Dear Friends:

I have now added a zitterbewegung analysis of the time operator. The first section is the same as in my most recent post.

1. The Differential Time Operator

Let's start with the Dirac Hamiltonian:

$$\hat{H}_0 = \gamma^0 m - \gamma^0 \gamma^k p_k = \gamma^0 m - \alpha^k p_k = \beta m + \boldsymbol{\alpha} \cdot \mathbf{p} \quad (1.1)$$

where the final sign reversal arises from a Minkowski tensor $\operatorname{diag}(\eta_{\mu\nu}) = (+1,-1,-1,-1)$. The "hat" designates this as a 4x4 Dirac operator, and the 0 subscript designates the Hamiltonian as being related to the "time" component of Dirac's equation written as $p_0 \psi = (\gamma^0 m - \gamma^0 \gamma^k p_k)\psi$.

Using the Heisenberg canonical commutation relation $\lfloor x_i, p_j \rfloor = -i\eta_{ij}$, one can show from (1.1) in the usual manner that:

$$\frac{d\hat{x}_k}{dt} = i\left[\hat{H}_0, x_k\right] = \gamma^0 \gamma_k = \alpha_k \quad (1.2)$$

Because the eigenvalues $\lambda(\alpha_k) = \pm 1 (= \pm c)$, this suggests that the instantaneous velocity of a fermion is equal to the speed of light, and zitterbewegung motion and Foldy-Wouthuysen transformations are generally employed to make sense of this result and obtain a velocity spectrum $-c \le v \le c$. This is all standard physics.

We may rewrite (1.2) above as the differential position operator:

$$d\hat{x}^k = \alpha^k dx^0 = \gamma^0 \gamma^k dx^0 \quad (1.3)$$

Now, let's *define* a differential time operator $d\hat{x}^0$ according to:

$$d\tau^2 \equiv d\hat{x}^\sigma d\hat{x}_\sigma = d\hat{x}^0 d\hat{x}_0 + d\hat{x}^k d\hat{x}_k = d\hat{x}_0{}^2 - d\hat{x}^{1\,2} - d\hat{x}^{2\,2} - d\hat{x}^{3\,2} \quad (1.4)$$

That is, *by definition*, we require that $d\hat{x}^\sigma = \left(d\hat{x}^0, d\hat{x}^1, d\hat{x}^2, d\hat{x}^3\right)$ form a Lorentz quadruplet of operators in combination with the $d\hat{x}^k$.

The $d\hat{x}^0$ which satisfies (1.4) is specified by:

$$\boxed{d\hat{x}_0 \equiv 2\gamma^0 d\tau - \sqrt{3}\gamma^0 \gamma^k dx_k = 2\gamma^0 d\tau - \sqrt{3}\alpha^k dx_k} \quad (1.5)$$

Note that this mirrors $\hat{H}_0$ in (1.1), but with additional constant factors. To see this, we may calculate out the following, with all steps shown:

$$
\begin{aligned}
& d\hat{x}_0{}^2 - d\hat{x}^{1\,2} - d\hat{x}^{2\,2} - d\hat{x}^{3\,2} \\
&= \left(2\gamma^0 d\tau - \sqrt{3}\gamma^0\gamma^k dx_k\right)^2 - \left(\gamma^0\gamma^1 dx^0\right)^2 - \left(\gamma^0\gamma^2 dx^0\right)^2 - \left(\gamma^0\gamma^3 dx^0\right)^2 \\
&= \left(2\gamma^0 d\tau - \sqrt{3}\gamma^0\gamma^k dx_k\right)\left(2\gamma^0 d\tau - \sqrt{3}\gamma^0\gamma^l dx_l\right) \\
&- \left(\gamma^0\gamma^1 dx^0\right)\left(\gamma^0\gamma^1 dx^0\right) - \left(\gamma^0\gamma^2 dx^0\right)\left(\gamma^0\gamma^2 dx^0\right) - \left(\gamma^0\gamma^3 dx^0\right)\left(\gamma^0\gamma^3 dx^0\right) \\
&= 4\gamma^0\gamma^0 d\tau^2 + 3\gamma^0\gamma^k\gamma^0\gamma^l dx_k dx_l - 2\sqrt{3}\gamma^0\gamma^k\gamma^0 dx_k d\tau - 2\sqrt{3}\gamma^0\gamma^0\gamma^l dx_l d\tau \\
&- \gamma^0\gamma^1\gamma^0\gamma^1 dx^0 dx^0 - \gamma^0\gamma^2\gamma^0\gamma^2 dx^0 dx^0 - \gamma^0\gamma^3\gamma^0\gamma^3 dx^0 dx^0 \\
&= 4d\tau^2 - 3\gamma^k\gamma^l dx_k dx_l - 3dx^0 dx^0 \\
&= 4d\tau^2 - 3dx^k dx_k - 3dx^0 dx_0 \\
&= 4d\tau^2 - 3d\tau^2 \\
&= d\tau^2
\end{aligned}
\qquad (1.6)
$$

