Why Pythagoras Unknowingly Anticipated Quantum Physics

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Two and a half millennia ago, Pythagoras established that the length *l* between any two points *A* and *B*, if one lays out this separation along *x*, *y*, *z* coordinates, is given by $l^2 = x^2 + y^2 + z^2$.

Suppose that at any time between 500 BC and 1901 when the quantum revolution began, someone had said "let's take the non-trivial square root of $l^2 = x^2 + y^2 + z^2$," that is, let's do something more interesting than merely writing $l = \pm\sqrt{x^2 + y^2 + z^2}$. If one had known the Pauli matrices σ^k, one could have used these to form $\not{X} \equiv \sigma^i x^i$ with $x^i = (x, y, z)$. Then, in the manner that Dirac's Minkowskian relation $\eta^{\mu\nu} = \frac{1}{2}\left(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\right) = \frac{1}{2}\left\{\gamma^\mu, \gamma^\nu\right\}$ which may be used in the special relativistic energy relationship $m^2 = \eta^{\mu\nu} p_\mu p_\nu$, one could have used $\delta^{ij} = \frac{1}{2}\left(\sigma^i\sigma^j + \sigma^j\sigma^i\right) = \frac{1}{2}\left\{\sigma^i, \sigma^j\right\}$ to write $l^2 = \delta^{ij} x^i x^j = \frac{1}{2}\left(\sigma^i\sigma^j + \sigma^j\sigma^i\right) x^i x^j = \not{X}^2$, and more explicitly:

$$\begin{pmatrix} l^2 & 0 \\ 0 & l^2 \end{pmatrix} = \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix}\begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix} = \begin{pmatrix} x^2 + y^2 + z^2 & 0 \\ 0 & x^2 + y^2 + z^2 \end{pmatrix}. \tag{1}$$

Then, similarly to Dirac's $m = \gamma^\sigma p_\sigma = \not{p}$ absent the spinors *u* which really require this to be an eigenvector relationship $mu = \not{p}u$, one would have found that the square root equation is simply $l = \not{X}$. But when written out explicitly, this would be:

$$\begin{pmatrix} l & 0 \\ 0 & l \end{pmatrix} = \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix} = \not{X}, \tag{2}$$

which, like $m = \gamma^\sigma p_\sigma = \not{p}$, is a mathematically-invalid equation. So we would be forced by Pythagoras, no less, to introduce a spinor which we shall call *p*, and to write (2) as the eigenvalue equation $(\not{X} - l)\, p = 0$, which in explicit form, is:

$$\begin{pmatrix} z - l & x - iy \\ x + iy & -z - l \end{pmatrix}\begin{pmatrix} p_A \\ p_B \end{pmatrix} = 0. \tag{3}$$

So $(\not{X} - l)\, p = 0$ above now tells us that the length *l* represents eigenvalues of $\not{X}$, with associated eigenstates p_A, p_B. So what are these eigenvalues? As with any eigenvalue equation, we determine these using

$$0 = \det(\not{X} - l) = (l - z)(l + z) - x^2 - y^2 = l^2 - x^2 - y^2 - z^2. \tag{4}$$

This now recovers the original Pythagorean relationship $l^2 = x^2 + y^2 + z^2$, and this means that the eigenvalues of $\not{X}$ are $l = \pm\sqrt{x^2 + y^2 + z^2}$.

But now we have eigenstates also. How do we interpret these? In the only sensible way possible: The length *l* represents the *magnitude* of the distance between two points *A* and *B*. But there is also a question of direction, and that is why physics uses vectors. If we start at point *A* and go to point *B*, which we denote as $A \to B$ with "$\to$" being a vector, then eigenstate p_A represents the circumstance where we start at *A* and go to *B*, i.e., in which the vector direction points from *A* to *B*, and its eigenvalue is $l = +\sqrt{x^2 + y^2 + z^2}$. Conversely, p_B represents $B \to A$, starting at *B* and going to *A*, and its eigenvalue is $l = -\sqrt{x^2 + y^2 + z^2}$. This is a two-valuedness, but contradicting Pauli, *it is perfectly classical*. It simply says that in going from point *A* to point *B* a person will traverse the same distance as when going from point *B* to point *A*, but will go in the *opposite direction*. The reason we use vectors, and not merely lines, in physics, is because direction, and not only magnitude, is important.

