An Extension of Rainich's Already Unified Field Theory to Electromagnetic Sources, In a Singly Connected Spacetime Topology

ABSTRACT

In this essay it will be shown how the source-free "already unified" field theory of Rainich, (Trans. Am. Math. Soc., Vol. 27, p. 106, (1925)), as exposited by Misner and Wheeler -1 and others, -2 may be expanded to encompass electromagnetic sources, strictly within the bounds of a simply connected spacetime topology. The result is a fully generalized, strictly classical, already unified field theory of electromagnetism and gravitation, which integrates fully into electrodynamics, the antisymmetric tensor formalism of Levi-Cevita. In particular, it is shown how the entirety of Maxwell's electrodynamics, including sources, may be directly consolidated into the geometrodynamic equation $R_{\mu\nu}=0$.

The introductory section presents a general overview of the classical already unified field theory with sources, to be developed herein. The [redacted] First six sections examine how this theory with sources may be developed out of Maxwell's classical electrodynamics and Rainich's classical source-free already unified field theory. In sections 7-9, a number of special cases of electrodynamics with sources, including source-free electrodynamics and electrodynamics in the extremal electric reference frame, are presented. The final section examines how the transition from a simply connected Euclidean spacetime topology to a multiply connected non-Euclidean topology, may lead to a particularly

natural basis for understanding non-classical, ie., quantized already unified field theory.

CONTENTS

INTRODUCTION- Geometrodynamic Electrodynamics with Sources: A General Overview

It is possible, as will be shown in detail herein, to consolidate the entirety of Maxwell's electrodynamics, including sources, into the single geometrodynamic equation:

$$R_{\mu\nu} = 0. \qquad (1)$$

It turns out, when all possible sources of electromagnetic field are considered, that the above may be translated into the electrodynamic equivalents:

$$\text{(A)}\quad F_{\alpha\sigma}(J^\sigma + {}^*J^\sigma) + {}^*F_{\alpha\sigma}(P^\sigma + {}^*P^\sigma) + \tfrac{1}{2}{}^*F^{\sigma\tau}(J_{\tau\sigma\alpha} + {}^*J_{\tau\sigma\alpha}) - \tfrac{1}{2}F^{\sigma\tau}(P_{\tau\sigma\alpha} + {}^*P_{\tau\sigma\alpha}) = 0$$

$$(2)$$

$$\text{(B)}\quad {}^*F_{\beta\sigma}(J^\sigma + {}^*J^\sigma) - F_{\beta\sigma}(P^\sigma + {}^*P^\sigma) - \tfrac{1}{2}F^{\sigma\tau}(J_{\tau\sigma\beta} + {}^*J_{\tau\sigma\beta}) - \tfrac{1}{2}{}^*F^{\sigma\tau}(P_{\tau\sigma\alpha} + {}^*P_{\tau\sigma\alpha}) = 0,$$

where charge J and monopole P are connected with the electromagnetic field F according to:

(a)

(i) $J^{\sigma} = -P^{*\sigma} = \frac{1}{4\pi} F^{\tau\sigma}{}_{;\tau}$

(ii) $J_{\tau\sigma\upsilon} = -P^{*}_{\tau\sigma\upsilon} = -\frac{1}{4\pi} {}^{*}\left({}^{*}F_{\tau\sigma;\upsilon} + {}^{*}F_{\sigma\upsilon;\tau} + {}^{*}F_{\upsilon\tau;\sigma}\right)$

(b)

(i) $P^{\sigma} = J^{*\sigma} = \frac{1}{4\pi} {}^{*}F^{\tau\sigma}{}_{;\tau}$

(ii) $P_{\tau\sigma\upsilon} = J^{*}{}_{\tau\sigma\upsilon} = \frac{1}{4\pi} {}^{*}\left(F_{\tau\sigma;\upsilon} + F_{\sigma\upsilon;\tau} + F_{\upsilon\tau;\sigma}\right)$

(3)

(c)

(i) ${}^{*}J^{\sigma} = -{}^{*}P^{*}_{\tau\sigma\upsilon} = \frac{1}{4\pi} {}^{*}\left(F^{\tau\sigma}{}_{;\tau}\right)$

(ii) ${}^{*}J_{\tau\sigma\upsilon} = -{}^{*}P^{*}{}_{\tau\sigma\upsilon} = -\frac{1}{4\pi}\left({}^{*}F_{\tau\sigma;\upsilon} + {}^{*}F_{\sigma\upsilon;\tau} + {}^{*}F_{\upsilon\tau;\sigma}\right)$

(d)

(i) ${}^{*}P^{\sigma} = {}^{*}J^{*\sigma} = \frac{1}{4\pi} {}^{*}\left({}^{*}F^{\tau\sigma}{}_{;\tau}\right)$

(ii) ${}^{*}P_{\tau\sigma\upsilon} = {}^{*}J^{*}_{\tau\sigma\upsilon} = \frac{1}{4\pi}\left(F_{\tau\sigma;\upsilon} + F_{\sigma\upsilon;\tau} + F_{\upsilon\tau;\sigma}\right).$

In the above, the "dual" of any anti-symmetric tensor **A** in four-space is constructed in the usual way, using the Levi-Cevita formalism, according to ${}^{*}A^{\sigma} = \frac{1}{3!}\epsilon^{\alpha\tau\gamma\sigma}A_{\alpha\tau\gamma}$, ${}^{*}A^{\sigma\tau} = \frac{1}{2!}\epsilon^{\delta\gamma\sigma\tau}A_{\delta\gamma}$ and ${}^{*}A^{\tau\sigma\upsilon} = \epsilon^{\gamma\tau\sigma\upsilon}A_{\gamma}$; from which follow respectively the usual identities ${}^{**}A^{\sigma} = A^{\sigma}$, ${}^{**}A^{\sigma\tau} = -A^{\sigma\tau}$ and ${}^{**}A^{\tau\sigma\upsilon} = A^{\tau\sigma\upsilon}$. It is also to be noted that the electrodynamical equations (2) are invariant with respect to all duality transformations of both the second-second rank form $F \rightarrow {}^{*}F$ for fields and the first-third rank form $J \rightarrow {}^{*}J$ and $P \rightarrow {}^{*}P$ for sources.

Now, while a good portion of the exposition herein will be devoted to the rigorous derivation and justification of eqs. (1), (2) and (3), it should be noted, when (1) is considered in light of the Einstein equation/Bianchi identity $-\chi T^{\mu}{}_{\nu;\mu} = (R^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}{}_{\nu}R)_{;\mu} = 0$ for energy conservation; and when (2) is considered in light of the equations $J^{\sigma}{}_{;\sigma} = 0$ and $J_{\tau\sigma\nu;\lambda} - J_{\sigma\nu\lambda;\tau} + J_{\nu\lambda\tau;\sigma} - J_{\lambda\tau\sigma;\nu} = 0$ for electromagnetic source conservation; that the strength of eq. (1) is identical with that of eqs. (2). In other words, eqs. (1) and (2), from a general relativistic standpoint, are absolutely compatable. While this does not prove the equivalence of eqs. (1) and (2), it does make this equivalence plausible. -3 The rigorous justification of this equivalence will be developed in sections 1-6.

Once the equations (1),(2) and (3) of "geometrodynamic electrodynamics" with sources have been developed, a number of specializations are possible. One may, in the usual manner, reduce to the extremal electric formulation of (2) and (3) by setting

$$F^{\mu\nu} = e^{*\alpha}\mathcal{E}^{\mu\nu}. \tag{4}$$

One may also, as will be discussed in depth herein, reduce from electrodynamics with sources to source-free electrodynamics, by constructing the field tensor $F^{\mu\nu}$ from the potential vector A^{μ} according to the familiar:

$$F^{\mu\nu} = A^{\nu;\mu} - A^{\mu;\nu}. \tag{5}$$

It is to be emphasized at the outset that eqs. (1)-(3), which describe electrodynamics with sources, do not in any way involve eq. (5) above. It is this supplementary condition (5) for the four potential which, as it turns out, is used to reduce from electrodynamics with sources, to source-free electrodynamics. In other words, eq. (5) above has absolutely nothing whatsoever to do with electrodynamics **with** sources, and as soon as one employs eq. (5), one is automatically talking about electrodynamics **without** souces, for it is this equation which directly causes the reduction to source-free electrodynamics.

Two further specializations of interest come about when:

(a) $$T_{(Max)}{}^{\mu}{}_{\nu} = -\frac{1}{8\pi}\left[F^{\mu\tau}F_{\nu\tau} + {}^{*}F^{\mu\tau}\,{}^{*}F_{\nu\tau}\right] = 0$$

and/or when: (6)

(b) $$\tilde{J}^{\sigma} = J^{\sigma} + {}^{*}J^{\sigma} = 0.$$

In the former case when the Maxwell tensor $T_{(Max)}{}^{\mu}{}_{\nu}$ is equal to zero, the second-second rank duality operator * becomes equivalent with ~~$i = \sqrt{-1}$~~, while in the latter case, the first-third rank duality operator * becomes equivalent with -1. That is, ${}^{*}F^{\mu\nu} = i F^{\mu\nu}$ **iff** (6)(a) is true, and ${}^{*}J^{\sigma} = -J^{\sigma}$ **iff** (6)(b) is true. These specializations (4)-(6) will be developed and examined in sections 7-9.

Finally, while the entire theory to be developed herein presupposes a simply connected Euclidean spacetime topology described in terms of the complexion gradient a_{μ} by:

(a) $$a^{\mu;\nu} - a^{\nu;\mu} = 0 , \qquad (7)$$

one would certainly consider the generalization of the above, sooner or later, to a multiply connected spacetime topology, given by:

(b) $$\oint a^{\sigma} dx_{\sigma} = 2\pi \cdot n \quad ; \; n = 0, 1, 2, 3 \ldots \qquad (7)$$

As it happens, this seems to lead toward a particularly natural approach for the quantization of classical already unified field theory, as will be examined in section 10.

The discussion in the first section begins with a brief review of Maxwell's classical electrodynamics and Rainich's classical, source-free, already unified field theory.

SECTION 1- A Review of Maxwell's Electrodynamics and Rainich's Source-Free Already Unified Field Theory

In their ordinary tensor formulation, the equations of Maxwell's electrodynamics take the form:

$$\text{(a)}\qquad J^{\tau} = \frac{1}{4\pi}\left(F^{\sigma\tau}{}_{;\sigma}\right)$$

$$\text{(b)}\qquad 0 = F_{\tau\sigma;\upsilon} + F_{\sigma\upsilon;\tau} + F_{\upsilon\tau;\sigma}\,. \tag{8}$$

It is however more appropriate for the subsequent discussion, borrowing from (3), to rewrite the above as:

$$\text{(a)}\qquad J^{\tau} = \frac{1}{4\pi}\left(F^{\sigma\tau}{}_{;\sigma}\right)$$

$$\text{(b)}\qquad {}^{*}P_{\tau\sigma\upsilon} = \frac{1}{4\pi}\left(F_{\tau\sigma;\upsilon} + F_{\sigma\upsilon;\tau} + F_{\upsilon\tau;\sigma}\right) \tag{9}$$

-where-

$$F^{\mu\upsilon} = A^{\upsilon;\mu} - A^{\mu;\upsilon}. \tag{10}$$

In this form, the dualistic symmetry between charge/monopole and first/third rank electromagnetic sources becomes more apparent, as does the dependence of the Maxwell equations (8), strictly speaking, upon the supplementary potential condition (10). This is because without (10), the right hand side of (8)(b) will no longer be identically equal to zero.

