Correcting a Long-Standing Error in the Newton-Wigner Velocity Operator, and Solving the "Speed of Light Fermion Perplexity"

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1. Introduction: The "Speed of Light Fermion Perplexity"

A "perplexity" long associated with Dirac's equation is the following: Start with Dirac's equation for a free fermion, written as:

$$m\psi = \gamma^\sigma p_\sigma \psi = \left(\gamma^0 p_0 + \gamma^k p_k\right)\psi \qquad (1.1)$$

The equation for the Hamiltonian operator $\hat{H}_0$ is specified by removing the ψ, and by employing $\hat{H}_0$ in place of p_0. With some rearrangement, and multiplying from the left by γ^0, (1.1) then becomes the familiar Hamiltonian operator equation:

$$\hat{H}_0 \equiv \gamma^0 m - \gamma^0\gamma^k p_k = \gamma^0 m - \alpha^k p_k = \beta m + \boldsymbol{\alpha}\cdot\mathbf{p} \qquad (1.2)$$

where $\gamma^0\gamma^k \equiv \alpha^k$, $\gamma^0 \equiv \beta$, and $-\alpha^k p_k = \alpha^1 p^1 + \alpha^2 p^2 + \alpha^3 p^3 \equiv \boldsymbol{\alpha}\cdot\mathbf{p}$ with the sign reversal originating in a Minkowski metric tensor for which $diag(\eta_{\mu\nu}) = (+1,-1,-1,-1)$. We place the covariant 0 index on $\hat{H}_0$ simply to keep track, for now, of its connection to the time component p_0 of the energy-momentum tensor in (1.1). With the exception of the unit vector in the direction of momentum, $\hat{\mathbf{p}} \equiv \mathbf{p}/|\mathbf{p}|$ below, we shall generally use the "hat" to designate operators.

From (1.2), one may immediately deduce the velocity operator:

$$\hat{\mathbf{v}} \equiv \frac{\partial \hat{H}_0}{\partial \mathbf{p}} = \boldsymbol{\alpha} \qquad (1.3)$$

The subject "perplexity," which is unresolved to this day and perplexed the likes of Dirac and Feynman and Fock, [need references] is that the eigenvalues λ of $\boldsymbol{\alpha}$ in units of $c = 1$ are $\lambda(\boldsymbol{\alpha}) = \pm 1$, which would suggest that the subject fermions must travel at the speed of light. We shall now show how to resolve this perplexity.

2. Why the Usual Newton-Wigner Velocity Operator is Wrong, and What it Should Be

There is yet another problem which arises when one applies a Foldy-Wouthuysen transformation to (1.2) and (1.3) to obtain the Newton-Wigner representation, and as we shall see, it is related to the foregoing perplexity. The Foldy-Wouthuysen transformation is specified by the unitary operator and its inverse:

$$\begin{cases} U \equiv e^{\beta\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}\theta} = \cos\theta + \beta\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}\sin\theta = e^{-\gamma^0\alpha^k\hat{p}_k\theta} = \cos\theta - \gamma^0\alpha^k\hat{p}_k\sin\theta \\ U^{-1} \equiv e^{-\beta\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}\theta} = \cos\theta - \beta\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}\sin\theta = e^{\gamma^0\alpha^k\hat{p}_k\theta} = \cos\theta + \gamma^0\alpha^k\hat{p}_k\sin\theta \end{cases}. \quad (2.1)$$

where $\hat{\mathbf{p}} \equiv \mathbf{p}/|\mathbf{p}|$. The above are easily confirmed by series expansion of the exponentials. The problem now to be considered, arises as follows:

First, using (2.1), we Foldy-Wouthuysen transform $\hat{H}_0 \to \hat{H}_0'$ from (1.2) in the usual way to obtain (we omit the full calculation, because this is well known, but see, e.g., [1] if the reader needs to review this):

$$\hat{H}_0 \to \hat{H}_0' = U\hat{H}_0U^{-1} = U(\beta m + \boldsymbol{\alpha}\cdot\mathbf{p})U^{-1} = \beta\sqrt{m^2+|\mathbf{p}|^2} = \beta p_0\ , \quad (2.2)$$

with the usual diagonalizing choice of $\tan 2\theta \equiv |\mathbf{p}|/m$, hence, by elementary trigonometry:

$$\sin 2\theta = |\mathbf{p}|/\sqrt{m^2+|\mathbf{p}|^2} \quad \text{and} \quad \cos 2\theta = m/\sqrt{m^2+|\mathbf{p}|^2}\ . \quad (2.3)$$