But, we are able to use the core language of quantum physics to discuss all of this. Specifically, we may write all of this in bras and kets by saying that $\not{X} \left| p \right\rangle = l \left| p \right\rangle$. More specifically, if we denote $\left| + \right\rangle \equiv \left| A \to B \right\rangle \equiv \left| p_A \right\rangle$ as the state with a forward vector from *A* to *B* and $\left| - \right\rangle \equiv \left| B \to A \right\rangle \equiv \left| p_B \right\rangle$ as the state with a "return trip" vector from *B* to *A*, then the "trip away" from *A* is denoted by $\not{X} \left| + \right\rangle = +\sqrt{x^2 + y^2 + z^2} \left| + \right\rangle$ and the trip back home to *A* is denoted as $\not{X} \left| - \right\rangle = -\sqrt{x^2 + y^2 + z^2} \left| - \right\rangle$. This is the Pythagorean theorem represented using quantum mechanical expressions *identical in form to those used for things such as Stern-Gerlach* (see, e.g. [1] sections 1.1 and 1.2). Yet there is nothing spooky or weird or in any way perplexing or disconcerting about the notion that a Pythagorean length has a two-fold degeneracy associated simply with whether one is "coming" or "going" over the distance described by $l^2 = x^2 + y^2 + z^2$. Physics is *always* not only about magnitude, but also about direction.

This analysis tells us something very profound about the nature of three-dimensional space as well as about four-dimensional spacetime. It tells us that a great deal of what we think of as "quantum mechanics" is actually hidden in the very nature and structure of a Pythagorean space. But this is not clearly seen until one takes the non-trivial square root of a Pythagorean relationship and naturally comes across matrix operators and eigenkets and eigenvalues with twofold or fourfold degeneracy.

In fact, let's carefully parse this out, from the geometric beginning: Start with a one-dimensional "straight" (Euclidean) line and let's label the axis for this line as "z." Along this line one may define a length *l* bounded by points *A* and *B*. But physics also calls for vectors, and in this one-dimensional space, there are two vectors, one with $A \to B$ direction, the other with $B \to A$ direction. So the length *l*, from the start, should really be thought of as the two-valued $\pm l$ in view of the vectorial aspects of physics.

Now let's propose to introduce the concept of *rotation*. All we have available for rotation is the *z* axis. But in order to do a rotation, we must introduce, not one, but two more axes, which we label *x* and *y*, and which now carve out and define an *x*-*y* plane. We now have a three dimensional space, with no concept of time yet introduced. What is the most economical way to talk about these rotations? In 1843, William Rowan Hamilton carved the quaternion relationships $i^2 = j^2 = k^2 = ijk = -1$ into the Brougham Bridge in Dublin Ireland. These quaternions *i*, *j*, *k* are the first known examples of *non-commuting* numbers, and their non-commuting nature was dictated not by any abstract mathematics, but by Hamilton's recognizing that in a three-dimensional space, rotation is *not commutative*. This later found expression in the Pauli matrices σ^k which $\sigma^{1\,2} = \sigma^{2\,2} = \sigma^{3\,2} = -i\sigma^1\sigma^2\sigma^3 = I$ which are the modern expression of Hamilton's quaternions. The non-commutation of rotations is then expressed very explicitly via the relationship $\frac{1}{2}\left[\sigma_i, \sigma_j\right] = i\varepsilon_{ijk}\sigma_k$, which itself becomes the model for extension into higher-rank Yang-Mills theories. And the anticommutator relation is $\delta^{ij} = \frac{1}{2}\left(\sigma^i\sigma^j + \sigma^j\sigma^i\right) = \frac{1}{2}\left\{\sigma^i, \sigma^j\right\}$. Still nothing spooky or quantum mechanical here: just an objective description of how rotations occur in a three-dimensional space.