Assuming that $F^{\mu\nu} = A^{\nu;\mu} - A^{\mu;\nu}$ and hence that $F_{\tau\sigma;\nu} + F_{\sigma\nu;\tau} + F_{\nu\tau;\sigma}$ in (8)(b) really is identically equal to zero, one may proceed to use energy conservation to connect (8)(b) with the geometrodynamic energy tensor $T^{\mu\nu}$, as such:

$$\begin{aligned}
-\kappa T^{\mu}{}_{\nu;\mu} &= \left(R^{\mu}{}_{\nu} - \tfrac{1}{2}\delta^{\mu}{}_{\nu} R \right)_{;\mu} \\
&= -\kappa \left(T^{\mu}{}_{\nu} + t^{\mu}{}_{\nu} \right)_{,\mu} \\
&= \frac{\kappa}{4\pi} \left[\tfrac{1}{2} F^{\mu\tau} \left(F_{\mu\nu;\tau} + F_{\nu\tau;\mu} + F_{\tau\mu;\nu} \right) \right] \\
&= \frac{\kappa}{4\pi} \left[F^{\mu\tau} F_{\nu\tau;\mu} - \tfrac{1}{2}\delta^{\mu}{}_{\nu} F^{\sigma\tau} F_{\sigma\tau;\mu} \right] \qquad (11) \\
&= \frac{\kappa}{4\pi} \left[\left(F^{\mu\tau} F_{\nu\tau} \right)_{;\mu} - \tfrac{1}{4}\delta^{\mu}{}_{\nu} \left(F^{\sigma\tau} F_{\sigma\tau} \right)_{;\mu} - F_{\nu\tau} F^{\sigma\tau}{}_{;\sigma} \right] \\
&= \frac{\kappa}{8\pi} \left[F^{\mu\tau} F_{\nu\tau} + {}^{*}F^{\mu\tau\,*}F_{\nu\tau} \right] - \kappa F_{\nu\tau} J^{\tau} \\
&= -\kappa \left(T_{(max)}{}^{\mu}{}_{\nu} + t_{(em)}{}^{\mu}{}_{\nu} \right)_{;\mu} \\
&= 0 ,
\end{aligned}$$

-where-

(a)
$$\begin{aligned}
T_{(max)}{}^{\mu}{}_{\nu} &= -\frac{1}{8\pi} \left[F^{\mu\tau} F_{\nu\tau} + {}^{*}F^{\mu\tau\,*}F_{\nu\tau} \right] \\
&= -\frac{1}{4\pi} \left[F^{\mu\tau} F_{\nu\tau} - \tfrac{1}{4}\delta^{\mu}{}_{\nu} F^{\sigma\tau} F_{\sigma\tau} \right]
\end{aligned}$$

-and- (12)

(b)
$$K_{(em)\nu} = t_{(em)}{}^{\mu}{}_{\nu;\mu} = F_{\nu\tau} J^{\tau} = \frac{1}{4\pi} F_{\nu\tau} F^{\sigma\tau}{}_{;\sigma}$$

are the Maxwell tensor and the gravitational energy components of the electromagnetic field, respectively. Therefore, neglecting henceforth any cosmological constants of integration:

$$T^{\mu}{}_{\nu} = T_{(MAN)}{}^{\mu}{}_{\nu} + \tau_{(EM)}{}^{\mu}{}_{\nu} . \qquad (13)$$

Now, by applying the potential condition (10) to (9)(b), one finds that ${}^{*}P_{\tau\sigma\nu}$, and all the other third rank sources of the electromagnetic field deduceable therefrom by any sort of duality rotation (see (3)), will be equal to zero. Employing the first-third rank Levi-Cevita formalisms ${}^{*}A^{\sigma} = \frac{1}{3!}\epsilon^{\alpha\tau\gamma\sigma}A_{\alpha\tau\gamma}$, ${}^{*}A^{\tau\sigma\nu} = \epsilon^{\alpha\tau\sigma\nu}A_{\nu}$, it is also apparant that the first rank sources of the electromagnetic field will reduce to zero, with the consequence that (9) ultimately reduces to:

(a) $$J^{\tau} = \frac{1}{4\pi}\left(F^{\sigma\tau}{}_{;\sigma}\right) = 0$$

(14)

(b) $${}^{*}P_{\tau\sigma\nu} = \frac{1}{4\pi}\left(F_{\tau\sigma;\nu} + F_{\sigma\nu;\tau} + F_{\nu\tau;\sigma}\right) = 0,$$

but <u>if and only if</u> the potential condition (10) holds true. This is just source-free electrodynamics.

Assuming that $F^{\mu\nu} = A^{\nu;\mu} - A^{\mu;\nu}$ does hold true, one may further use (14)(a) above to set $J^{\tau} = 0$ in (11), which is tantamount to setting $K_{(EM)\nu} = \tau_{(EM)}{}^{\mu}{}_{\nu;\mu} = F_{\nu\tau}J^{\tau} = 0$ in (12)(b). As a consequence one may extract from (11), the source-free energy tensor and trace equation: (eq. (13) with $K_{(EM)\nu} = \tau_{(EM)}{}^{\mu}{}_{\nu;\mu} = F_{\nu\tau}J^{\tau} = 0$, and no cosmological integration constant) ~~(neglecting any cosmological constants of integration)~~

(a) $$T^{\mu}{}_{\nu} = T_{(MAN)}{}^{\mu}{}_{\nu} = -\frac{1}{8\pi}\left[F^{\mu\tau}F_{\nu\tau} + {}^{*}F^{\mu\tau}{}^{*}F_{\nu\tau}\right]$$

(15)

(b) $$T = T^{\sigma}{}_{\sigma} = 0 .$$

(14) and (15) of course, are the primary equations of source-free

electrodynamics,[4] (14) being the source-free Maxwell equations, and (15) being the usual expressions for the Maxwell tensor and its trace. It is important to note the main point, which is that these equations all come about by virtue of the supplementary potential condition (10).

One may of course derive from (15), in the usual way:

$$\text{(a)} \qquad R^{\mu\sigma}R_{\nu\sigma} - \frac{1}{4}\delta^{\mu}{}_{\nu}\left(R^{\sigma\tau}R_{\sigma\tau}\right) = 0$$

$$\text{(b)} \qquad R^{\sigma}{}_{\sigma} = 0. \qquad (16)$$

If one further defines the duality rotation operation:

$$F^{\mu\nu} = e^{*a}\mathcal{E}^{\mu\nu} \qquad (17)$$

in the usual way, and supplements this with the requirement: $(a^{\mu} = a^{;\mu} = a^{,\mu})$

$$a^{\mu;\nu} - a^{\nu;\mu} = 0 \qquad (18)$$

that spacetime topology be simply connected, then one finds in (16) and (18) the crux of Rainich's classical, source-free, already unified field theory.[5]

Thus, it is the condition $F^{\mu\nu} = A^{\nu;\mu} - A^{\mu;\nu}$ which, in the final analysis, is responsible for reducing all the sources of the electromagnetic field to zero; ie., it is this condition which brings about all the primary equations of source-free electrodynamics and already unified field theory. It is therefore apparant, to develop a more general theory of electrodynamics <u>with</u> sources, that this condition must be discarded, and the theory with sources developed such that there is no dependence upon this condition at any point in the theory. Conversely it is necessary that the theory with sources be developed in such a way, when the condition $F^{\mu\nu} = A^{\nu;\mu} - A^{\mu;\nu}$ is finally added as a form of specialization, tnat the theory with sources will reduce in this special case to the classical source-free theory discussed above.

It is therefore our objective from here on, to extend classical electrodynamics and source-free already unified field theory, to include all forms of electromagnetic source. This will be the primary theme of sections 2-6.

SECTION 2- Difficulties Encountered in the Transition from Source-Free Already Unified Field Theory, to Already Unified Field Theory with Sources

To develop an "already unified" field theory of electromagnetism and gravitation that includes sources, one must first forego any dependence upon the vector potential equation $F^{\mu\nu} = A^{\nu;\mu} - A^{\mu;\nu}$, as shown above; since it is this equation which, of its very nature, brings about source-free electrodynamics. Thus, source-free electrodynamics must come to be viewed as a specialization of electrodynamics with sources, in the [redacted] special case where the condition $F^{\mu\nu} = A^{\nu;\mu} - A^{\mu;\nu}$ has been added. In abandoning the use ~~of~~ the equation $F^{\mu\nu} = A^{\nu;\mu} - A^{\mu;\nu}$ for electrodynamics with sources, a number of related difficulties arise, all of which may be illuminated by a closer examination of eq. (11).

First, and of paramount importance, in the absence of $F^{\mu\nu} = A^{\nu;\mu} - A^{\mu;\nu}$, the term $F_{\mu\nu;\tau} + F_{\nu\tau;\mu} + F_{\tau\mu;\nu}$ in eq. (11) will no longer be identically equal to zero, which indicates further that the conservation of energy is, according to (11), no longer identically assured. This is a serious problem, which forces one to consider appropriate generalizations of (11) that do assure the conservation of energy, [redacted] but that do not rely upon the condition $F^{\mu\nu} = A^{\nu;\mu} - A^{\mu;\nu}$ for this assurance. Thus, one must look for some sort of extended electrodynamic term that can [redacted] plays the same role in energy conservation as does the term $F_{\mu\nu;\tau} + F_{\nu\tau;\mu} + F_{\tau\mu;\nu}$ (see (11)) currently, but which is identically equal to zero without dependence upon the condition $F^{\mu\nu} = A^{\nu;\mu} - A^{\mu;\nu}$. In short, one must look for an electrodynamic analog of the Bianchi identity $(R^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}{}_{\nu} R)_{;\mu} = 0$, which can [redacted] be connected with $T^{\mu}{}_{\nu;\mu}$ to assure identically that $T^{\mu}{}_{\nu;\mu} = 0$, but which does not in any way rely upon the supplementary condition (10) for the four-potential. This is the primary route along which the development of electrodynamics with sources must take place.

Second, the fact that the non-symmetric energy components $K_{(em)\nu} = F_{\nu\tau} J^{\tau}$ of the electromagnetic field, in a theory with sources, is no longer equal to zero, causes the energy tensor $T^{\mu\nu}$ ~~in~~ (11) to lose its transposition symmetry, through reintroduction of the non-symmetric components $T_{(em)}$. (See also (12))

Third, and also because $K_{(em)\nu}$ is no longer equal to zero in a theory with sources, the energy tensor in (11) will no longer be invariant under a duality rotation. Note in the source-free theory, since $K_{(em)\nu} = F_{\nu\tau} J^{\tau} = 0$, that $T^{\mu}{}_{\nu}$ becomes equal to the Maxwell tensor $T^{\mu}_{(max)\nu}$, (see (15)(a)) which tensor is of course dualistically invariant. This too, needs to be corrected.

Finally, once the term $F_{\tau\sigma;\nu} + F_{\sigma\nu;\tau} + F_{\nu\tau;\sigma}$ on the right hand side of (8)(b) and (11) is no longer identically equal to zero, one must begin to consider both charge and monopole sources, of both first and third rank (see (9)); and to develop a theory that reflects the necessary [redacted] symmetries [redacted] among all these varying types of electromagnetic source.

As will be seen, it is indeed possible to resolve all of these difficulties, all at once, by the development of an appropriate electrodynamic theory with sources. Further, it will be shown how the development of such a theory, as an added bonus, leads directly to the consolidation of Maxwell's entire electrodynamics, including sources, into the simple non-linear geometrodynamic equation $R_{\mu\nu} = 0$.

SECTION 3- Preliminary Development of Already Unified Field Theory with Sources

In Einstein's geometrodynamics, the field equation $-\kappa T^{\mu}{}_{\nu;\mu} = (R^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}{}_{\nu}R)_{;\mu} = 0$ expresses, among other things, the fact that energy must be conserved; and it assures one that energy will in fact be conserved, by associating the contracted covariant derivative of the energy tensor, $T^{\mu}{}_{\nu;\mu}$, with the mathematical Bianchi identity $(R^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}{}_{\nu}R)_{;\mu} = 0$, which is known in all cases to be identically satisfied.

For electrodynamics with sources, it is necessary to follow a similar path, by associating $T^{\mu}{}_{\nu;\mu}$ with some form of (dualistically invariant) electrodynamic term which, like the Bianchi identity, is known in all cases to be identically equal to zero. In eq. (11) of course, $T^{\mu}{}_{\nu;\mu}$ is associated with the term $F^{\mu\tau}(F_{\mu\nu;\tau} + F_{\nu\tau;\mu} + F_{\tau\mu;\nu})$; however, the identity of this term with zero is assured only when one has added the supplementary condition $F^{\mu\nu} = A^{\nu;\mu} - A^{\mu;\nu}$ for the four-potential, which condition applies only to source-free electrodynamics. Hence, this is not an acceptable route to follow if one wishes to describe electrodynamics with sources and to simultaneously conserve energy. Instead, it is necessary to base the development of electrodynamics with sources upon the association of $T^{\mu}{}_{\nu;\mu}$ with some electrodynamic analog of the Bianchi identities, which does not in any way depend upon $F^{\mu\nu} = A^{\nu;\mu} - A^{\mu;\nu}$, but which nevertheless insures the identical conservation of energy in all cases.

To construct a generalization of electrodynamics that includes sources, and which does not therefore rely upon the condition $F^{\mu\nu} = A^{\nu;\mu} - A^{\mu;\nu}$ to assure the conservation of energy, it is helpful to begin by examining the identity:

$$\frac{1}{2} A^{\sigma\tau}\left(B_{\sigma\nu;\tau} + B_{\nu\tau;\sigma} + B_{\tau\sigma;\nu}\right) - {}^{*}A_{\nu\sigma}\,{}^{*}B^{\tau\sigma}{}_{;\tau} = 0, \qquad (19)$$

which of course, holds true for any two antisymmetric tensors A and B in four space. -6 In the further case where B is defined out of a four vector V according to:

$$B^{\mu\nu} = V^{\nu;\mu} - V^{\mu;\nu}, \qquad (20)$$

but only in this case, the identity (19) will yield the two sub-identities:

$$\text{(a)} \qquad {}^{*}B^{\tau\sigma}{}_{;\tau} = 0$$

$$\text{(b)} \qquad B_{\sigma\nu;\tau} + B_{\nu\tau;\sigma} + B_{\tau\sigma;\nu} = 0, \qquad (21)$$

which of course, are of the same form as the usual equations of source-free electrodynamics. (see eqs. (14)). Thus, the identity (19), which holds true even in the absence of an equation of the form (20); and which further, in the presence of eq. (20) reduces to the form (21) that characterizes source-free electrodynamics, will provide the starting point for the development of electrodynamics with sources.