We have also in (2.2) used $p_0 = \sqrt{m^2+|\mathbf{p}|^2}$, which, with $|\mathbf{p}|^2 = -p^k p_k$, is another way of saying that $m^2 = p^\sigma p_\sigma$ for an on-shell mass. One would thereby expect from (2.2) that the transformed velocity operator should be:

$$\hat{\mathbf{v}}' = \frac{\partial \hat{H}_0'}{\partial \mathbf{p}} = \beta\frac{|\mathbf{p}|}{\sqrt{m^2+|\mathbf{p}|^2}} = \beta\frac{|\mathbf{p}|}{p_0} \equiv \frac{d\hat{\mathbf{x}}'}{dt}\ , \quad (2.4)$$

which further serves to define the transformed (Newton-Wigner) position operator $\hat{\mathbf{x}}'$. In fact, (2.4) is generally taken to be the velocity operator of Newton-Wigner theory following a Foldy-Wouthuysen transformation, see for example, [2] at page 3.

This is all well and good, except for two problems which, carefully considered, demonstrate not only that *(2.4) is wrong and must be corrected*, but also solve the perplexity introduced in the preceding section. The first problem is this: Using (1.2) written in terms of the Dirac gamma matrices as $\hat{H}_0 \equiv \gamma^0 m - \gamma^0\gamma^k p_k$, we write the transformation (2.2) as:

$$
\begin{aligned}
\hat{H}_0' &= U\hat{H}_0U^{-1} = U\left(\gamma^0 m - \gamma^0\gamma^k p_k\right)U^{-1} = U\gamma^0U^{-1}m - U\gamma^0\gamma^kU^{-1}p_k \\
&= \left(U\gamma^0U^{-1}\right)m - \left(U\gamma^0U^{-1}\right)\left(U\gamma^kU^{-1}\right)p_k \qquad (2.5)\\
&\equiv \gamma'^0 m - \gamma'^0\gamma'^k p_k \equiv \beta' m - \alpha'^k p_k = \beta' m + \boldsymbol{\alpha}'\cdot\mathbf{p}
\end{aligned}
$$

where we have defined the transformed Dirac gamma as $\gamma'^0 \equiv U\gamma^0U^{-1}$ and $\gamma'^k \equiv U\gamma^kU^{-1}$, as well as $\beta' \equiv \gamma'^0$ and $\alpha'^k \equiv \gamma'^0\gamma'^k$. *This is exactly the same transformation as (2.2), but we have not yet made the particular choice of diagonalizing representation given in (2.3) which leads to* $\hat{H}_0' = \beta\sqrt{m^2+|\mathbf{p}|^2} = \beta p_0$ *in (2.2).* In effect, as soon as we impose (2.3), we choose a particular representation, and if we do this too soon, as we shall see, we lose a critical term from the Newton-Wigner velocity operator.

Thus, using (2.5) as $\hat{H}_0' = \beta' m + \boldsymbol{\alpha}'\cdot\mathbf{p}$, we do the exact same calculation as in (2.4):

$$
\hat{\mathbf{v}}' = \frac{\partial \hat{H}_0'}{\partial \mathbf{p}} = \boldsymbol{\alpha}' = \gamma'^0\boldsymbol{\gamma}' = \left(U\gamma^0U^{-1}\right)\left(U\boldsymbol{\gamma}U^{-1}\right) = U\gamma^0\boldsymbol{\gamma}U^{-1} = U\boldsymbol{\alpha}U^{-1} \equiv \frac{d\hat{\mathbf{x}}'}{dt} \qquad (2.6)
$$

However, when we calculate out $\hat{\mathbf{v}}' = \boldsymbol{\alpha}' = U\boldsymbol{\alpha}U^{-1}$ in the above, we obtain a different result than in (2.4). *That is,* $\hat{\mathbf{v}}'$ *in (2.6) is not the same as* $\hat{\mathbf{v}}'$ *in (2.4), and this is because we have not yet chosen a representation by applying (2.3).*

