Following some earlier feedback from KP, I would like feedback on whether the particular calculation (6)-(9) shown below for a Gaussian Integral is correct. First, let me start with a calculation I am pretty sure is correct, then ask about the calculation (6)-(9) of interest.

The calculation I am pretty sure about is as follows: Start with the well-known Gaussian integral:

$$\int e^{-\frac{1}{2}Ax^2+Bx}dx = \sqrt{\frac{2\pi}{A}}e^{\frac{B^2}{2A}}. \quad (1)$$

Next, obtain a closed for expression for the related integral $\int e^{-\frac{1}{2}Ax^2-V(x)+Bx}dx$, where

$$V(x) \equiv \sum_{n=0}^{\infty} C^{(n)}x^n \ , \quad (2)$$

is a an unspecified, completely-general polynomial in *x*, and the $C^{(n)}$ represents an infinite set of coefficients corresponding to each order of *x*.

Substituting (2) into $\int e^{-\frac{1}{2}Ax^2-V(x)+Bx}dx$ then allows us to write:

$$\begin{aligned}\int e^{-\frac{1}{2}Ax^2-V(x)+Bx}dx &= \int\left(1-\sum_{n=0}^{\infty} C^{(n)}x^n + \frac{1}{2!}\left(\sum_{n=0}^{\infty} C^{(n)}x^n\right)^2 - \ldots\right)e^{-\frac{1}{2}Ax^2+Bx}dx \\ &= \int\left(1-\sum_{n=0}^{\infty} C^{(n)}\left(\frac{d}{dB}\right)^n + \frac{1}{2!}\left(\sum_{n=0}^{\infty} C^{(n)}\left(\frac{d}{dB}\right)^n\right)^2 - \ldots\right)e^{-\frac{1}{2}Ax^2+Bx}dx \quad . \quad (3) \\ &= \int e^{-\sum_{n=0}^{\infty} C^{(n)}\left(\frac{d}{dB}\right)^n} e^{-\frac{1}{2}Ax^2+Bx}dx = e^{-\sum_{n=0}^{\infty} C^{(n)}\left(\frac{d}{dB}\right)^n}\int e^{-\frac{1}{2}Ax^2+Bx}dx\end{aligned}$$

Between the first two lines, the polynomial in *x* becomes a polynomial in the operator d/dB. In the final line, we are able to move $e^{-\sum_{n=0}^{\infty} C^{(n)}\left(\frac{d}{dB}\right)^n}$ outside the integral, because it is no longer a direct function of *x*. Now, we define the polynomial *V* as a function of d/dB:

$$V\left(\frac{d}{dB}\right) \equiv \sum_{n=0}^{\infty} C^{(n)}\left(\frac{d}{dB}\right)^n \ , \quad (4)$$

so that (3), using the Gaussian integral (1), may finally be rewritten as:

$$\int e^{-\frac{1}{2}Ax^2-V(x)+Bx}dx = e^{-V\left(\frac{d}{dB}\right)}\int e^{-\frac{1}{2}Ax^2+Bx}dx = e^{-V\left(\frac{d}{dB}\right)}\sqrt{\frac{2\pi}{A}}e^{\frac{B^2}{2A}}. \quad (5)$$

This is the integral expression which underlies what Zee [i] at 460 refers to as the "Central Identity of Quantum Field Theory."

Now, my question: With the above backdrop, I'd like to know the closed form integral for $\int e^{-\frac{1}{2}Ax^2 - V(x)}dx$. That is, for (5) above, but in the circumstance where $B = 0$, *and is taken to be a constant coefficient*. I think perhaps this will work (in contrast to the post earlier today):

Reverting to (3), and observing that $\frac{d}{dB}e^{-\frac{1}{2}Ax^2+Bx} = xe^{-\frac{1}{2}Ax^2+Bx}$ and that $-2\frac{d}{dA}e^{-\frac{1}{2}Ax^2+Bx} = x^2 e^{-\frac{1}{2}Ax^2+Bx}$, can we perhaps also write $\left(-2\frac{d}{dA}\right)^{.5} e^{-\frac{1}{2}Ax^2+Bx} = xe^{-\frac{1}{2}Ax^2+Bx}$, taking the square root of the operator d/dA independently of the operand $\exp\left(-\frac{1}{2}Ax^2 + Bx\right)$, so that an alternative to (3) is:

$$\int e^{-\frac{1}{2}Ax^2 - V(x) + Bx}dx = \int\left(1 - \sum_{n=0}^{\infty} C^{(n)}x^n + \frac{1}{2!}\left(\sum_{n=0}^{\infty} C^{(n)}x^n\right)^2 - \ldots\right)e^{-\frac{1}{2}Ax^2+Bx}dx$$

$$= \int\left(1 - \sum_{n=0}^{\infty} C^{(n)}\left(-2\frac{d}{dA}\right)^{.5n} + \frac{1}{2!}\left(\sum_{n=0}^{\infty} C^{(n)}\left(-2\frac{d}{dA}\right)^{.5n}\right)^2 - \ldots\right)e^{-\frac{1}{2}Ax^2+Bx}dx \text{ ?} \quad (6)$$

$$= \int e^{-\sum_{n=0}^{\infty} C^{(n)}\left(-2\frac{d}{dA}\right)^{.5n}} e^{-\frac{1}{2}Ax^2+Bx}dx = e^{-\sum_{n=0}^{\infty} C^{(n)}\left(-2\frac{d}{dA}\right)^{.5n}} \int e^{-\frac{1}{2}Ax^2+Bx}dx$$

Then, akin to (4), we define a new polynomial:

$$V\left(\frac{d}{dA}\right) \equiv \sum_{n=0}^{\infty} C^{(n)}\left(-2\frac{d}{dA}\right)^{.5n}, \quad (7)$$

so that using (6) and (7), and akin to (5), we may write:

$$\int e^{-\frac{1}{2}Ax^2 - V(x) + Bx}dx = e^{-V\left(\frac{d}{dA}\right)}\int e^{-\frac{1}{2}Ax^2+Bx}dx = e^{-V\left(\frac{d}{dA}\right)}\sqrt{\frac{2\pi}{A}}e^{\frac{B^2}{2A}}. \quad (8)$$

Then, if *B*=0, this simply reduces to:

$$\int e^{-\frac{1}{2}Ax^2 - V(x)}dx = e^{-V\left(\frac{d}{dA}\right)}\int e^{-\frac{1}{2}Ax^2+Bx}dx = e^{-V\left(\frac{d}{dA}\right)}\sqrt{\frac{2\pi}{A}}, \quad (9)$$

using the new polynomial (7). Does that look right?

Thanks,

Jay R. Yablon, May 15, 2008

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[i] Zee, A., *Quantum Field Theory in a Nutshell*, Princeton (2003)