In particular, applying (19) above to the field tensor $F^{\mu\nu}$, it is possible to form the four identities:

$$\text{(A)}\begin{cases}\text{(i)} & \frac{1}{2}F^{\sigma\tau}(F_{\sigma\nu;\tau} + F_{\nu\tau;\sigma} + F_{\tau\sigma;\nu}) - {}^*F_{\nu\sigma}{}^*F^{\tau\sigma}{}_{;\tau} = 0 \\ \text{(ii)} & \frac{1}{2}{}^*F^{\sigma\tau}({}^*F_{\sigma\nu;\tau} + {}^*F_{\nu\tau;\sigma} + {}^*F_{\tau\sigma;\nu}) - F_{\nu\sigma}F^{\tau\sigma}{}_{;\tau} = 0\end{cases}$$

(22)

$$\text{(B)}\begin{cases}\text{(i)} & \frac{1}{2}{}^*F^{\sigma\tau}(F_{\sigma\nu;\tau} + F_{\nu\tau;\sigma} + F_{\tau\sigma;\nu}) + F_{\nu\sigma}{}^*F^{\tau\sigma}{}_{;\tau} = 0 \\ \text{(ii)} & -\frac{1}{2}F^{\sigma\tau}({}^*F_{\sigma\nu;\tau} + {}^*F_{\nu\tau;\sigma} + {}^*F_{\tau\sigma;\nu}) - {}^*F_{\nu\sigma}F^{\tau\sigma}{}_{;\tau} = 0,\end{cases}$$

which, on application of the condition

$$F^{\mu\nu} = A^{\nu;\mu} - A^{\mu;\nu} \tag{23}$$

of source-free electrodynamics, will further reduce to the equations of source-free electrodynamics:

(A) $${}^*F_{\sigma\nu;\tau} + {}^*F_{\nu\tau;\sigma} + {}^*F_{\tau\sigma;\nu} = 0$$

(24.1)

(B) $$F_{\sigma\nu;\tau} + F_{\nu\tau;\sigma} + F_{\tau\sigma;\nu} = 0$$

in third rank form, or alternatively:

(A) $$F^{\tau\sigma}{}_{;\tau} = 0$$

$$^*F^{\tau\sigma}{}_{;\tau} = 0 \qquad (24.2)$$

(B)

in first rank form, as expected.

Now while the above identities [redacted] (22) may appear at first glance to consist of sixteen independent equations, it is clear on further examination that the two identities [redacted] (A) differ from one another merely by a duality rotation of the second rank field tensor $F^{\mu\nu}$, as do the two identities [redacted] (B). Hence, the two equations (A), and also the two equations (B), are really just a renaming of one another. Equations (A) and (B) however cannot be derived from one another simply by a duality rotation of the field tensor $F^{\mu\nu}$, hence these equations, (A) and (B), are indeed truly independent. Therefore, eqs. (22) really contain a total of eight independent equations. Fortunately, this is precisely the number of independent equations that is needed in electrodynamic theory, and these equations will therefore be taken as the starting point for considering electrodynamics with sources.

Now, because each of (A), and each of (B), differs from the other by a mere duality rotation, it is possible to combine each of (A), and each of (B), into a pair of single identities; and, as a natural consequence of this procedure, the resulting identities will themselves exhibit invariance under duality transformations $F^{\mu\nu} \rightarrow {}^*F^{\mu\nu}$ of the electromagnetic field. (This "dualistic" invariance is also sometimes called "phase" or "gauge" invariance, though "gauge" is misleading for a number of reasons, including the fact that it is easily confused with the very different invariance of $F^{\mu\nu} = A^{\nu;\mu} - A^{\mu;\nu}$, under the "gauge" transformation $A^{\mu} \rightarrow A^{\mu} + \Lambda^{;\mu}$.) Further, contrasting (22) above with the source-free (11), it is apparant for the development of electrodynamics with sources, hence without $F^{\mu\nu} = A^{\nu;\mu} - A^{\mu;\nu}$, that these identities (22) may [redacted] be connected, with an appropriate constant factor, to the contracted covariant energy tensor derivative $T^{\mu}{}_{\nu;\mu}$, to assure the identical conservation of energy. In other words, (22) readily [redacted] furnish[redacted] the electrodynamic analogs of the Bianchi identities, which are required for the development of electrodynamics with sources, and the identical conservation of energy, without $F^{\mu\nu} = A^{\nu;\mu} - A^{\mu;\nu}$. This may all be achieved by connecting $T^{\mu}{}_{\nu;\mu}$ with the identities (22), appropriately combined, as follows: (contrast with the source-free (11))

$$
\text{(A)}\quad
\begin{aligned}
-\chi T^{\sigma}{}_{a;\sigma} &= \left(R^{\sigma}{}_{a} - \tfrac{1}{2}\delta^{\sigma}{}_{a} R\right)_{;\sigma} \\
&= \frac{\chi}{4\pi}\Big[\tfrac{1}{2} F^{\sigma\tau}\left(F_{\sigma a;\tau} + F_{a\tau;\sigma} + F_{\tau\sigma;a}\right) - {}^{*}F_{a\sigma}{}^{*}F^{\tau\sigma}{}_{;\tau} \\
&\qquad + \tfrac{1}{2}{}^{*}F^{\sigma\tau}\left({}^{*}F_{\sigma a;\tau} + {}^{*}F_{a\tau;\sigma} + {}^{*}F_{\tau\sigma;a}\right) - F_{a\sigma} F^{\tau\sigma}{}_{;\tau}\Big] \\
&= 0
\end{aligned}
\qquad (25)
$$

$$
\text{(B)}\quad
\begin{aligned}
-\chi T^{\sigma}{}_{B;\sigma} &= \left(R^{\sigma}{}_{B} - \tfrac{1}{2}\delta^{\sigma}{}_{B} R\right)_{;\sigma} \\
&= \frac{\chi}{4\pi}\Big[\tfrac{1}{2}{}^{*}F^{\sigma\tau}\left(F_{\sigma B;\tau} + F_{B\tau;\sigma} + F_{\tau\sigma;B}\right) + F_{B\sigma}{}^{*}F^{\tau\sigma}{}_{;\tau} \\
&\qquad - \tfrac{1}{2} F^{\sigma\tau}\left({}^{*}F_{\sigma B;\tau} + {}^{*}F_{B\tau;\sigma} + {}^{*}F_{\tau\sigma;B}\right) - {}^{*}F_{B\sigma} F^{\tau\sigma}{}_{;\tau}\Big] \\
&= 0 .
\end{aligned}
$$

Note again, as illustrated in eqs. (19) - (21) and (22)-(24), that these identities hold true independently of the four-potential condition $F^{\mu\nu} = A^{\nu;\mu} - A^{\mu;\nu}$; and, in the presence of this condition , that these equations reduce to the corresponding equations of source-free electrodynamics. Particularly important [redacted], because of the connection of these identities to $T^{\mu}{}_{\nu;\mu}$ in the above, ~~we are assured simultaneously~~ is the simultaneous assurance that energy will be identically conserved ~~of the identical conservation of energy~~, independent of the condition $F^{\mu\nu} = A^{\nu;\mu} - A^{\mu;\nu}$ for the four-potential. This is precisely what was required earlier for the proper development of electrodynamics with sources. Hence we shall regard eqs. (25) as our preliminary dynamical equations of electrodynamics with sources.

The only limitation on the above equations (25), while they account for second-second rank duality <u>field</u> transformations of the form $F \rightarrow {}^{*}F$, is that they do not yet account for first-third rank <u>source</u> transformations, for example, of the form $J \rightarrow {}^{*}J$ or $P \rightarrow {}^{*}P$. In Section 5, we shall examine how (25) may be further extended to account for such first-third rank duality transformations, and how one may derive from this extension the general connections between second rank electromagnetic fields, and the first/third rank sources ~~of~~ these fields. The net result of this will be the full integration of the Levi-Cevita formalism into Maxwell's electrodynamics with sources.

For the moment however, let us return to the above equations (25), to show how these equations may in fact be consolidated into the single non-linear geometrodynamic equation $R_{\mu\nu} = 0$.

SECTION 4- Consolidation of Electrodynamics with Sources, into the Geometrodynamic Equation $R_{\mu\nu}=0$

To show how the electrodynamic equations (25) may be consolidated into the geometrodynamic equation $R_{\mu\nu}=0$, begin by rewriting the identities (22) in the following manner:

$$
\text{(A)}\begin{cases}
\text{(i)} & F^{\sigma\tau}F_{\sigma\nu;\tau} - {}^{*}F_{\nu\sigma}{}^{*}F^{\tau\sigma}{}_{;\tau} + \tfrac{1}{2}F^{\sigma\tau}F_{\tau\sigma;\nu} = 0 \\
\text{(ii)} & {}^{*}F^{\sigma\tau\,*}F_{\sigma\nu;\tau} - F_{\nu\sigma}F^{\tau\sigma}{}_{;\tau} + \tfrac{1}{2}{}^{*}F^{\sigma\tau\,*}F_{\tau\sigma;\nu} = 0
\end{cases}
\qquad (26)
$$

$$
\text{(B)}\begin{cases}
\text{(i)} & {}^{*}F^{\sigma\tau}F_{\sigma\nu;\tau} + F_{\nu\sigma}{}^{*}F^{\tau\sigma}{}_{;\tau} + \tfrac{1}{2}{}^{*}F^{\sigma\tau}F_{\tau\sigma;\nu} = 0 \\
\text{(ii)} & -F^{\sigma\tau\,*}F_{\sigma\nu;\tau} - {}^{*}F_{\nu\sigma}F^{\tau\sigma}{}_{;\tau} - \tfrac{1}{2}F^{\sigma\tau\,*}F_{\tau\sigma;\nu} = 0 .
\end{cases}
$$

Since each of (A), and each of (B), differs from the other by a simple duality rotation, it is appropriate to combine each of (A), and each of (B), into a single pair of equations, as was done above for eqs. (22). Once this has been done, using the identity $F^{\sigma\tau}F_{\tau\sigma} + {}^{*}F^{\sigma\tau\,*}F_{\tau\sigma} = 0$,-8 it is possible to derive from (26)(A) the following identity:

(A) $$\frac{1}{2}\left[F^{\sigma\tau}F_{\sigma\nu} + {}^*F^{\sigma\tau}{}^*F_{\sigma\nu}\right]_{;\tau} = F^{\sigma\tau}F_{\sigma\nu;\tau} + {}^*F^{\sigma\tau}{}^*F_{\sigma\nu;\tau} = F_{\sigma\nu}F^{\sigma\tau}{}_{;\tau} + {}^*F_{\sigma\nu}{}^*F^{\sigma\tau}{}_{;\tau} \,. \tag{27}$$

It is further possible, directly from (26)(B) above, to write down the identity:

(B) $$\frac{1}{2}\left[{}^*F^{\sigma\tau}F_{\tau\sigma;\nu} - F^{\sigma\tau}{}^*F_{\tau\sigma;\nu}\right] = \left[F^{\sigma\tau}{}^*F_{\sigma\nu} - {}^*F^{\sigma\tau}F_{\sigma\nu}\right]_{;\tau} \,. \tag{27}$$

Then using the identity (27)(A) to reduce (25)(A); and the identity (27)(B) to reduce (25)(B), (25)(A) may be reduced as follows: (contrast with the source-free (11))

(A) $$\begin{aligned} T^{\sigma}{}_{\alpha;\sigma} &= -\frac{1}{4\pi}\Big[\frac{1}{2}F^{\sigma\tau}(F_{\sigma\alpha;\tau} + F_{\alpha\tau;\sigma} + F_{\tau\sigma;\alpha}) - {}^*F_{\alpha\sigma}{}^*F^{\tau\sigma}{}_{;\tau} \\ &\quad + \frac{1}{2}{}^*F^{\sigma\tau}({}^*F_{\sigma\alpha;\tau} + {}^*F_{\alpha\tau;\sigma} + {}^*F_{\tau\sigma;\alpha}) - F_{\alpha\sigma}F^{\tau\sigma}{}_{;\tau}\Big] \\ &= -\frac{1}{4\pi}\Big[F^{\sigma\tau}F_{\alpha\tau;\sigma} - {}^*F_{\alpha\sigma}{}^*F^{\tau\sigma}{}_{;\tau} + \frac{1}{2}F^{\sigma\tau}F_{\tau\sigma;\alpha} \\ &\quad + {}^*F^{\sigma\tau}{}^*F_{\alpha\tau;\sigma} - F_{\alpha\sigma}F^{\tau\sigma}{}_{;\tau} + \frac{1}{2}{}^*F^{\sigma\tau}{}^*F_{\tau\sigma;\alpha}\Big] \\ &= -\frac{1}{4\pi}\Big[F^{\sigma\tau}F_{\alpha\tau;\sigma} + {}^*F^{\sigma\tau}{}^*F_{\alpha\tau;\sigma} \\ &\quad - F_{\alpha\tau}F^{\sigma\tau}{}_{;\sigma} - {}^*F_{\alpha\tau}{}^*F^{\sigma\tau}{}_{;\sigma}\Big] \\ &= -\frac{1}{4\pi}\Big[\frac{1}{2}\left[F^{\sigma\tau}F_{\alpha\tau} + {}^*F^{\sigma\tau}{}^*F_{\alpha\tau}\right] \\ &\quad - \frac{1}{2}\left[F^{\sigma\tau}F_{\alpha\tau} + {}^*F^{\sigma\tau}{}^*F_{\alpha\tau}\right]\Big]_{;\sigma} \\ &= [0]_{;\sigma} = 0 \;; \end{aligned} \tag{28}$$

and (25)(B) may be reduced as follows:

(B) $$\begin{aligned} T^{\sigma}{}_{\beta;\sigma} &= -\frac{1}{4\pi}\Big[\frac{1}{2}{}^*F^{\sigma\tau}(F_{\sigma\beta;\tau} + F_{\beta\tau;\sigma} + F_{\tau\sigma;\beta}) + F_{\beta\sigma}{}^*F^{\tau\sigma}{}_{;\tau} \\ &\quad - \frac{1}{2}F^{\sigma\tau}({}^*F_{\sigma\beta;\tau} + {}^*F_{\beta\tau;\sigma} + {}^*F_{\tau\sigma;\beta}) - {}^*F_{\beta\sigma}F^{\tau\sigma}{}_{;\tau}\Big] \\ &= -\frac{1}{4\pi}\Big[{}^*F^{\sigma\tau}F_{\beta\tau;\sigma} + F_{\beta\tau}{}^*F^{\sigma\tau}{}_{;\sigma} + \frac{1}{2}{}^*F^{\sigma\tau}F_{\tau\sigma;\beta} \\ &\quad - F^{\sigma\tau}{}^*F_{\beta\tau;\sigma} - {}^*F_{\beta\tau}F^{\sigma\tau}{}_{;\sigma} - \frac{1}{2}F^{\sigma\tau}{}^*F_{\tau\sigma;\beta}\Big] \\ &= -\frac{1}{4\pi}\Big[({}^*F^{\sigma\tau}F_{\beta\tau})_{;\sigma} - (F^{\sigma\tau}{}^*F_{\beta\tau})_{;\sigma} \\ &\quad + \frac{1}{2}{}^*F^{\sigma\tau}F_{\tau\sigma;\beta} - \frac{1}{2}F^{\sigma\tau}{}^*F_{\tau\sigma;\beta}\Big] \\ &= -\frac{1}{4\pi}\Big[\left[{}^*F^{\sigma\tau}F_{\beta\tau} - F^{\sigma\tau}{}^*F_{\beta\tau}\right] \\ &\quad - \left[{}^*F^{\sigma\tau}F_{\beta\tau} - F^{\sigma\tau}{}^*F_{\beta\tau}\right]\Big]_{;\sigma} \\ &= [0]_{;\sigma} = 0 \,. \end{aligned} \tag{28}$$

The net effect of all this, is that each of (28)(A) and (B) above may covariantly integrated, and then, via the Einstein equation $-\varkappa T^{\mu}{}_{\nu} = R^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}{}_{\nu} R$, combined into the single non-linear geometrodynamic equation:

$$R_{\mu\nu} = 0. \tag{29}$$

Thus, by basing energy conservation in electrodynamics with sources directly upon the the connection of $T^{\mu}{}_{\nu;\mu}$ with the identities (22) appropriately combined, as shown in (25), in the very same manner that energy conservation in geometrodynamics is based upon connection of $T^{\mu}{}_{\nu;\mu}$ with the Bianchi identity $(R^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}{}_{\nu} R)_{;\mu} = 0$, one is assured, even without the supplemenatary condition $F^{\mu\nu} = A^{\nu;\mu} - A^{\mu;\nu}$, that energy is identically conserved; and as an added feature, that the electrodynamic energy tensor itself is identically equal to zero, ie., that $R_{\mu\nu} = 0$.

Thus, all at once, one can assure energy conservation in the theory of electrodynamics with sources, without depending on the condition $F^{\mu\nu} = A^{\nu;\mu} - A^{\mu;\nu}$ involving the four-potential; one can restore the dualistic invariance of the energy tensor $T^{\mu\nu}$ under second rank duality transformations $F^{\mu\nu} \rightarrow {}^*F^{\mu\nu}$ of the electromagnetic field; one can thereby deduce the appropriate symmetries between charges and monopoles, and between first and third rank electromagnetic field sources; one can restore the transposition symmetry of $T^{\mu\nu}$; and as an added bonus, one finds that this symmetric energy tensor $T^{\mu\nu}$ may be described by the very simple geometrodynamic equation $R_{\mu\nu} = 0$. Aside from the fact that it still remains to consider first-third rank duality source transformations a bit more thoroughly, this solves directly all of the difficulties noted in section 2, for the transition from source free electrodynamics to electrodynamics with sources.

The consolidation of electrodynamics with sources into the single relativistic equation $R_{\mu\nu} = 0$, thereby yields a significant non-linear simplification of electrodynamics.

SECTION 5- Integration of Levi-Cevita Duality Source Transformations into Electrodynamics with Sources; and Consequent Derivation of the Connection between Fields and Sources

At this point, while the electrodynamic equations: (see (25))

$$\text{(A)}\quad \tfrac{1}{2}F^{\mu\tau}\left(F_{\mu\alpha;\tau}+F_{\alpha\tau;\mu}+F_{\tau\mu;\alpha}\right)-{}^{*}F_{\alpha\tau}{}^{*}F^{\sigma\tau}{}_{;\sigma} + \tfrac{1}{2}{}^{*}F^{\mu\tau}\left({}^{*}F_{\mu\alpha;\tau}+{}^{*}F_{\alpha\tau;\mu}+{}^{*}F_{\tau\mu;\alpha}\right)-F_{\alpha\tau}F^{\sigma\tau}{}_{;\sigma}=0$$

$$\text{(B)}\quad \tfrac{1}{2}{}^{*}F^{\mu\tau}\left(F_{\mu\beta;\tau}+F_{\beta\tau;\mu}+F_{\tau\mu;\beta}\right)+F_{\beta\tau}{}^{*}F^{\sigma\tau}{}_{;\sigma} - \tfrac{1}{2}F^{\mu\tau}\left({}^{*}F_{\mu\beta;\tau}+{}^{*}F_{\beta\tau;\mu}+{}^{*}F_{\tau\mu;\beta}\right)-{}^{*}F_{\beta\tau}F^{\sigma\tau}{}_{;\sigma}=0 \qquad (30)$$

imply the geometrodynamic equation:

$$R_{\mu\nu}=0, \qquad (31)$$

as shown above, the above equations (30) are not yet the most general electrodynamical equations which in fact imply $R_{\mu\nu}=0$. In other words, if one reverses the logic used to this point, and now takes $R_{\mu\nu}=0$ as the starting point for describing electrodynamics with sources, then the equation $R_{\mu\nu}=0$ itself implies a set of electrodynamic equations which is actually broader than that shown in (30). This is particularly because, as was noted earlier, while (30) indeed accounts for all second rank duality __field__ transformations of the form $F \rightarrow {}^*F$, it does not in any way account for first or third rank duality __source__ transformations of the forms $J \rightarrow {}^*J$ and $P \rightarrow {}^*P$. In other words, if one considers the more general set of duality field and source transformations:

$$
\begin{aligned}
&(a) \qquad {}^*B_\tau = \frac{1}{3!}\epsilon_{\sigma\gamma\lambda\tau}B^{\sigma\gamma\lambda} \\
&(b) \qquad {}^*A_{\nu\tau} = \frac{1}{2!}\epsilon_{\delta\lambda\nu\tau}A^{\delta\lambda} \qquad (32) \\
&(c) \qquad {}^*B_{\tau\mu\nu} = \frac{1}{1!}\epsilon_{\sigma\tau\mu\nu}B^{\sigma}
\end{aligned}
$$

associated with the Levi-Civita formalism,[9] then while (30) above does account fully for duality transformations of the form (32)(b), it does not account in any way for duality transformations of the form shown in either of (32)(a) or (c). Further, while eqs. (30) do describe electrodynamics with sources in terms of the field tensor $F^{\mu\nu}$, there has not yet been any connection drawn between fields and sources; ie., there has not yet been deduced a general way to describe electromagnetic sources in terms of electromagnetic fields.

To rectify these difficulties, consider the above transformations (32), along with the various relationships: (A is an arbitrary antisymmetric tensor)

$$
(a) \qquad
\begin{aligned}
&-\epsilon^{\alpha\beta\sigma\lambda}\epsilon_{\mu\nu\tau\nabla} = 0!\,\delta^{\alpha\beta\sigma\lambda}{}_{\mu\nu\tau\nabla} \\
&-\epsilon^{\alpha\beta\sigma\delta}\epsilon_{\mu\nu\tau\delta} = 0!\,\delta^{\alpha\beta\sigma\delta}{}_{\mu\nu\tau\delta} = 1!\,\delta^{\alpha\beta\sigma}{}_{\mu\nu\tau} \\
&-\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\mu\nu\gamma\delta} = 0!\,\delta^{\alpha\beta\gamma\delta}{}_{\mu\nu\gamma\delta} = 1!\,\delta^{\alpha\beta\gamma}{}_{\mu\nu\gamma} = 2!\,\delta^{\alpha\beta}{}_{\mu\nu} \\
&-\epsilon^{\alpha\tau\gamma\delta}\epsilon_{\mu\tau\gamma\delta} = 0!\,\delta^{\alpha\tau\gamma\delta}{}_{\mu\tau\gamma\delta} = 1!\,\delta^{\alpha\tau\gamma}{}_{\mu\tau\gamma} = 2!\,\delta^{\alpha\tau}{}_{\mu\tau} = 3!\,\delta^{\alpha}{}_{\mu} \\
&-\epsilon^{\sigma\tau\gamma\delta}\epsilon_{\sigma\tau\gamma\delta} = 0!\,\delta^{\sigma\tau\gamma\delta}{}_{\sigma\tau\gamma\delta} = 1!\,\delta^{\sigma\tau\gamma}{}_{\sigma\tau\gamma} = 2!\,\delta^{\sigma\tau}{}_{\sigma\tau} = 3!\,\delta^{\sigma}{}_{\sigma} = 4!\,\delta ,
\end{aligned}
$$

(33)

$$\frac{1}{4!}\,\delta^{\alpha\beta\sigma\lambda}{}_{\mu\nu\tau\nabla}\,A_{\alpha\beta\sigma\lambda} = A_{\mu\nu\tau\Delta}$$

(b)
$$\frac{1}{3!}\,\delta^{\alpha\beta\sigma}{}_{\mu\nu\tau}\,A_{\alpha\beta\sigma} = A_{\mu\nu\tau}$$
$$\frac{1}{2!}\,\delta^{\alpha\beta}{}_{\mu\nu}\,A_{\alpha\beta} = A_{\mu\nu}$$
$$\frac{1}{1!}\,\delta^{\alpha}{}_{\mu}\,A_{\alpha} = A_{\mu}$$
$$\frac{1}{0!}\,\delta A = A$$

and hence

(c)
$$\delta = 1\,,$$

of the Levi-Cevita formalism.-10 From (32) and (33) above, it is possible to deduce the generalized identity:

$$A_{\nu\tau}B^{\tau} + \tfrac{1}{2}\,{}^*A^{\mu\tau}\,{}^*B_{\tau\mu\nu} = 0\,, \tag{34}$$

which applies to any arbitrary antisymmetric first, second and third rank tensors, A and B, in spacetime.

From this identity, if one substitutes **F** for **A** and **J** for **B** in (34), and then subjects (34) to all possible duality transformations of both the second-second rank form $F \rightarrow {}^*F$, and the first-third rank form $J \rightarrow {}^*J$, then it is possible [redacted] to deduce From (34) the following set of identities:

(A)

(a)
$$\text{(i)}\quad F_{\alpha\tau}J^{\tau} + \tfrac{1}{2}\,{}^*F^{\mu\tau}\,{}^*J_{\tau\mu\alpha} = 0$$
$$\text{(ii)}\quad F_{\alpha\tau}{}^*J^{\tau} + \tfrac{1}{2}\,{}^*F^{\mu\tau}J_{\tau\mu\alpha} = 0$$

(b)
$$\text{(i)}\quad {}^*F_{\alpha\tau}J^{*\tau} - \tfrac{1}{2}\,F^{\mu\tau}\,{}^*J^{*}{}_{\tau\mu\alpha} = 0$$
$$\text{(ii)}\quad {}^*F_{\alpha\tau}{}^*J^{*\tau} - \tfrac{1}{2}\,F^{\mu\tau}J^{*}{}_{\tau\mu\alpha} = 0$$

(35)

$$(B)\begin{cases}(a)\begin{cases}(i)\ \ {}^*F_{\beta\tau}J^{\tau} - \frac{1}{2}F^{\mu\tau *}J_{\tau\mu\beta} = 0\\(ii)\ \ {}^*F_{\beta\tau}{}^*J^{\tau} - \frac{1}{2}F^{\mu\tau}J_{\tau\mu\beta} = 0\end{cases}\\(b)\begin{cases}(i)\ \ -F_{\beta\tau}J^{*\tau} - \frac{1}{2}{}^*F^{\mu\tau *}J^{*}{}_{\tau\mu\beta} = 0\\(ii)\ \ -F_{\beta\tau}{}^*J^{*\tau} - \frac{1}{2}{}^*F^{\mu\tau}J^{*}{}_{\tau\mu\beta} = 0\,.\end{cases}\end{cases}$$

Note in the above, because $J^{\tau} = \frac{1}{4\pi}F^{\sigma\tau}{}_{;\sigma}$ (half of Maxwell's classical equations – see (B)(a)), that we may also define $J^{*\tau} \equiv \frac{1}{4\pi}{}^*F^{\sigma\tau}{}_{;\sigma}$, such that the transformation $F \rightarrow {}^*F$ implies a simultaneous transformation of the form $J \rightarrow J^*$. It is very important to realize however, that J^* is not the same as ${}^*J$, since ${}^*F^{\sigma\tau}{}_{;\sigma}$ is not the same as ${}^*(F^{\sigma\tau}{}_{;\sigma})$. (${}^*F^{\sigma\tau}{}_{;\sigma}$ actually is the same as $({}^*F^{\sigma\tau})_{;\sigma}$, though the parenthesis in this case are excluded for notational convenience.) Therefore, for instance, note that ${}^{**}J = J$, but that $J^{**} = -J$.