For example, consider α^3. We find using (2.1) and commuting α^3 to the left, that:

$$
\begin{aligned}
\hat{v}'^3 &= \alpha'^3 = U\alpha^3U^{-1} = \left(\cos\theta + \beta\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}\sin\theta\right)\alpha^3\left(\cos\theta - \beta\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}\sin\theta\right) \\
&= \left(\cos\theta - \beta\left(\alpha^1\hat{p}^1 + \alpha^2\hat{p}^2 + \alpha^3\hat{p}^3\right)\sin\theta\right)\alpha^3\left(\cos\theta + \beta\left(\alpha^1\hat{p}^1 + \alpha^2\hat{p}^2 + \alpha^3\hat{p}^3\right)\sin\theta\right) \qquad (2.7)\\
&= \alpha^3\left(\cos\theta + \beta\left(-\alpha^1\hat{p}^1 - \alpha^2\hat{p}^2 + \alpha^3\hat{p}^3\right)\sin\theta\right)\left(\cos\theta + \beta\left(\alpha^1\hat{p}^1 + \alpha^2\hat{p}^2 + \alpha^3\hat{p}^3\right)\sin\theta\right)
\end{aligned}
$$

To simplify further, let's select the "3" axis to align with the motion of the subject fermion, so that that $\hat{\mathbf{p}} = (0,0,1)$ and $\boldsymbol{\alpha}\cdot\hat{\mathbf{p}} = -\alpha^3\hat{p}^3$ and $|\mathbf{p}| = p^3$. Now, we finally use (2.3) to choose a representation, and the above reduces to:

$$
\begin{aligned}
\hat{v}'^3 &= \alpha'^3 = \alpha^3\left(\cos\theta + \beta\alpha^3\hat{p}^3\sin\theta\right)^2 = \alpha^3 e^{2\beta\alpha^3\hat{p}^3\theta} = \alpha^3\left(\cos 2\theta + \beta\alpha^3\hat{p}^3\sin 2\theta\right) \\
&= \alpha^3\left(\frac{m}{\sqrt{m^2+|\mathbf{p}|^2}} - \beta\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}\frac{|\mathbf{p}|}{\sqrt{m^2+|\mathbf{p}|^2}}\right) = \alpha^3\left(\frac{m - \beta\boldsymbol{\alpha}\cdot|\mathbf{p}|}{\sqrt{m^2+|\mathbf{p}|^2}}\right) = \alpha^3\left(\frac{m + \boldsymbol{\alpha}\cdot\beta|\mathbf{p}|}{p_0}\right) \quad . \qquad (2.8)\\
&= \alpha^3\frac{m}{p_0} + \beta\frac{|\mathbf{p}|}{p_0} = \alpha^3\frac{m}{p_0} + \beta\frac{p^3}{p_0}
\end{aligned}
$$

To compare apples-to-apples, we return to (2.4) and similarly align the $3 = z$ axis with the fermion motion, so that (2.4) yields:

$$\hat{v}'^3 = \beta \frac{|\mathbf{p}|}{p_0} = \beta \frac{p^3}{p_0} \quad , \quad (2.9)$$

So, the first problem is that we have arrived at two different expressions for the $\hat{v}'^3$ operator. Is the correct operator the usual $\hat{v}'^3 = \beta \frac{p^3}{p_0}$ of (2.9), or is it the operator $\hat{v}'^3 = \alpha^3 \frac{m}{p_0} + \beta \frac{p^3}{p_0}$ of (2.8)? Each operator contains the identical term $\beta p^3 / p_0$, *but there is an additional term* $\alpha^3 m / p_0$ *in (2.8) which does not appear in (2.9).* So, we need to figure out whether the usual Newton-Wigner velocity operators (2.4), (2.9) are really correct, *or whether they are in fact missing a term* $\alpha^3 m / p_0$. This inconsistency originates from whether one chooses a specific representation *before or after defining the velocity operator.* If we impose (2.3) *before* we define the velocity operator, we end up with (2.4), (2.9). If we impose (2.3) *after* defining the velocity operator, then we end up with (2.8) and the extra term $\alpha^3 m / p_0$.