where in the reduction, we have employed:

$$
\begin{aligned}
\gamma^k\gamma^l dx_k dx_l &= \left(-2i\sigma^{kl} + \gamma^l\gamma^k\right)dx_k dx_l = \gamma^l\gamma^k dx_k dx_l \\
&= \tfrac{1}{2}\left\{\gamma^k\gamma^l + \gamma^l\gamma^k\right\}dx_k dx_l = \eta^{kl} dx_k dx_l = dx^k dx_k
\end{aligned}
\qquad (1.7)
$$

which in turn employs:

$$\sigma^{kl} = \tfrac{1}{2} i\left[\gamma^k\gamma^l - \gamma^l\gamma^k\right] \quad (1.8)$$

and, because $\sigma^{kl} = -\sigma^{lk}$:

$$\sigma^{kl} dx_k dx_l = 0 \quad (1.9)$$

2. The Zitterbewegung of the Differential Time Operator

Now let's take the time operator (1.5) and divide through by $dt = dx_0$, as:

$$\dot{\hat{x}}_0 \equiv \frac{d\hat{x}_0}{dt} = 2\gamma^0 \frac{d\tau}{dt} - \sqrt{3}\gamma^0\gamma^k \frac{dx_k}{dt} \quad (2.1)$$

We are now going to consider the behavior of the time *operator* over time. Specifically, let us now calculate:

$$\frac{d^2\hat{x}_0}{dt^2} = \frac{d\dot{\hat{x}}_0}{dt} = i\left[\hat{H}_0, \dot{\hat{x}}_0\right] \quad (2.2)$$

The calculation takes place in several steps, using $\hat{H}_0 = \gamma^0 m - \gamma^0\gamma^k p_k$ from (1). First, we use (2.1) and (2.2) to set up the calculation:

$$
\begin{aligned}
-i\frac{d^2\hat{x}_0}{dt^2} &= -i\frac{d\dot{\hat{x}}_0}{dt} = \left[\hat{H}_0, \dot{\hat{x}}_0\right] = \hat{H}_0\dot{\hat{x}}_0 - \dot{\hat{x}}_0\hat{H}_0 \\
&= \left(\gamma^0 m - \gamma^0\gamma^k p_k\right)\left(2\gamma^0\frac{d\tau}{dt} - \sqrt{3}\gamma^0\gamma^l\frac{dx_l}{dt}\right) - \left(2\gamma^0\frac{d\tau}{dt} - \sqrt{3}\gamma^0\gamma^k\frac{dx_k}{dt}\right)\left(\gamma^0 m - \gamma^0\gamma^l p_l\right)
\end{aligned}
\tag{2.3}
$$

This expands and then partially reduces to:

$$
\begin{aligned}
-i\frac{d^2\hat{x}_0}{dt^2} &= 2m\gamma^0\gamma^0\frac{d\tau}{dt} + \sqrt{3}\gamma^0\gamma^k\gamma^0\gamma^l p_k\frac{dx_l}{dt} - 2\gamma^0\gamma^k\gamma^0\frac{d\tau}{dt}p_k - \sqrt{3}m\gamma^0\gamma^0\gamma^l\frac{dx_l}{dt} \\
&\quad - 2m\gamma^0\gamma^0\frac{d\tau}{dt} - \sqrt{3}\gamma^0\gamma^k\gamma^0\gamma^l p_l\frac{dx_k}{dt} + 2\gamma^0\gamma^0\gamma^l\frac{d\tau}{dt}p_l + \sqrt{3}m\gamma^0\gamma^k\gamma^0\frac{dx_k}{dt} \\
&= \sqrt{3}\left[\gamma^l\gamma^k - \gamma^k\gamma^l\right]p_k\frac{dx_l}{dt} + 4\gamma^k\frac{d\tau}{dt}p_k - 2\sqrt{3}m\gamma^k\frac{dx_k}{dt} \\
&= \left[i2\sqrt{3}\sigma^{kl}p_k + \left(4 - 2\sqrt{3}\right)\gamma^l m\right]\frac{dx_l}{dt}
\end{aligned}
\quad , \tag{2.4}
$$

using $\sigma^{kl} = \frac{1}{2}i\left[\gamma^k\gamma^l - \gamma^l\gamma^k\right]$ and $p^\mu = mdx^\mu / d\tau$.