Now while (35) above may appear at first sight to present 32 independent identities, one should realize on closer examination that there are really only eight independent equations in (35). This is because, within each of (A) and (B), the subset (a) differs from the subset (b) by the second-second rank duality field transformation $F \rightarrow {}^*F$ (hence $J \rightarrow J^*$); while each equation within the subset (a), and within the subset (b), differs from the other equation in the subset merely by the first/third rank duality source transformation $J \rightarrow {}^*J$. The equations in the major sets (A) and (B) however, cannot be directly derived from one another by any form of duality transformation. Hence these equations are truly independent of one another, and one therefore finds a total of eight independent equations embodied in (35) as a whole. This is encouraging, since we know again, that eight is precisely the number of independent equations required for a complete electrodynamic theory. Further, since (30) also consists of eight independent equations, but since (30) is written solely in terms of fields while (35) includes sources, one would expect some sort of connection to exist between (30) and (35); and from this connection, that it should be possible to derive the more general relationships between electromagnetic fields and sources.

Consequently, following the path pursued earlier when only second-second rank duality field transformations were considered, one may combine all four of (35)(A) together, and all four of (35)(B) together, to form two primary equations, each with four independent components, in the following manner:

$$(A)\quad F_{\alpha\tau}(J^{\tau} + {}^{*}J^{\tau}) + {}^{*}F_{\alpha\tau}(J^{*\tau} + {}^{*}J^{*\tau}) + \tfrac{1}{2}{}^{*}F^{\mu\tau}(J_{\tau\mu\alpha} + {}^{*}J_{\tau\mu\alpha}) - \tfrac{1}{2}F^{\mu\tau}(J^{*}_{\tau\mu\alpha} + {}^{*}J^{*}_{\tau\mu\alpha}) = 0$$

(36)

$$(B)\quad {}^{*}F_{\beta\tau}(J^{\tau} + {}^{*}J^{\tau}) - F_{\beta\tau}(J^{*\tau} + {}^{*}J^{*\tau}) - \tfrac{1}{2}F^{\mu\tau}(J_{\tau\mu\beta} + {}^{*}J_{\tau\mu\beta}) - \tfrac{1}{2}{}^{*}F^{\mu\tau}(J^{*}_{\tau\mu\beta} + {}^{*}J^{*}_{\tau\mu\beta}) = 0 .$$

Once again, as a natural consequence of this procedure, the equations (36) themselves will be fully dualistically invariant, this time, under all of first, second and third rank duality transformations, of both fields and sources. These are also the equations which, through the conservation of energy, one would expect to ultimately link with the geometrodynamic conservation equation $T^{\mu}{}_{\nu;\mu} = 0$, as is normally done with the Bianchi identity $(R^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}{}_{\nu}R)_{;\mu} = 0$.

If we now go back to (30) and recognize that this equation does not yet account for first-third rank source transformations of the form $J \rightarrow {}^*J$, and from (36), that such an accounting ultimately requires (30) to be rewritten such that all terms of the form $J^\tau = \frac{1}{4\pi} F^{\sigma\tau}{}_{;\sigma}$ are replaced by terms of the form $(J^\tau + {}^*J^\tau) = \frac{1}{4\pi}(F^{\sigma\tau}{}_{;\sigma} + {}^*(F^{\sigma\tau}{}_{;\sigma}))$, then one is led to rewrite (30) in the extended form:

$$\text{(A)}\quad F_{\alpha\tau}(F^{\sigma\tau}{}_{;\sigma} + {}^*(F^{\sigma\tau}{}_{;\sigma})) + {}^*F_{\alpha\tau}({}^*F^{\sigma\tau}{}_{;\sigma} + {}^*({}^*F^{\sigma\tau}{}_{;\sigma}))$$
$$-\tfrac{1}{2}{}^*F^{\mu\tau}[({}^*F_{\mu\alpha;\tau} + {}^*F_{\alpha\tau;\mu} + {}^*F_{\tau\mu;\alpha}) + {}^*({}^*F_{\mu\alpha;\tau} + {}^*F_{\alpha\tau;\mu} + {}^*F_{\tau\mu;\alpha})]$$
$$-\tfrac{1}{2}F^{\mu\tau}[(F_{\mu\alpha;\tau} + F_{\alpha\tau;\mu} + F_{\tau\mu;\alpha}) + {}^*(F_{\mu\alpha;\tau} + F_{\alpha\tau;\mu} + F_{\tau\mu;\alpha})] = 0$$

(37)

$$\text{(B)}\quad {}^*F_{\beta\tau}(F^{\sigma\tau}{}_{;\sigma} + {}^*(F^{\sigma\tau}{}_{;\sigma})) - F_{\beta\tau}({}^*F^{\sigma\tau}{}_{;\sigma} + {}^*({}^*F^{\sigma\tau}{}_{;\sigma}))$$
$$+\tfrac{1}{2}F^{\mu\tau}[({}^*F_{\mu\beta;\tau} + {}^*F_{\beta\tau;\mu} + {}^*F_{\tau\mu;\beta}) + {}^*({}^*F_{\mu\beta;\tau} + {}^*F_{\beta\tau;\mu} + {}^*F_{\tau\mu;\beta})]$$
$$-\tfrac{1}{2}{}^*F^{\mu\tau}[(F_{\mu\beta;\tau} + F_{\beta\tau;\mu} + F_{\tau\mu;\beta}) + {}^*(F_{\mu\beta;\tau} + F_{\beta\tau;\mu} + F_{\tau\mu;\beta})] = 0.$$

Since eq. (37) is an identity based strictly on the second-second rank duality transformation properties of $F^{\mu\nu}$; and since (36) is an independently derived identity based on both the second-second rank duality field transformation properties of $F^{\mu\nu}$ and the first-third rank duality source transformation properties of J^τ , $J_{\tau\mu\alpha}$; it should now be possible, given the appropriate initial connection between fields and sources (such as Maxwell's classical "charge" equation $J^\tau = \frac{1}{4\pi}F^{\sigma\tau}{}_{;\sigma}$,) to derive a full set of connections between fields and sources generally, from a direct comparison of (36) with (37).

Particularly, if one considers the Maxwell equation $J^\tau = \frac{1}{4\pi}F^{\sigma\tau}{}_{;\sigma}$, and if one defines the monopole four-vector according to $P^\tau \equiv J^{*\tau} = \frac{1}{4\pi}{}^*F^{\sigma\tau}{}_{;\sigma}$, which means that monopoles are formed by a

90 degree counterclockwise second rank duality rotation of the fields from which one constructs charges, then the direct comparison of (37) with (36) allows one to derive the full set of connections between the first and third rank sources **J** and **P** of the electromagnetic field, and the second rank electromagnetic field **F** itself, as such:

(a)

(i) $$J^{\tau} = -P^{*\tau} = \frac{1}{4\pi} F^{\sigma\tau}{}_{;\sigma}$$

(ii) $$J_{\tau\mu\nu} = -P^{*}_{\tau\mu\nu} = -\frac{1}{4\pi} {}^{*}\left({}^{*}F_{\tau\mu;\nu} + {}^{*}F_{\mu\nu;\tau} + {}^{*}F_{\nu\tau;\mu}\right)$$

(b)

(i) $$P^{\tau} = J^{*\tau} = \frac{1}{4\pi} {}^{*}F^{\sigma\tau}{}_{;\sigma}$$

(ii) $$P_{\tau\mu\nu} = J^{*}_{\tau\mu\nu} = \frac{1}{4\pi} {}^{*}\left(F_{\tau\mu;\nu} + F_{\mu\nu;\tau} + F_{\nu\tau;\mu}\right)$$

(c) (38)

(i) $${}^{*}J^{\tau} = -{}^{*}P^{*\tau} = \frac{1}{4\pi} {}^{*}\left(F^{\sigma\tau}{}_{;\sigma}\right)$$

(ii) $${}^{*}J_{\tau\mu\nu} = -{}^{*}P^{*}_{\tau\mu\nu} = -\frac{1}{4\pi} \left({}^{*}F_{\tau\mu;\nu} + {}^{*}F_{\mu\nu;\tau} + {}^{*}F_{\nu\tau;\mu}\right)$$

(d)

(i) $${}^{*}P^{\tau} = {}^{*}J^{*\tau} = \frac{1}{4\pi} {}^{*}\left({}^{*}F^{\sigma\tau}{}_{;\sigma}\right)$$

(ii) $${}^{*}P_{\tau\mu\nu} = {}^{*}J^{*}_{\tau\mu\nu} = \frac{1}{4\pi} \left(F_{\tau\mu;\nu} + F_{\mu\nu;\tau} + F_{\nu\tau;\mu}\right).$$

From here, one may work backwards using (38), to rewrite (36) as such:

(A)

$$F_{a\tau}\left(J^{\tau} + {}^{*}J^{\tau}\right) + {}^{*}F_{a\tau}\left(P^{\tau} + {}^{*}P^{\tau}\right)$$
$$+ \frac{1}{2} {}^{*}F^{\mu\tau}\left(J_{\tau\mu a} + {}^{*}J_{\tau\mu a}\right) - \frac{1}{2} F^{\mu\tau}\left(P_{\tau\mu a} + {}^{*}P_{\tau\mu a}\right) = 0$$

(39)

(B)

$$*F_{\beta\tau}\left(J^{\tau} + *J^{\tau}\right) - F_{\beta\tau}\left(P^{\tau} + *P^{\tau}\right)$$
$$-\tfrac{1}{2}F^{\mu\tau}\left(J_{\tau\mu\beta} + *J_{\tau\mu\beta}\right) - \tfrac{1}{2}*F^{\mu\tau}\left(P_{\tau\mu\beta} + *P_{\tau\mu\beta}\right) = 0 .$$

As the above accounts fully for both second-second rank and also first-third rank duality transformations, it is the above that is in fact the most general set of electrodynamic equations consistent with $R_{\mu\nu} = 0$. That the above equations (39) and (38) really do imply $R_{\mu\nu} = 0$ can be shown through the same line of reasoning developed in section 4, wherein all terms of the form J are simply replaced by terms of the form $J + *J$.

Therefore, as shown earlier one may proceed to consolidate both of the above electrodynamic equations (39), into the simple non-linear geometrodynamic equation:

$$R_{\mu\nu} = 0 . \qquad (40)$$

The respective equations (40), (39) and (38) above, are simply the equations (1), (2) and (3) which we posed at the outset of the introductory discussion, as the primary equations of already unified Field Theory [redacted] with sources.

SECTION 6- On the Strength and Compatability of Pure Gravitational Geometrodynamics and Electrodynamics with Sources

In the introduction it was indicated briefly, when eq. (40) ie., (1) is considered in light of geometrodynamic energy conservation, and eq. (39) ie., (2) is considered in light of electrodynamic source conservation, that these equations determine their respective $g_{\mu\nu}$ and $F^{\mu\nu}$ fields with precisely equal strength, ie., that these equations (1) and (2) are absolutely compatable.

While the mathematical derivation of the equivalence of these equations is now complete, it is important to show from a more general standpoint, why these two equations (1) and (2) truly can be regarded as formal equivalents of one another. To do this, we shall digress here to compute the "strength" of each of these equations, following the identical sort of calculation used by Einstein in his final published paper, Relativistic Theory of the Non Symmetric Field,-II which, in part, demonstrated the compatability of $R_{\mu\nu}=0$ with the source free Maxwell equations $F^{\sigma\tau}{}_{;\sigma}=0$; $F_{\mu\nu;\tau}+F_{\nu\tau;\mu}+F_{\tau\mu;\nu}=0$. We do this because, regardless of any derivations given to this point, one cannot really regard (1) and (2) as equivalent equations until it has been shown, with all factors considered, that these equations really do determine their fields with exactly the same strength.