So then, one might take these quaternions / spin matrices and revisit Pythagoras and form $\not{X} \equiv \sigma^i x^i$ and discover that the non-trivial square root equations use to describe the Pythagorean Theorem in three dimensions are $\not{X}\left|+\right\rangle = +\sqrt{x^2 + y^2 + z^2}\left|+\right\rangle$ with $\left|+\right\rangle$ representing "going" from $A \to B$, and $\not{X}\left|-\right\rangle = -\sqrt{x^2 + y^2 + z^2}\left|-\right\rangle$ with $\left|-\right\rangle$ representing "coming" (returning) from $B \to A$. Quantum language for sure. And classical Pythagorean geometry for sure.

Now let's ask about time, which is a fourth dimension. Classical relativistic physics tells us that flat spacetime has a Minkowski metric signature $\mathrm{diag}\left(\eta^{\mu\nu}\right) = \left(1, -1, -1, -1\right)$. So now, in contrast to $\delta^{ij} = \frac{1}{2}\left\{\sigma^i, \sigma^j\right\}$ which was used to deconstruct Pythagoras in three dimensions, we deconstruct Minkowski with $\eta^{\sigma\tau} = \frac{1}{2}\left\{\gamma^\sigma, \gamma^\tau\right\}$. The "nuts and bolts" of how Dirac discovered his famous Dirac equation, is by finding that there is no way to reproduce $\mathrm{diag}\left(\eta^{\mu\nu}\right) = \left(1, -1, -1, -1\right)$ with a 2x2 matrix, and that one must now go to the 4x4 γ^μ matrices. Now, the eigenvalue equation is $\left(\not{p} - m\right)u = 0$, and the eigenvalue / eigenstate associations uncovered are $\not{p}\left|u\right\rangle = +m\left|u\right\rangle$ and $\not{p}\left|v\right\rangle = -m\left|v\right\rangle$. Is there some way to understand these in a similarly "classical" way in terms of a vectorial "coming and going"? The added dimension is time, and now the two "points" *A* and *B* are the two "events" *A* and *B*. But if we trace this back to the Pythagorean roots, we now have coming and going in *space* <u>and coming and going in *time*</u>. So if we place *A* in the past and *B* in the future, then $A \to B$ represents a future-oriented vector and $B \to A$ represents a past-oriented vector. So $\left|u\right\rangle$ represents a particle with a vector in a past-to-future orientation and $\left|v\right\rangle$ a particle with a vector in a future-to-past orientation.

We then turn to Feynman-Stückelberg [2] and reinterpret the past-oriented negative energy particle as a future-oriented positive energy antiparticle. So again, what Dirac introduces via the deconstruction of $\eta^{\sigma\tau} = \frac{1}{2}\{\gamma^\sigma, \gamma^\tau\}$ into 4x4 matrices, is both a "coming and going in space" (as specified in (3)) and a "coming and going in time," with the coming and going in time reinterpreted via Feynman-Stückelberg into "always going forward in time" wherein negative energy particles back-travelling in time become positive energy antiparticles forward-travelling in time. Pauli incorrectly asserted that the two-valuedness of spin is *non-classical*, which has had the unfortunate impact of making quantum mechanics more opaque than it ought to be. In fact, if the non-trivial square root of Pythagoras tells us that all lengths must have a twofold degeneracy of either $A \to B$ or $B \to A$, then when we start to look at spins in three dimensions, spin up simply has its spin axis oriented along the Pythagorean $A \to B$ eigenstate axis, and spin down then orients oppositely around $B \to A$ axis. The fourfold degeneracy of Dirac's equation is then seen as a direct consequence of three space dimensions and one time dimension, where a twofold degeneracy of coming and going in time is multiplied by the further twofold degeneracy of coming and going in space. To further demonstrate how these ostensibly "non-classical" "mysterious" phenomena can be seen through a fully classical lens, Ohanian in [3] establishes how the flow of energies associated with so-called intrinsic spin are entirely classical circular energy flows in the electron (Fermion) wave field.