Next, we substitute (1.1) written in terms of the mass as $m = \gamma^0\hat{H}_0 + \gamma^k p_k$, and multiply through by $\frac{1}{2}$. This further reduces to:

$$
\begin{aligned}
-\frac{1}{2}i\frac{d^2\hat{x}_0}{dt^2} &= \left[i\sqrt{3}\sigma^{kl}p_k + \left(2 - \sqrt{3}\right)\gamma^l m\right]\frac{dx_l}{dt} \\
&= \left[i\sqrt{3}\sigma^{kl}p_k + \left(2 - \sqrt{3}\right)\gamma^l\left(\gamma^0\hat{H}_0 + \gamma^k p_k\right)\right]\frac{dx_l}{dt} \\
&= \left[i\sqrt{3}\sigma^{kl}p_k - \sqrt{3}\gamma^l\gamma^k p_k - \sqrt{3}\gamma^l\gamma^0\hat{H}_0 + 2\left(\gamma^l\gamma^0\hat{H}_0 + \gamma^l\gamma^k p_k\right)\right]\frac{dx_l}{dt} \\
&= \left[-\sqrt{3}\tfrac{1}{2}\left[\gamma^k\gamma^l + \gamma^l\gamma^k\right]p_k - \sqrt{3}\gamma^l\gamma^0\hat{H}_0 + 2\left(\gamma^l\gamma^0\hat{H}_0 + \gamma^l\gamma^k p_k\right)\right]\frac{dx_l}{dt} \\
&= \left[-\sqrt{3}\eta^{kl}p_k + 2\gamma^l\gamma^k p_k - \sqrt{3}\gamma^l\gamma^0\hat{H}_0 + 2\gamma^l\gamma^0\hat{H}_0\right]\frac{dx_l}{dt}
\end{aligned}
\quad , \tag{2.5}
$$

using $\sigma^{kl} = \frac{1}{2}i\left[\gamma^k\gamma^l - \gamma^l\gamma^k\right]$ and $\eta^{kl} = \frac{1}{2}\left[\gamma^k\gamma^l + \gamma^l\gamma^k\right]$.

Now, we distribute the dx_l / dt and use $p^\mu = mdx^\mu / d\tau$ and (1.7) to write:

$$
\begin{aligned}
-\frac{1}{2}i\frac{d^2\hat{x}_0}{dt^2} &= \left[-\sqrt{3}\eta^{kl}p_k + 2\gamma^l\gamma^k p_k - \sqrt{3}\gamma^l\gamma^0\hat{H}_0 + 2\gamma^l\gamma^0\hat{H}_0\right]\frac{dx_l}{dt} \\
&= \left[-\sqrt{3}\eta^{kl}m\frac{dx_k}{d\tau}\frac{dx_l}{dt} + 2\gamma^l\gamma^k m\frac{dx_k}{d\tau}\frac{dx_l}{dt} - \sqrt{3}\gamma^l\gamma^0\hat{H}_0\frac{dx_l}{dt} + 2\gamma^l\gamma^0\hat{H}_0\frac{dx_l}{dt}\right] \\
&= \left[-\sqrt{3}\eta^{kl}m\frac{dx_k}{d\tau}\frac{dx_l}{dt} + 2\eta^{kl}m\frac{dx_k}{d\tau}\frac{dx_l}{dt} - \sqrt{3}\gamma^l\gamma^0\hat{H}_0\frac{dx_l}{dt} + 2\gamma^l\gamma^0\hat{H}_0\frac{dx_l}{dt}\right] \quad , \ (2.6) \\
&= \left(2-\sqrt{3}\right)\left[+\eta^{kl}m\frac{dx_k}{d\tau}\frac{dx_l}{dt} + \gamma^l\gamma^0\hat{H}_0\frac{dx_l}{dt}\right] \\
&= \left(2-\sqrt{3}\right)\left[p^l - \alpha^l\hat{H}_0\right]\frac{dx_l}{dt}
\end{aligned}
$$