We begin by examining the geometrodynamic equation (1) for the pure gravitational field:

$$R_{\mu\nu} = 0. \tag{41}$$

In a general relativistic theory, one starts the computation of strength with the symmetric metric tensor $g_{\mu\nu}$, which furnishes ten independent components of the gravitational potential, of zero order in the metric, and hence $10\cdot\binom{4}{n}$ independent coefficients of nth order, where $\binom{4}{n} = \frac{(n+3)!}{n!\,3!}$. The Christoffel objects $\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\tau}(g_{\tau\mu,\nu} + g_{\nu\tau,\mu} - g_{\mu\nu,\tau})$ furnish forty more independent components of first metric order, and hence $40\cdot\binom{4}{n-1}$ independent nth order coefficients. This is to be reduced however by the $4\cdot\binom{4}{n+1}$ free nth order coefficients associated with the permitted transformations $g_{\mu\nu}' = \frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial x^{\beta}}{\partial x'^{\nu}} g_{\alpha\beta}$ of the four coordinates, in a generally covariant theory.

At this point, the dynamical equation $R_{\mu\nu} = 0$, which contains ten independent component equations of second metric order, reduces the number of free nth order coefficients by $10\cdot\binom{4}{n-2}$; while the equation $g_{\mu\nu;\sigma} = 0$, which contains forty independent components of first metric order, reduces the number of free nth order coefficients by $40\cdot\binom{4}{n-1}$, thereby cancelling precisely the freedom of the $\Gamma^{\sigma}_{\mu\nu}$.

Finally, the Bianchi identity/energy conservation equation:

$$-\kappa T^{\mu}{}_{\nu;\mu} = (R^{\mu}{}_{\nu} - \tfrac{1}{2}\delta^{\mu}{}_{\nu} R)_{;\mu} = 0, \tag{42}$$

which contains four independent conditions of third metric order, is already included within $R_{\mu\nu} = 0$. Therefore, the $4\cdot\binom{4}{n-3}$ free nth order coefficients associated with this identity are to be subtracted from those associated with $R_{\mu\nu}=0$.

The net result is a total of

$$Z = \left[10\cdot\binom{4}{n} + 40\cdot\binom{4}{n-1} - 4\cdot\binom{4}{n+1} \right] - \left[10\cdot\binom{4}{n-2} + 40\cdot\binom{4}{n-1} - 4\cdot\binom{4}{n-3} \right] \quad (43)$$

free coefficients associated with $R_{\mu\nu}=0$ [redacted] in a generally relativistic theory. Setting $Z \sim \binom{4}{n}\left[F + \frac{S}{n}\right]$, where F is the number of free functions of the spacetime coordinates and S is the strength of any given system of equations, one finds asymptotically, for large n, that:

$$\begin{aligned} Z \sim &\binom{4}{n}\left[10 + 40\cdot\left(1-\tfrac{3}{n}\right) - 4\cdot\left(1+\tfrac{3}{n}\right)\right] \\ &- \binom{4}{n}\left[10\left(1-\tfrac{6}{n}\right) + 40\cdot\left(1-\tfrac{3}{n}\right) - 4\left(1-\tfrac{9}{n}\right)\right] \\ &= \binom{4}{n}\left[0 + \tfrac{12}{n}\right]. \end{aligned} \quad (44)$$

F must of course be equal to zero in any "complete" system of equations.
Thus [redacted], the strength of $R_{\mu\nu}=0$, in a general relativistic framework, is:

$$S = 12. \quad (45)$$

We turn now to the electrodynamic equations (39), originally (2), which are reproduced here for convenience:

$$\text{(A)} \quad \begin{aligned} &F_{\alpha\tau}\left(J^{\tau} + {}^{*}J^{\tau}\right) + {}^{*}F_{\alpha\tau}\left(P^{\tau} + {}^{*}P^{\tau}\right) \\ &+ \tfrac{1}{2}{}^{*}F^{\mu\tau}\left(J_{\tau\mu\alpha} + {}^{*}J_{\tau\mu\alpha}\right) - \tfrac{1}{2}F^{\mu\tau}\left(P_{\tau\mu\alpha} + {}^{*}P_{\tau\mu\alpha}\right) = 0 \end{aligned}$$

(46)

(B)
$$ {}^*F_{\beta\tau}(J^{\tau} + {}^*J^{\tau}) - F_{\beta\tau}(P^{\tau} + {}^*P^{\tau}) $$
$$ - \tfrac{1}{2} F^{\mu\tau}(J_{\tau\mu\beta} + {}^*J_{\tau\mu\beta}) - \tfrac{1}{2} {}^*F^{\mu\tau}(P_{\tau\mu\beta} + {}^*P_{\tau\mu\beta}) = 0 . $$

While the above certainly do not look at all like $R_{\mu\nu} = 0$, it turns out that the above are of precisely the same strength as $R_{\mu\nu} = 0$, as will be demonstrated below.

Particularly, the antisymmetric field tensor $F^{\mu\nu}$ furnishes six independent components of zero order in the electromagnetic field, which, in cartesian coordinates, are simply the x,y,z components of each of the electric and magnetic fields. Thus, one begins with $6\cdot\binom{4}{n}$ free nth order coefficients. The eight equations (46), which, because of (38), (original (3)), are of first field order, reduce by $8\cdot\binom{4}{n-1}$ the free coefficients of the field itself. This equation however, already contains the two scalar conservation equations: (contrast with (9))

(a)
$$ J^{\sigma}{}_{;\sigma} = 0 $$

(47)

(b)
$$ {}^*P_{\alpha\mu\nu;\beta} - {}^*P_{\mu\nu\beta;\alpha} + {}^*P_{\nu\beta\alpha;\mu} - {}^*P_{\beta\alpha\mu;\nu} = 0 $$

for first and third rank sources, (note that there are a number of alternatives to (47) differing from the above by a mere duality rotation of either first-third or second-second rank, and which need not therefore be added to the above for computing strength) and [redacted] these are of second field order. Hence, the nth order free coefficients associated with (46) are further reduced, by the amount $2\cdot\binom{4}{n-2}$.

All tolled, one therefore arrives at:

$$Z = 6\binom{4}{n} - \left[8\binom{4}{n-1} - 2\binom{4}{n-2}\right] \qquad (48)$$

free nth order coefficients associated with (46), which may be reduced asymptotically, for large n, to:

$$\begin{aligned} Z &\sim \binom{4}{n}\left[6 - 8\left(1-\frac{3}{n}\right) + 2\left(1-\frac{6}{n}\right)\right] \\ &= \binom{4}{n}\left[0 + \frac{12}{n}\right]. \end{aligned} \qquad (49)$$

Hence, for the system of equations that includes (46), the strength is also given by:

$$S = 12. \qquad (50)$$

Consequently, when the geometrodynamic equation (41) is considered in light of geometrodynamic energy conservation as given in (42); and when the electrodynamic equation (46) is considered in light of electrodynamic source conservation as given in (47); it is found that these two equations, (41) and (46), hence (40) and (39), hence (1) and (2), are of exactly the same strength. In other words, from a general relativistic standpoint, these equations are absolutely compatable. This points up the true significance of Einstein's "surprising" finding, in <u>Relativistic Theory of the Non Symmetric Field</u>, "...that the gravitational equations for empty space determine their field just as strongly as do Maxwell's [source-free] equations in the case of the electromagnetic field." -12

The above computation of "strengths," in addition to the derivations given in sections 1-5, furnishes convincing evidence that equations (1) and (2) really can be regarded as the same dynamical equation, and it allows one to conclude with confidence that the entirety of electrodynamics with sources can indeed be consolidated into the single non-linear geometrodynamic equation $R_{\mu\nu}=0$. As a result, $R_{\mu\nu}=0$ becomes the primary dynamical equation of already unified field theory including electromagnetic sources; and it provides the global starting point for all subsequent discussions of "geometrodynamic electrodynamics."

This concludes the derivation of electrodynamics and already unified field theory with sources. Attention will now be turned, in the next three sections, to examining various specializations of this theory. In section 7, we examine the electrodynamical version (2) of $R_{\mu\nu}=0$, transformed into the extremal electric frame of reference via eq. (4). In section 8, we examine the source-free reduction of (2), via the potential condition $F^{\mu\nu}=A^{\nu;\mu}-A^{\mu;\nu}$ of eq. (5). In section 9, we examine some other special cases of interest, as given by eqs. (6). Section 10 is devoted to a brief discussion of topology, (eqs. (7)) and of possible topological approaches toward the quantization of this classical already unified field theory.

SECTION 7- Reduction of Electrodynamics with Sources to the Extremal Electric Frame of Reference

To arrive at the extremal electric formulation of the electrodynamic equation (2) with sources, one substitutes into (2), via (3), the usual transformation equation:

$$F^{\mu\nu} = e^{*a}\mathcal{E}^{\mu\nu} \tag{51}$$

given in eq. (4).

Following this prescription, eq. (3) may be reduced to:

(a)

(i) $$J^{\sigma} = e^{*a}\left[\mathcal{J}^{\sigma} + \frac{1}{4\pi}{}^{*}\mathcal{E}^{\tau\sigma}a_{\tau}\right]$$

(ii) $$J_{\tau\sigma\nu} = e^{*a}\left[\mathcal{J}_{\tau\sigma\nu} + \frac{1}{4\pi}{}^{*}\left(\mathcal{E}_{\tau\sigma}a_{\nu} + \mathcal{E}_{\sigma\nu}a_{\tau} + \mathcal{E}_{\nu\tau}a_{\sigma}\right)\right]$$

(b)

(i) $$P^{\sigma} = e^{*a}\left[\mathcal{P}^{\sigma} - \frac{1}{4\pi}\mathcal{E}^{\tau\sigma}a_{\tau}\right]$$

(ii) $$P_{\tau\sigma\nu} = e^{*a}\left[\mathcal{P}_{\tau\sigma\nu} + \frac{1}{4\pi}{}^{*}\left({}^{*}\mathcal{E}_{\tau\sigma}a_{\nu} + {}^{*}\mathcal{E}_{\sigma\nu}a_{\tau} + {}^{*}\mathcal{E}_{\nu\tau}a_{\sigma}\right)\right]$$

(52)

(c)

(i) $${}^{*}J^{\sigma} = e^{*a}\left[{}^{*}\mathcal{J}^{\sigma} + \frac{1}{4\pi}{}^{*}\left({}^{*}\mathcal{E}^{\tau\sigma}a_{\tau}\right)\right]$$

(ii) $${}^{*}J_{\tau\sigma\nu} = e^{*a}\left[{}^{*}\mathcal{J}_{\tau\sigma\nu} + \frac{1}{4\pi}\left(\mathcal{E}_{\tau\sigma}a_{\nu} + \mathcal{E}_{\sigma\nu}a_{\tau} + \mathcal{E}_{\nu\tau}a_{\sigma}\right)\right]$$

(d)

(i) $$^*p^{\sigma} = e^{*a}\left[{}^*\mathcal{P}^{\sigma} - \frac{1}{4\pi}{}^*(\varepsilon^{\tau\sigma} a_\tau)\right]$$

(ii) $$^*p_{\tau\sigma\nu} = e^{*a}\left[{}^*p_{\tau\sigma\nu} + \frac{1}{4\pi}\left({}^*\varepsilon_{\tau\sigma} a_\nu + {}^*\varepsilon_{\sigma\nu} a_\tau + {}^*\varepsilon_{\nu\tau} a_\sigma\right)\right],$$

where:

(a)

(i) $$\mathcal{J}^{\sigma} = -\mathcal{P}^{*\sigma} = \frac{1}{4\pi}\varepsilon^{\tau\sigma}{}_{;\tau}$$

(ii) $$\mathcal{J}_{\tau\sigma\nu} = -\mathcal{P}^{*}_{\tau\sigma\nu} = -\frac{1}{4\pi}{}^*\left({}^*\varepsilon_{\tau\sigma;\nu} + {}^*\varepsilon_{\sigma\nu;\tau} + {}^*\varepsilon_{\nu\tau;\sigma}\right)$$

(b)

(i) $$\mathcal{P}^{\sigma} = \mathcal{J}^{*\sigma} = \frac{1}{4\pi}{}^*\varepsilon^{\tau\sigma}{}_{;\tau}$$

(ii) $$\mathcal{P}_{\tau\sigma\nu} = \mathcal{J}^{*}_{\tau\sigma\nu} = \frac{1}{4\pi}{}^*\left(\varepsilon_{\tau\sigma;\nu} + \varepsilon_{\sigma\nu;\tau} + \varepsilon_{\nu\tau;\sigma}\right)$$

(53)

(c)

(i) $$^*\mathcal{J}^{\sigma} = -{}^*\mathcal{P}^{*\sigma} = \frac{1}{4\pi}{}^*\left(\varepsilon^{\tau\sigma}{}_{;\tau}\right)$$

(ii) $$^*\mathcal{J}_{\tau\sigma\nu} = -{}^*\mathcal{P}^{*}_{\tau\sigma\nu} = -\frac{1}{4\pi}\left({}^*\varepsilon_{\tau\sigma;\nu} + {}^*\varepsilon_{\sigma\nu;\tau} + {}^*\varepsilon_{\nu\tau;\sigma}\right)$$

(d)

(i) $$^*\mathcal{P}^{\sigma} = {}^*\mathcal{J}^{*\sigma} = \frac{1}{4\pi}{}^*\left({}^*\varepsilon^{\tau\sigma}{}_{;\tau}\right)$$

