I would like feedback on whether the particular calculation (6) shown below for a Gaussian Integral is correct. First, let me start with a calculation I am pretty sure is correct, then ask about the calculation (6) of interest.

The calculation I am pretty sure about is as follows: Start with the well-known Gaussian integral:

$$\int e^{-\frac{1}{2}Ax^2+Bx}dx = \sqrt{\frac{2\pi}{A}}e^{\frac{B^2}{2A}}. \quad (1)$$

Next, obtain a closed for expression for the related integral $\int e^{-\frac{1}{2}Ax^2-V(x)+Bx}dx$, where

$$V(x) \equiv \sum\nolimits_{n=0}^{\infty} C^{(n)}x^n, \quad (2)$$

is a an unspecified, completely-general polynomial in *x*, and the $C^{(n)}$ represents an infinite set of coefficients corresponding to each order of *x*.

Substituting (2) into $\int e^{-\frac{1}{2}Ax^2-V(x)+Bx}dx$ then allows us to write:

$$\begin{aligned}
\int e^{-\frac{1}{2}Ax^2-V(x)+Bx}dx &= \int\left(1-\sum\nolimits_{n=0}^{\infty} C^{(n)}x^n + \frac{1}{2!}\left(\sum\nolimits_{n=0}^{\infty} C^{(n)}x^n\right)^2 - \ldots\right)e^{-\frac{1}{2}Ax^2+Bx}dx \\
&= \int\left(1-\sum\nolimits_{n=0}^{\infty} C^{(n)}\left(\frac{d}{dB}\right)^n + \frac{1}{2!}\left(\sum\nolimits_{n=0}^{\infty} C^{(n)}\left(\frac{d}{dB}\right)^n\right)^2 - \ldots\right)e^{-\frac{1}{2}Ax^2+Bx}dx \qquad . \quad (3) \\
&= \int e^{-\sum_{n=0}^{\infty} C^{(n)}\left(\frac{d}{dB}\right)^n} e^{-\frac{1}{2}Ax^2+Bx}dx = e^{-\sum_{n=0}^{\infty} C^{(n)}\left(\frac{d}{dB}\right)^n}\int e^{-\frac{1}{2}Ax^2+Bx}dx
\end{aligned}$$

Between the first two lines, the polynomial in *x* becomes a polynomial in the operator $d/dB$. In the final line, we are able to move $e^{-\sum_{n=0}^{\infty} C^{(n)}\left(\frac{d}{dB}\right)^n}$ outside the integral, because it is no longer a direct function of *x*. Now, we define the polynomial *V* as a function of $d/dB$:

$$V\left(\frac{d}{dB}\right) \equiv \sum\nolimits_{n=0}^{\infty} C^{(n)}\left(\frac{d}{dB}\right)^n, \quad (4)$$

so that (3), using the Gaussian integral (1), may finally be rewritten as:

$$\int e^{-\frac{1}{2}Ax^2-V(x)+Bx}dx = e^{-V\left(\frac{d}{dB}\right)}\int e^{-\frac{1}{2}Ax^2+Bx}dx = e^{-V\left(\frac{d}{dB}\right)}\sqrt{\frac{2\pi}{A}}e^{\frac{B^2}{2A}} = \sqrt{\frac{2\pi}{A}}e^{-V\left(\frac{d}{dB}\right)+\frac{B^2}{2A}}. \quad (5)$$

This is the integral expression which underlies what Zee [i] at 460 refers to as the "Central Identity of Quantum Field Theory."

Now, my question: With the above backdrop, I'd like to know the closed form integral for $\int e^{-\frac{1}{2}Ax^2 - V(x)} dx$. That is, for (5) above, but in the circumstance where $B = 0$, *and is taken to be a constant coefficient*. One can set $B = 0$ above to arrive at the right-hand side $\exp[-V(d/dB)]\sqrt{2\pi / A}$ directly, but it is prudent to be certain by calculating this integral explicitly. If we start with $\int e^{-\frac{1}{2}Ax^2 - V(x)} dx$, then by the substitution of variables $x \rightarrow x - B/A$, and moving terms out from the integral which are not functions of *x*, we may write out:

$$\begin{aligned}
&\int e^{-\frac{1}{2}Ax^2 - V(x)} dx = \int e^{-V(x)} e^{-\frac{1}{2}Ax^2} dx = \int e^{-V(x)} e^{-\frac{1}{2}A\left(x - \frac{B}{A}\right)^2} dx = \int e^{-V(x)} e^{-\frac{1}{2}Ax^2 + Bx - \frac{1}{2}\frac{B^2}{A}} dx \\
&= e^{-\frac{1}{2}\frac{B^2}{A}} \int e^{-V(x)} e^{-\frac{1}{2}Ax^2 + Bx} dx = e^{-\frac{1}{2}\frac{B^2}{A}} \int e^{-V\left(\frac{d}{dB}\right)} e^{-\frac{1}{2}Ax^2 + Bx} dx = e^{-\frac{1}{2}\frac{B^2}{A}} e^{-V\left(\frac{d}{dB}\right)} \int e^{-\frac{1}{2}Ax^2 + Bx} dx \quad . \quad (6) \\
&= e^{-\frac{1}{2}\frac{B^2}{A}} e^{-V\left(\frac{d}{dB}\right)} \sqrt{\frac{2\pi}{A}} e^{\frac{1}{2}\frac{B^2}{A}} = e^{-V\left(\frac{d}{dB}\right)} \sqrt{\frac{2\pi}{A}}
\end{aligned}$$

where we also employ (1) in the final line. So, it appears that when $B = 0$, we may simply substitute $B = 0$ into (5), and still retain the operator term $\exp[-V(d/dB)]$ on the right hand side, multiplying the basic Gaussian $\sqrt{2\pi / A}$.

What makes me unsure, is that with $B = 0$, $V\left(\frac{d}{dB}\right) = \sum_{n=0}^{\infty} C^{(n)} \left(\frac{d}{dB}\right)^n = \sum_{n=0}^{\infty} C^{(n)} \left(\frac{d}{d0}\right)^n$, see (4). The expression $d/d0$ is divergent, which makes $\exp[-V(d/dB)] = \exp[-\infty] = 0$, and thus makes (6)=0. Now, one can of course always talk sensibly about $d/dx\,|^{x=0}$ in circumstances where *x* varies – that is, we can always talk about the "slope" at the point where $x = 0$, but if $B = 0$ is a coefficient fixed at zero, does that makes sense? I am having a mental block here, and could use some help to straighten out my thinking about this.

Thanks,

Jay R. Yablon, May 15, 2008

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[i] Zee, A., *Quantum Field Theory in a Nutshell*, Princeton (2003)