There are now two forms in which we shall wish to write the above. First, we make note of the fact that the known Zitterbewegung acceleration operator is specified by:

$$
\hat{z}^l \equiv \frac{d\alpha^l}{dt} = \frac{d\hat{v}^l}{dt} = \frac{d^2\hat{x}^l}{dt^2} = i\left[\hat{H}_0, \alpha^l\right] = 2i\left[p^l - \alpha^l\hat{H}_0\right] \quad (2.7)
$$

where $\hat{v}^l = \alpha^l$ is the velocity operator. Therefore, from the last line of the (2.6), we may write:

$$
\boxed{\frac{d^2\hat{x}_0}{dt^2} = \frac{d\dot{\hat{x}}_0}{dt} = 2i\left(2-\sqrt{3}\right)\left[p^l - \alpha^l\hat{H}_0\right]\frac{dx_l}{dt} = \left(2-\sqrt{3}\right)\hat{z}^l\,\frac{dx_l}{dt}} \ , \ (2.8)
$$

which establishes a direct relationship between $d\dot{\hat{x}}_0 / dt$ and the Zitterbewegung acceleration.

Second, from the next-to-last line of (2.6), we may also reduce using $v_l = dx_l / dt$, to:

$$
\begin{aligned}
-\frac{1}{2}i\frac{d^2\hat{x}_0}{dt^2} &= \left(2-\sqrt{3}\right)\left[+\eta^{kl}m\frac{dx_k}{d\tau}\frac{dx_l}{dt} + \gamma^l\gamma^0\hat{H}_0\frac{dx_l}{dt}\right] \\
&= \left(2-\sqrt{3}\right)\left[+m\frac{dt}{d\tau}\frac{dx^l}{dt}\frac{dx_l}{dt} + \gamma^l\gamma^0\hat{H}_0\frac{dx_l}{dt}\right] \ . \ (2.9) \\
&= -\left(2-\sqrt{3}\right)\left[\frac{mv^2}{\sqrt{1-v^2}} + \alpha^l v_l\hat{H}_0\right]
\end{aligned}
$$

This admits to a further reduction if we employ $\hat{H}_0 = \gamma^0 m - \gamma^0\gamma^k p_k$. Then, this becomes:

$$
\begin{aligned}
-\frac{1}{2}i\frac{d^2\hat{x}_0}{dt^2} &= -\left(2-\sqrt{3}\right)\left[\frac{mv^2}{\sqrt{1-v^2}} + \alpha^l v_l\left(\gamma^0 m - \gamma^0\gamma^k p_k\right)\right] \\
&= -\left(2-\sqrt{3}\right)\left[\frac{mv^2}{\sqrt{1-v^2}} - \gamma^0\gamma^l\gamma^0\gamma^k v_l p_k + \gamma^0\gamma^l\gamma^0 m v_l\right] \\
&= -\left(2-\sqrt{3}\right)\left[\frac{mv^2}{\sqrt{1-v^2}} + m\gamma^l\gamma^k\frac{v_l v_k}{\sqrt{1-v^2}} - m\gamma^l v_l\right] \\
&= -\left(2-\sqrt{3}\right)\left[\frac{mv^2}{\sqrt{1-v^2}} + m\frac{\eta^{lk} v_l v_k}{\sqrt{1-v^2}} - m\gamma^l v_l\right] \\
&= -\left(2-\sqrt{3}\right)\left[\frac{mv^2}{\sqrt{1-v^2}} - \frac{mv^2}{\sqrt{1-v^2}} - m\gamma^l v_l\right] \\
&= \left(2-\sqrt{3}\right)m\gamma^l v_l
\end{aligned}
\quad . \quad (2.10)
$$

where in the fourth line we again make use of (1.7). The upshot of (2.10), is that:

$$\boxed{\frac{1}{2}\frac{d^2\hat{x}_0}{dt^2} = \frac{1}{2}\frac{d\dot{\hat{x}}_0}{dt} = i\left(2-\sqrt{3}\right)m\gamma^l v_l = i\left(2-\sqrt{3}\right)m\gamma^l\frac{dx_l}{dt} = -i\left(2-\sqrt{3}\right)m\boldsymbol{\gamma}\cdot\mathbf{v} = -i\left(2-\sqrt{3}\right)m\boldsymbol{\gamma}\cdot\frac{\mathbf{dx}}{dt}} \quad . \quad (2.11)$$

This is a very direct relationship between $d\dot{\hat{x}}_0 / dt$ and the scalar velocity v_l. But what is of even more interest, is that (2.11) is easily integrated, twice, to obtain the time operator $\hat{x}_0$.