(ii) $$^*\mathcal{P}_{\tau\sigma\nu} = {}^*\mathcal{J}^{*}_{\tau\sigma\nu} = \frac{1}{4\pi}\left(\varepsilon_{\tau\sigma;\nu} + \varepsilon_{\sigma\nu;\tau} + \varepsilon_{\nu\tau;\sigma}\right).$$

Then, substituting (52) above back into eq. (2), and recognizing that the first rank duelity operator * may be distributed according to $^*({}^*\varepsilon^{\tau\sigma} a_\tau) = {}^*\varepsilon^{\tau\sigma}\,{}^*a_\tau$; $^*(\varepsilon^{\tau\sigma} a_\tau) = \varepsilon^{\sigma\tau}\,{}^*a_\tau$; and $^*(\varepsilon_{\tau\sigma} a_\nu + \varepsilon_{\sigma\nu} a_\tau + \varepsilon_{\nu\tau} a_\sigma) = (\varepsilon_{\tau\sigma}{}^*a_\nu + \varepsilon_{\sigma\nu}{}^*a_\tau + \varepsilon_{\nu\tau}{}^*a_\sigma)$;

$*({}^*\epsilon_{\tau\sigma}a_\nu + {}^*\epsilon_{\sigma\nu}a_\tau + {}^*\epsilon_{\sigma\tau}a_\nu) = ({}^*\epsilon_{\tau\sigma}{}^*a_\nu + {}^*\epsilon_{\sigma\nu}{}^*a_\tau + {}^*\epsilon_{\nu\tau}{}^*a_\sigma)$, where ${}^*a_\tau = {}^*(a_\tau)$, (as opposed to $({}^*a)_\tau$), one arrives at:

$$
\begin{aligned}
(\mathrm{A})\quad & \epsilon_{a\sigma}(\mathfrak{g}^\sigma + {}^*\mathfrak{g}^\sigma) + {}^*\epsilon_{a\sigma}(\mathcal{P}^\sigma + {}^*\mathcal{P}^\sigma) \\
& + \tfrac{1}{2}{}^*\epsilon^{\sigma\tau}(\mathfrak{g}_{\tau\sigma a} + {}^*\mathfrak{g}_{\tau\sigma a}) - \tfrac{1}{2}\epsilon^{\sigma\tau}(\mathcal{P}_{\tau\sigma a} + {}^*\mathcal{P}_{\tau\sigma a}) \\
& + \tfrac{1}{4\pi}\epsilon_{a\sigma}{}^*\epsilon^{\tau\sigma}(a_\tau + {}^*a_\tau) - \tfrac{1}{4\pi}{}^*\epsilon_{a\sigma}\epsilon^{\tau\sigma}(a_\tau + {}^*a_\tau) \\
& + \tfrac{1}{8\pi}{}^*\epsilon^{\sigma\tau}\left[\epsilon_{\tau\sigma}(a_a + {}^*a_a) + \epsilon_{\sigma a}(a_\tau + {}^*a_\tau) + \epsilon_{a\tau}(a_\sigma + {}^*a_\sigma)\right] \\
& - \tfrac{1}{8\pi}\epsilon^{\sigma\tau}\left[{}^*\epsilon_{\tau\sigma}(a_a + {}^*a_a) + {}^*\epsilon_{\sigma a}(a_\tau + {}^*a_\tau) + {}^*\epsilon_{a\tau}(a_\sigma + {}^*a_\sigma)\right] = 0
\end{aligned}
\qquad (54)
$$

$$
\begin{aligned}
(\mathrm{B})\quad & {}^*\epsilon_{B\sigma}(\mathfrak{g}^\sigma + {}^*\mathfrak{g}^\sigma) - \epsilon_{B\sigma}(\mathcal{P}^\sigma + {}^*\mathcal{P}^\sigma) \\
& - \tfrac{1}{2}\epsilon^{\sigma\tau}(\mathfrak{g}_{\tau\sigma B} + {}^*\mathfrak{g}_{\tau\sigma B}) - \tfrac{1}{2}{}^*\epsilon^{\sigma\tau}(\mathcal{P}_{\tau\sigma B} + {}^*\mathcal{P}_{\tau\sigma B}) \\
& + \tfrac{1}{4\pi}{}^*\epsilon_{B\sigma}{}^*\epsilon^{\tau\sigma}(a_\tau + {}^*a_\tau) + \tfrac{1}{4\pi}\epsilon_{B\sigma}\epsilon^{\tau\sigma}(a_\tau + {}^*a_\tau) \\
& - \tfrac{1}{8\pi}\epsilon^{\sigma\tau}\left[\epsilon_{\tau\sigma}(a_B + {}^*a_B) + \epsilon_{\sigma B}(a_\tau + {}^*a_\tau) + \epsilon_{B\tau}(a_\sigma + {}^*a_\sigma)\right] \\
& - \tfrac{1}{8\pi}{}^*\epsilon^{\sigma\tau}\left[{}^*\epsilon_{\tau\sigma}(a_B + {}^*a_B) + {}^*\epsilon_{\sigma B}(a_\tau + {}^*a_\tau) + {}^*\epsilon_{B\tau}(a_\sigma + {}^*a_\sigma)\right] = 0,
\end{aligned}
$$

which are the extremal electric formulations of [redacted] (2)(A) and (B) respectively. It is interesting to note the presence of terms with the form $a_\tau + {}^*a_\tau$ in the above, in addition to the expected terms with form $\mathfrak{g} + {}^*\mathfrak{g}$ and $\mathcal{P} + {}^*\mathcal{P}$.

In (54)(A), each term on the third, fourth and fifth lines vanishes identically, due to the extremal identity ${}^*\epsilon^{\tau\sigma}\epsilon_{a\sigma} = 0$.[13] In (54)(B), the term on the third line is cancelled identically by the combination of terms on the fourth and fifth lines, with further assistance from the identity $\epsilon^{\sigma\tau}\epsilon_{\tau\sigma} + {}^*\epsilon^{\sigma\tau}{}^*\epsilon_{\tau\sigma} = 0$. Therefore, (54)(A) and (B) reduce simply to:

(A) $$\mathcal{E}_{a\sigma}(\mathcal{J}^{\sigma} + {}^*\mathcal{J}^{\sigma}) + {}^*\mathcal{E}_{a\sigma}(\mathcal{P}^{\sigma} + {}^*\mathcal{P}^{\sigma}) + \frac{1}{2}{}^*\mathcal{E}^{\sigma\tau}(\mathcal{J}_{\tau\sigma a} + {}^*\mathcal{J}_{\tau\sigma a}) - \frac{1}{2}\mathcal{E}^{\sigma\tau}(\mathcal{P}_{\tau\sigma a} + {}^*\mathcal{P}_{\tau\sigma a}) = 0$$

(55)

(B) $${}^*\mathcal{E}_{B\sigma}(\mathcal{J}^{\sigma} + {}^*\mathcal{J}^{\sigma}) - \mathcal{E}_{B\sigma}(\mathcal{P}^{\sigma} + {}^*\mathcal{P}^{\sigma}) - \frac{1}{2}\mathcal{E}^{\sigma\tau}(\mathcal{J}_{\tau\sigma a} + {}^*\mathcal{J}_{\tau\sigma a}) - \frac{1}{2}{}^*\mathcal{E}^{\sigma\tau}(\mathcal{P}_{\tau\sigma a} + {}^*\mathcal{P}_{\tau\sigma a}) = 0,$$

which is of precisely the same form as the original equation (2), but with each of F, J and P replaced by their extremal electric counterparts $\mathcal{E}$, $\mathcal{J}$ and $\mathcal{P}$.

Thus we find, in electrodynamic theory with sources, that all terms involving the complexion gradient a_μ cancel identically in the extremal electric formulation of both equations (2)(A) and (2)(B). In source-frée electrodynamics, as we shall see, the situation is different. While all terms involving a_μ will continue to cancel from (2)(A), these terms will <u>not</u> cancel from (2)(B). Thus it will be possible in the source-free theory to obtain an expression for a_μ in the expected manner. As we shall see however, there are some important differences between the usual source-free expression for a_μ and the expression for a_μ to be obtained here.

We turn then, to the source-free, extremal electric formulation of the electrodynamic equations (2). It is here that we may directly compare what has been derived here, with the currently existing results of source-free already unified field theory.

SECTION 8- The Source-Free External Electric Formulation of Electrodynamics with Sources

At the very outset of the discussion, it was indicated how the supplementary condition $F^{\mu\nu} = A^{\nu;\mu} - A^{\mu;\nu}$ (56) for the four-potential is directly responsible for bringing about the vanishing of sources in source-free electrodynamics. Consequently, to assure energy conservation, and for other reasons, it was necessary in the development of electrodynamics with sources to discard this condition; and to develop a theory which could stand independently of this condition. It was however noted that the real use of this condition, once an appropriate theory with sources had been developed, should be for the purpose of reducing back from electrodynamics with sources, to source-free electreodynamics. This of course, would furnish a ready comparison between the theory with sources developed here, and the pre-existing source free already unified field theory of Rainich. It is this reduction that is the object of the discussion following.

The actual reduction that takes place in eqs. (2) and (3) by virtue of (56) above (originally eq. (5)) is really quite simple. Substituting (5) into (3), all of the third rank sources reduce immediately to zero, as is easily seen; and as was snown earlier, (see eqs. (19)-(21), also eqs. (22)-(24)) this forces all the first rank sources to zero as well. Thus, through this condition (56) for the four potential, one arrives immediately at source-free electrodynamics. As a result, the bottom lines of (2)(A) and (B) may each be eliminated, to yield:

$$\text{(A)} \quad F_{\alpha\sigma}(J^{\sigma} + {}^{*}J^{\sigma}) + {}^{*}F_{\alpha\sigma}(P^{\sigma} + {}^{*}P^{\sigma}) = 0$$

$$\text{(B)} \quad {}^{*}F_{\beta\sigma}(J^{\sigma} + {}^{*}J^{\sigma}) - F_{\beta\sigma}(P^{\sigma} + {}^{*}P^{\sigma}) = 0, \tag{57.1}$$

or alternatively, if one wishes to work with third rather than first rank sources, one may write:

$$\text{(A)} \quad \tfrac{1}{2}\,{}^{*}F^{\sigma\tau}(J_{\tau\sigma\alpha} + {}^{*}J_{\tau\sigma\alpha}) - \tfrac{1}{2}F^{\sigma\tau}(P_{\tau\sigma\alpha} + {}^{*}P_{\tau\sigma\alpha}) = 0$$

$$\text{(B)} \quad -\tfrac{1}{2}F^{\sigma\tau}(J_{\tau\sigma\beta} + {}^{*}J_{\tau\sigma\beta}) - \tfrac{1}{2}{}^{*}F^{\sigma\tau}(P_{\tau\sigma\beta} + {}^{*}P_{\tau\sigma\beta}) = 0. \tag{57.2}$$

The advantage of this procedure, while each of the 16 sources in (57) is individually reduced to zero by $F^{\mu\nu} = A^{\nu;\mu} - A^{\mu;\nu}$, is that (57) nevertheless shows a total of only eight independent electrodynamical equations, which is as it should be. This is because (57) already combines all of these sources in such a way as to eliminate any duplications due to a mere duality renaming of fields and/or sources. In other words, dualistic invariance is already built into eqs. (57), hence there is no further need to concern ourselves with the duplications due to duality renaming.

The extremal electric formulation of the above eqs. (57) may be deduced with the help of eqs. (51)-(54) from the prior section. For first rank sources, from (57.1), this is shown to be:

(A) $\mathcal{E}_{a\sigma}(\mathcal{J}^{\sigma} + {}^*\mathcal{J}^{\sigma}) + {}^*\mathcal{E}_{a\sigma}(\mathcal{P}^{\sigma} + {}^*\mathcal{P}^{\sigma}) = 0$

(58.1)

(B) $T_{(MAX)}{}^{\tau}{}_{\beta\tau}(a_\tau + {}^*a_\tau) = \frac{1}{2}\left[{}^*\mathcal{E}_{\beta\sigma}(\mathcal{J}^{\sigma} + {}^*\mathcal{J}^{\sigma}) - \mathcal{E}_{\beta\sigma}(\mathcal{P}^{\sigma} + {}^*\mathcal{P}^{\sigma})\right],$

and alternatively, for third rank sources, from (57.2) this is shown to be:

(A) $\frac{1}{2}{}^*\mathcal{E}^{\sigma\tau}(\mathcal{J}_{\tau\sigma a} + {}^*\mathcal{J}_{\tau\sigma a}) - \frac{1}{2}\mathcal{E}^{\sigma\tau}(\mathcal{P}_{\tau\sigma a} + {}^*\mathcal{P}_{\tau\sigma a}) = 0$

(58.2)

(B) $T_{(MAX)}{}^{\tau}{}_{\beta}(a_\tau + {}^*a_\tau) = \frac{1}{4}\left[\mathcal{E}^{\sigma\tau}(\mathcal{J}_{\tau\sigma\beta} + {}^*\mathcal{J}_{\tau\sigma\beta}) + {}^*\mathcal{E}^{\sigma\tau}(\mathcal{P}_{\tau\sigma\beta} + {}^*\mathcal{P}_{\tau\sigma\beta})\right].$

In eitner case:

$$T_{(MAX)}{}^{\tau}{}_{\beta} = -\frac{1}{8\pi}\left[F^{\tau\sigma}F_{\beta\sigma} + {}^*F^{\tau\sigma}{}^*F_{\beta\sigma}\right] = -\frac{1}{8\pi}\left[\mathcal{E}^{\tau\sigma}\mathcal{E}_{\beta\sigma} + {}^*\mathcal{E}^{\tau\sigma}{}^*\mathcal{E}_{\beta\sigma}\right], \quad (59)$$

because of the well know invariance of the Maxwell tensor $T_{(MAX)}^{\mu\nu}$ under duality field transformations of the forn $F \rightarrow {}^*F$.[-14]

Note, unlike what happened in the extremal electric formulation of electrodynamics with sources, that the complexion gradient a_μ does not drop out of both equations for source-free electrodynamics. While all terms involving a_μ do continue to cancel out of (58)(A) as expected, these terms do not cancel from (58)(B). Thus, in the source-free case, one can indeed deduce a definitive expression for the complexion gradient, as expected.