The second problem, which will help us answer the foregoing, is this: Consider the unitary Foldy-Wouthuysen operator of (2.1). For a fermion "at rest," $m = p_0$ and $|\mathbf{p}| = 0$, so from (2.3), $\cos 2\theta = 1$ so that $\theta = \pm 0, \pi, 2\pi, 3\pi \ldots$ and $\cos\theta = \pm 1$ and $\sin\theta = 0$. This means, for a fermion "at rest," that:

$$U = \pm I \quad , \quad (2.10)$$

where *I* is a 4x4 identity matrix. Therefore, "at rest," the velocity operators (and indeed all operators $\hat{O}$) in the Newton-Wigner representation (2.3), become synonymous with their counterparts in the Pauli-Dirac or Weyl representations, because for any operator $\hat{O}$, the transformation is given by $\hat{O} \to \hat{O}' = U\hat{O}U^{-1} = \pm I \cdot \hat{O} \cdot \pm I = \hat{O}$. Thus, for an "at rest" fermion, all operators $\hat{O}' = \hat{O}$. This should help us choose between $\hat{v}'^3 = \beta \frac{|\mathbf{p}|}{p_0}$ of (2.9), and $\hat{v}'^3 = \alpha^3 \frac{m}{p_0} + \beta \frac{|\mathbf{p}|}{p_0}$ of (2.8).

In the Pauli-Dirac or Weyl representations, (1.3) states that $\hat{\mathbf{v}} = \boldsymbol{\alpha}$. For an "at rest" fermion, we should obtain the same result in the Newton-Wigner representation. Yet, from (2.4) and (2.9), if $\mathbf{p} = 0$, then $\hat{\mathbf{v}}' = 0$, rather than the expected $\hat{\mathbf{v}}' = \boldsymbol{\alpha}$. This tells us that there is indeed something wrong with the usual Newton-Wigner velocity operator. In contrast, (2.8) does not have this problem, and turns out just as it should. That is, for $\mathbf{p} = 0$ and $m = p_0$, (2.8) indeed reduces precisely to $\hat{v}'^3 = \alpha^3 = \hat{v}^3$, and more generally, to $\hat{\mathbf{v}}' = \boldsymbol{\alpha} = \hat{\mathbf{v}}$.

So, the conflict as between the usual velocity operator (2.4) and the operator (2.8) with an additional $\alpha^3 m / p_0$ term, is actually resolved in favor of (2.8), for this exhibits the proper correspondence to the Pauli-Dirac or Weyl representations when the fermion is at rest and the Foldy-Wouthuysen transformation becomes an identity. The usual Newton-Wigner velocity (2.4) operator is wrong, and for a fermion moving along the "3" axis, it must be replaced by (2.8), which is $\hat{v}'^3 = \alpha^3 \dfrac{m}{p_0} + \beta \dfrac{|\mathbf{p}|}{p_0}$.

3. The Solution to the "Speed of Light Fermion Perplexity"

Of equal importance, if we closely study (2.8) written as:

$$\frac{d\hat{x}'^3}{dt} \equiv \hat{v}'^3 = \alpha'^3 = \alpha^3 \frac{m}{p_0} + \beta \frac{|\mathbf{p}|}{p_0} \qquad (3.1)$$

where we have *redefined* the differential position operator $d\hat{x}'^3$ as $d\hat{x}'^3 / dt \equiv \hat{v}'^3$, we see that this allows us to solve the perplexity introduced in section 1.

In order to see this on the simplest footing possible, let us begin by considering the metric interval in Minkowski spacetime, specified by:

$$d\tau^2 = \eta_{\mu\nu} dx^\mu dx^\nu = dx^\sigma dx_\sigma \ , \quad (3.2)$$

where $d\tau$ is the differential proper time invariant as measured in the rest frame of the subject fermion. This of course, has a plainly geometric interpretation, in which dx^μ are infinitesimal geometric displacements along time and space coordinates. In the customary manner, we multiply through by $m^2 / d\tau^2$ to obtain the on-shell mass condition $m^2 = p^\sigma p_\sigma$ wherein one concomitantly defines the usual energy-momentum four-vector:

$$p^{\mu} \equiv \left(p^0, p^1, p^2, p^3\right) = \left(\frac{m}{\sqrt{1-v^2}}, \frac{mv^1}{\sqrt{1-v^2}}, \frac{mv^2}{\sqrt{1-v^2}}, \frac{mv^3}{\sqrt{1-v^2}}\right) = (E, \mathbf{p}) = \left(E, \frac{m\mathbf{v}}{\sqrt{1-v^2}}\right) \\ = m\left(\frac{dt}{d\tau}, \frac{d\mathbf{x}}{d\tau}\right) . \tag{3.3}$$