First, we write the above as:

$$\frac{d}{dt}\frac{d\hat{x}_0}{dt} = 2i\left(2-\sqrt{3}\right)m\gamma^l\frac{dx_l}{dt} \quad . \quad (2.12)$$

Then, we multiply through by dt and apply an indefinite integral to each side to write:

$$\int d\frac{d\hat{x}_0}{dt} = \frac{d\hat{x}_0}{dt} = \dot{\hat{x}}_0 = 2i\left(2-\sqrt{3}\right)m\gamma^l\int dx_l = 2i\left(2-\sqrt{3}\right)m\gamma^l x_l + \dot{\hat{x}}_0(0) \quad . \quad (2.13)$$

where $\dot{\hat{x}}_0(0)$ is a constant of integration matrix. We rely on the fact that $m\gamma^l$ is a constant. Note that $\dot{\hat{x}}_0$ is a function only of space position, not of time. Then, we again multiply through by dt and apply an indefinite integral to each side to write:

$$\int d\hat{x}_0 = \hat{x}_0 = \int\left[2i\left(2-\sqrt{3}\right)m\gamma^l x_l + \dot{\hat{x}}_0(0)\right]dt = \left[2i\left(2-\sqrt{3}\right)m\gamma^l x_l + \dot{\hat{x}}_0(0)\right]t + \hat{x}_0(0) \quad . \quad (2.14)$$

Here, we use the fact that $2i\left(2-\sqrt{3}\right)m\gamma^l x_l + \dot{\hat{x}}_0(0)$ is independent of time, and now, the result is a linear function of time. Thus, the final version of the time operator is:

$$\boxed{\hat{t} = \hat{t}(0) + \dot{\hat{t}}(0)\cdot t + 2i\left(2-\sqrt{3}\right)m\gamma^l x_l\cdot t} \quad . \quad (2.15)$$

Finally, we may combine the alternative expressions (2.8) and (2.11) to write:

$$\frac{d^2\hat{x}_0}{dt^2} = \left(2-\sqrt{3}\right)\hat{z}^l \frac{dx_l}{dt} = 2i\left(2-\sqrt{3}\right)m\gamma^l \frac{dx_l}{dt} \quad (2.16)$$

Factoring out common terms then yields the very simple expression:

$$\boxed{\frac{1}{2}\hat{z}^l = \frac{1}{2}\frac{d^2\hat{x}^l}{dt^2} = i\,m\gamma^l} \quad (2.17)$$

The Zitterbewegung acceleration is now seen to be directly proportional to the mass of the fermion. Moreover, this expression is easily integrated to obtain the position operator as a function of time. First, we write:

$$\int d\frac{d\hat{x}^l}{dt} = \frac{d\hat{x}^l}{dt} = 2i\,m\gamma^l \int dt = 2i\,m\gamma^l t + \dot{\hat{x}}^l(0) \quad (2.18)$$

Then, we integrate up one last time to obtain:

$$\int d\hat{x}^l = \hat{x}^l = \int\left(2i\,m\gamma^l t + \dot{\hat{x}}^l(0)\right)dt = i\,m\gamma^l t^2 + \dot{\hat{x}}^l(0)\cdot t + \hat{x}^l(0) \quad (2.19)$$

In sum:

$$\boxed{\hat{x}^l = i\,\gamma^l mt^2 + \dot{\hat{x}}^l(0)\cdot t + \hat{x}^l(0)} \quad (2.20)$$

Reduced in this way, the Zitterbewegung is actually a constant (operator) acceleration *proportional to the fermion mass*. So, if we select $\dot{\hat{x}}^l(0)=0$ and $\hat{x}^l(0)=0$ to start, the space position operator $\hat{x}^l = i\,\gamma^l mt^2$ varies with mt^2, times the related Dirac $i\,\gamma^l$.

That's all for now.

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