor, which effectively uses the particle mass as the mass divisor. So, up to a constant of proportionality, we set:

$\mu_B E_0 \propto g_{\text{ext}}$ (10.7)

in (10.6), with g_{ext} given by (10.3), and driven by the extrinsic potential V_{ext}. The question then becomes experimental: does the foregoing formulation, or a related formulation using g_{ext} from (10.3) in connection with Paschen-Back, in fact yield the observed spectral lines?

WORK IN PROGRESS, TO BE CONTINUED.

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