Using the above, and setting $p^1 = p^2 = 0$ and $|\mathbf{p}| = p^3$ so that the momentum is again directed along the "3" axis, we now rewrite (3.1) as:

$$\frac{d\hat{x}'^3}{dt} = \hat{v}'^3 = \alpha'^3 = \alpha^3 \frac{m}{p_0} + \beta \frac{|\mathbf{p}|}{p_0} \quad = \alpha^3 \frac{d\tau}{dt} + \beta \frac{dx^3}{dt} = \alpha^3 \sqrt{1 - v^{3^2}} + \beta v^3 \tag{3.4}$$

It is *very important* in the above to distinguish the *physical* velocity $v = v^3$ from the velocity *operator* $\hat{v}'^3$ or the *eigenvalues* of this operator.

With (3.4) most simply expressed as $\hat{v}'^3 = \alpha^3 \sqrt{1 - v^{3^2}} + \beta v^3$, consider a fermion at rest, now specified by $v^1 = v^2 = v^3 = 0$. Then, although this fermion is at rest, the above reduces to $\hat{v}'^3 = \alpha^3$, and more generally, to $\hat{\mathbf{v}}' = \boldsymbol{\alpha}$. This is identical to the velocity operator of the Pauli-Dirac or Weyl representation, as it should be. Further, although this operator still has eigenvalues $\lambda(\boldsymbol{\alpha}) = \lambda(\hat{\mathbf{v}}') = \pm 1$, the physical *fermion is at rest, not travelling at the speed of light.* In contrast, now consider a fermion which is travelling at the speed of light, along the "3" axis, with $v^3 = 1$. Here, (3.4) reduces to $\hat{v}'^3 = \beta$. Again, the eigenvalues $\lambda(\beta) = \lambda(\hat{\mathbf{v}}') = \pm 1$. From here, one realizes that the eigenvalues of this corrected Newton-Wigner velocity operator will *always* be given by $\lambda(\hat{\mathbf{v}}') = \pm 1$. One can calculate this explicitly from $\hat{v}'^3 = \alpha^3 \sqrt{1 - v^{3^2}} + \beta v^3$ in which α^3 and components β are linearly mixed in proportion to $\sqrt{1 - v^{3^2}}$ and v^3, respectively, and will see that this proportionate mixing is precisely the mixing needed to maintain the eigenvalues $\lambda(\hat{\mathbf{v}}') = \pm 1$ *independently of the physical* v^3. As a result of this, the eigenvalues $\lambda(\hat{\mathbf{v}}') = \pm 1$ of the velocity operator are seen to be *invariant under Lorentz boosts*. The physical velocity $\mathbf{v}$ can and does change, and in response to a change in velocity the components of the velocity operator $\hat{\mathbf{v}}'$ can and do mix $\boldsymbol{\alpha}$ and β components. But, the eigenvalues of $\hat{\mathbf{v}}'$ remain invariant under Lorentz transformations, and are *always equal to the speed of light*. This is the solution to the "Speed of Light Fermion Perplexity," and it is based on correcting the Newton-Wigner velocity operator to contain an additional term $\alpha^3 \sqrt{1 - v^2}$.

In hindsight, what becomes clear through the corrected Newton-Wigner velocity operator $\hat{v}'^{3} = \alpha^{3}\sqrt{1-v^{3^{2}}} + \beta v^{3}$ for a fermion moving along the "3" axis, is that the eigenvalues of the velocity operator are *not* the same thing as the actual physical velocity itself. It is this misconception, in particular, which has caused many, including Dirac and Feynman, to scratch their heads over why Dirac's equation might be seeming to suggest that fermions must travel at the speed of light. Now we see that all is well: all Dirac's equation is telling us is that the eigenvalues of the velocity operator are Lorentz invariants, and that these Lorentz invariants are equal to the speed of light. Fermions are still free to travel with sub-luminous velocity, and for fermions transformed to "at rest," the Pauli-Dirac or Weyl representations are identical with the Newton-Wigner representation. All that happens when a fermion is boosted, is that the $\boldsymbol{\alpha}$ and β components of the velocity operator $\hat{\mathbf{v}}'$ will mix, while the eigenvalues of $\hat{\mathbf{v}}'$ will remain unchanged and always equal in magnitude to the speed of light.

References

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[1] http://www.physics.ucdavis.edu/~cheng/230A/RQM7.pdf.

[2] http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.27.3209&rep=rep1&type=pdf.