So, what is the point of all this? The human race has spent a century plus a decade in collective hand-wringing over the "weird," "mysterious," "strange," "non-classical," "counterintuitive," "non-local," "entangled," "why would God play dice?" nature of quantum mechanics. But in fact, there are many salient aspects of quantum mechanics that present directly and inexorably out of the simple exercise of taking the non-trivial square root of the relativistic mass-energy relation $p_\sigma p^\sigma = m^2$. Because the Pythagorean metricity of flat spacetime is based on the Minkowksi tensor $\eta^{\sigma\tau}$, the Dirac deconstruction of this into $\eta^{\sigma\tau} = \frac{1}{2}\{\gamma^\sigma, \gamma^\tau\}$ explicitly introduces a fourfold Dirac degeneracy of eigenstates that represent "coming and going vectors" in both space and in time. Although this degeneracy – which leads to such ostensibly-quantum mechanical phenomena as the electron magnetic moment and "intrinsic" spin – is in fact entirely understandable classically, it has often not been understood as such, which has made the logical consequence of this degeneracy much more difficult to take in stride than need be.

Wheeler's "geometrodynamic" program [4] seeks to understand all of physics as emanating from the properties of the spacetime stage in which that physics occurs. What the foregoing shows is that spacetime itself – even the Euclidean Phythagorean four-space of Minkowski – already contains many features brought about through the metric deconstruction $\eta^{\sigma\tau} = \frac{1}{2}\{\gamma^\sigma, \gamma^\tau\}$ which one associates with quantum theory, but which is in fact traceable to the very fact of living in a four-dimensional universe with one time and three space dimensions. And these in turn trace back to Pythagoras, which when analyzed using the Pauli matrices, teaches the very simple physics notion that when you ascribe a length *l* to the separation between two points *A* and *B*, and if you then wish to talk about that length in a physics context, that you also need to establish a vector of length *l* which – as a vector – must also have direction in addition to magnitude. So we use $\left|+\right\rangle$ as the eigenstate of $\not{X}\left|+\right\rangle=+\sqrt{x^2+y^2+z^2}\left|+\right\rangle$ which represents the $A\rightarrow B$ direction and $\left|-\right\rangle$ as the eigenstate of $\not{X}\left|-\right\rangle=-\sqrt{x^2+y^2+z^2}\left|-\right\rangle$ which represents the $B\rightarrow A$ direction.

One might, *a priori*, simply regard the deconstruction of $p_\sigma p^\sigma - m^2 = 0$ into $(\not{p}-m)u=0$ as a mathematical convenience and nothing more, and simply view our spacetime surroundings in the usual Pythagorean way, and no more. But the fact is, that when we do experiments, and observe electrons with all of the properties predicted by $(\not{p}-m)u=0$ which are hidden from view in the Pythagorean square equation $p_\sigma p^\sigma = m^2$, we understand that the deconstruction $\eta^{\sigma\tau}=\frac{1}{2}\left\{\gamma^\sigma,\gamma^\tau\right\}$ is more than just a mathematical curiosity. We realize that it reveals something about the intrinsic nature of our physical spacetime that is simply not apparent if one sticks only to the equation $p_\sigma p^\sigma = m^2$. But what it reveals are phenomena which we often think of as "quantum mechanical" and thus as counterintuitive, etc., when in fact these phenomena are endemic to the very nature of spacetime and thus fully geometrodynamic. When we ask "who is the culprit?" responsible for at least some fair share of quantum physics, the answer is now clear: Pythagorean spacetime itself!

One wonders from all this, how differently our understanding of quantum reality might have evolved if Pythagoras had himself discovered quaternions, and 2500 years ago, had taught that $\not{X}\left|+\right\rangle=+\sqrt{x^2+y^2+z^2}\left|+\right\rangle$ and $\not{X}\left|-\right\rangle=-\sqrt{x^2+y^2+z^2}\left|-\right\rangle$, in contrast to the historical accident whereby all we had to go on for 2500 years was $l=\pm\sqrt{x^2+y^2+z^2}$.

References

[1] J. J. Sakurai, *Modern Quantum Mechanics*, Addison Wesley (1993)
[2] E. Stueckelberg, *Un nouveau modele de l'electron ponctuel en theorie classique*, Helvetica Physica Acta, Vol.14, 1941, pp.51-80
[3] Ohanian, H. C., *What is spin?*, Am. J. Phys. 54 (6), 500-505 (June 1986)
[4] Wheeler, J. A., *Geometrodynamics*, Academic Press, pp. 225-253 (1962).