From here, it should be possible to compare the source-free extremal electric equations (58) with the known extremal electric equations of Rainich's source-free already unified field theory. Particularly, using the equations (58.1) containing first rank sources, it is expected that (58.1)(A) should correspond with eq. (67) on pg. 251 of Wheeler's Geometrodynamics; and that (58.1)(B) should correspond with eq. (70) on pg. 252 of the same reference. The only difference should be the fact that first-third rank as well as second-second rank duality transformations have been accounted for here, with the consequence that $\mathcal{J}^\sigma$ is replaced by $\mathcal{J}^\sigma + {}^*\mathcal{J}^\sigma$, $\mathcal{P}^\sigma$ by $\mathcal{P}^\sigma + {}^*\mathcal{P}^\sigma$ and a_τ by $a_\tau + {}^*a_\tau$.

Starting therefore with (58.1)(A) and ignoring the terms ${}^*\mathcal{J}^\sigma$, ${}^*\mathcal{P}^\sigma$ and ${}^*a_\sigma$ that have been added here, one finds that the corresponding equation (67) from pg. 251 of Geometrodynamics is rooted ultimately in the non-extremal equation:

$$F_{\alpha\sigma} J^\sigma + {}^*F_{\alpha\sigma} P^\sigma = \frac{1}{4\pi}\left[F_{\alpha\sigma} F^{\nu\sigma}{}_{;\nu} + {}^*F_{\alpha\sigma}{}^*F^{\nu\sigma}{}_{;\nu}\right]. \qquad (60)$$

Particularly, using $F^{\mu\nu} = e^{*\alpha}\mathcal{E}^{\mu\nu}$, one may deduce the extremal electric formulation of the above thus: (Recall, $\mathcal{E}_{\alpha\sigma}{}^*\mathcal{E}^{\tau\sigma} = 0$, and make use of (27) and (53))

$$\begin{aligned} &\frac{1}{8\pi}\left(\mathcal{E}^{\nu\sigma}\mathcal{E}_{\alpha\beta} + {}^*\mathcal{E}^{\nu\sigma}{}^*\mathcal{E}_{\alpha\beta}\right)_{;\nu} \\ &= \frac{1}{4\pi}\left(\mathcal{E}_{\alpha\sigma}\mathcal{E}^{\nu\sigma}{}_{;\nu} + {}^*\mathcal{E}_{\alpha\sigma}{}^*\mathcal{E}^{\nu\sigma}{}_{;\nu}\right) \qquad (61) \\ &= \mathcal{E}_{\alpha\sigma}\mathcal{J}^\sigma + {}^*\mathcal{E}_{\alpha\sigma}\mathcal{P}^\sigma = 0. \end{aligned}$$

The first line of the above is indeed equivalent with Wheeler's eq. (67) on pg. 251 of the cited reference, while the third line, once $\mathcal{J}^\sigma + {}^*\mathcal{J}^\sigma$ and $\mathcal{P}^\sigma + {}^*\mathcal{P}^\sigma$ have been substituted for $\mathcal{J}^\sigma$ and $\mathcal{P}^\sigma$ respectively, is equivalent with (58.1)(A) derived here. Hence, for this equation at least, one finds a complete agreement between the results here, and the results of Rainich's source-free already unified field theory. The above states, in essence, for source-free electrodynamics, that the Maxwell tensor (59) is conserved seperately from all other energy tensors.

Moving now to (58.1)(B), and again ignoring ${}^*\theta^\sigma$, ${}^*P^\sigma$ and ${}^*a_\tau$, one finds that the corresponding equation (70) on pg. 252 of <u>**Geometrodynamics**</u> is rooted in the non-extremal:

$$\begin{aligned} &{}^*F_{\beta\sigma} J^\sigma + F_{\beta\sigma} P^\sigma \\ &= \tfrac{1}{4\pi}\left[{}^*F_{\beta\sigma} F^{\nu\sigma}{}_{;\nu} + F_{\beta\sigma}{}^*F^{\nu\sigma}{}_{;\nu}\right] = 0. \end{aligned} \qquad (62)$$

Using $F^{\mu\nu} = e^{*\alpha}\mathcal{E}^{\mu\nu}$ to deduce the extremal electric formulation of the above, on arrives at:

$$\begin{aligned} &-\tfrac{1}{4\pi}\left({}^*\mathcal{E}^{\nu\sigma}\,{}^*\mathcal{E}_{\beta\sigma} - \mathcal{E}^{\nu\sigma}\mathcal{E}_{\beta\sigma}\right) a_\nu \\ &= -\tfrac{1}{8\pi}\,\delta^\nu{}_\beta\,\mathcal{E}^{\sigma\tau}\mathcal{E}_{\sigma\tau}\, a_\nu \\ &= -\tfrac{1}{8\pi}\,\mathcal{E}^{\sigma\tau}\mathcal{E}_{\sigma\tau}\, a_\beta \quad = \tfrac{1}{4\pi}\left({}^*\mathcal{E}_{\beta\sigma}\mathcal{E}^{\nu\sigma}{}_{;\nu} + \mathcal{E}_{\beta\sigma}{}^*\mathcal{E}^{\nu\sigma}{}_{;\nu}\right) \\ &\qquad\qquad\qquad\qquad\quad = {}^*\mathcal{E}_{\beta\sigma}\theta^\sigma + \mathcal{E}_{\beta\sigma}P^\sigma. \end{aligned} \qquad (63)$$

while the third line of the above is equivalent with Wheeler's eq. (70) on pg. 252, the above eq. (63) is <u>**not**</u> in any way equivalent with eq. (58.1) derived here.

In particular, while the source-free extremal electric equation (63) is rooted in the non-extremal (62), the eq. (58.1)(B) derived here, neglecting ${}^*\theta^\sigma$, ${}^*P^\sigma$ and ${}^*a^\sigma$, is rooted in the non-extremal: (contrast with (62))

$$\begin{aligned} &{}^*F_{\beta\sigma} J^\sigma - F_{\beta\sigma} P^\sigma \\ &= \tfrac{1}{4\pi}\left[{}^*F_{\beta\sigma} F^{\nu\sigma}{}_{;\nu} - F_{\beta\sigma}{}^*F^{\nu\sigma}{}_{;\nu}\right] = 0. \end{aligned} \qquad (64)$$

The extremal electric formulation of the aoove, using $F^{\mu\nu} = e^{*\alpha}\mathcal{E}^{\mu\nu}$, is therefore: (contrast with (63))

$$-\frac{1}{4\pi}\left({}^*\mathcal{E}^{\nu\sigma}\,{}^*\mathcal{E}_{\beta\sigma} + \mathcal{E}^{\nu\sigma}\mathcal{E}_{\beta\sigma}\right) a_\nu$$

$$= 2T^{\nu}{}_{\beta}\, a_\nu = \frac{1}{4\pi}\left({}^*\mathcal{E}_{\beta\sigma}\mathcal{E}^{\nu\sigma}{}_{;\nu} - \mathcal{E}_{\beta\sigma}{}^*\mathcal{E}^{\nu\sigma}{}_{;\nu}\right) \qquad (65)$$

$$= {}^*\mathcal{E}_{\beta\sigma}\mathcal{J}^{\sigma} - \mathcal{E}_{\beta\sigma}\mathcal{P}^{\sigma}.$$

The only difference therefore, between the known equations (62),(63) of Rainich's source-free already unified field theory; and the eqs. (64),(65) derived here, is the fact that the term $F_{\beta\sigma}P^{\sigma}$ is subtracted from ${}^*F_{\beta\sigma}J^{\sigma}$ in (64); but added to This Term in (62). All other differences are a direct consequence of this sign difference.

The question then arises whether it is eqs. (62),(63), or eqs. (64),(65), that are really the correct equations.

One of the requirements that has been pursued throughout the development here, is that the equations of electrodynamics retain their invariance under duality rotations of both fields and sources. In this manner, one may eliminate any duplication resulting from mere duality renamings of fields and/or sources; and one's attention is thereby restricted only to those equations that truly are independent.

Eqs. (60),(61), which are in agreement with both the known equations of source-free already unified field theory, and with the results derived here, are rooted ultimately in $F_{\alpha\sigma}J^{\sigma} + {}^*F_{\alpha\sigma}P^{\sigma} = 0$. Upon the duality field transformation $F^{\mu\nu} \rightarrow {}^*F^{\mu\nu}$, one finds that this equation transforms into itself, ie., that one recovers the original equation $F_{\alpha\sigma}J^{\sigma} + {}^*F_{\alpha\sigma}P^{\sigma} = 0$. Consequently, we say that this equation is "dualistically invariant," at least under second-second rank duality field transformations. If, per chance, one were to replace this equation with $F_{\alpha\sigma}J^{\sigma} - {}^*F_{\alpha\sigma}P^{\sigma} = 0$, and then transform it by $F^{\mu\nu} \to {}^*F^{\mu\nu}$, this equation would not transform into itself. Rather, it would transform into its own negative, ie., into $-F_{\alpha\sigma}J^{\sigma} + {}^*F_{\alpha\sigma}P^{\sigma} = 0$, and in any more general context, this means that the dualistic invariance has been lost. Such a circumstance is quite undesirable, and it indicates a proper preference for the equation $F_{\alpha\sigma}J^{\sigma} + {}^*F_{\alpha\sigma}P^{\sigma} = 0$ over $F_{\alpha\sigma}J^{\sigma} - {}^*F_{\alpha\sigma}P^{\sigma} = 0$, which preference is confirmed by known theory, and also, by the results derXived here.

Eqs. (61),(62) on the other hand, which are the known equations of source-free already unified field theory, are of course rooted in ${}^*F_{\beta\sigma}J^{\sigma} + F_{\beta\sigma}P^{\sigma} = 0$. If however one subjects this to the duality transformation $F^{\mu\nu} \to {}^*F^{\mu\nu}$, this equation will not transform into itself, but will instead transform into its own negative, ie., into $-{}^*F_{\beta\sigma}J^{\sigma} - F_{\beta\sigma}P^{\sigma} = 0$. As indicated above, this loss of dualistic invariance is not a desirable feature. Eqs. (63),(64) on the other hand, as proposed here, are rooted in ${}^*F_{\beta\sigma}J^{\sigma} - F_{\beta\sigma}P^{\sigma} = 0$. Here, if one transforms the field by $F^{\mu\nu} \to {}^*F^{\mu\nu}$, one will indeed recover the original equation precisely, without any change in sign. That is, one will get back ${}^*F_{\beta\sigma}J^{\sigma} - F_{\beta\sigma}P^{\sigma} = 0$. For this reason, eqs. (63),(64) as proposed here, are to be preferred to the known equations (61),(62) of currently accepted already unified field theory.

The difficulties cited above with the known eqs. (61),(62) appear to be due to an oversight in the development of source-free already unified field theory. Particularly, to construct the extremal electric formulation of source-free electrodynamics directly, one starts with a particular non-extremal term, one finds all terms that differ from the original term by a mere duality rotation, one combines all these terms together to avoid duplication, and then one transforms into an extremal electric reference frame. The equation that results from such combination therefore has the property of dualistic invariance, as a direct consequence of this construction process. For example, one may start in source-free theory with the equation $F_{\alpha\sigma} J^{\sigma} = 0$, and by a duality transformation $F^{\mu\nu} \rightarrow {}^{*}F^{\mu\nu}$, derive ${}^{*}F_{\alpha\sigma} P^{\sigma} = 0$. Another transformation of the form $F^{\mu\nu} \rightarrow {}^{*}F^{\mu\nu}$ recovers the original equation. Therefore the work is done, and one simply combines these two equations, to avoid the duplication of a duality renaming, and thereby arrives at $F_{\alpha\sigma} J^{\sigma} + {}^{*}F_{\alpha\sigma} P^{\sigma} = 0$, which is eq. (60). At this point, one reduces to the extremal electric equation. If one starts however with ${}^{*}F_{\beta\sigma} J^{\sigma} = 0$ and transforms this according to $F^{\mu\nu} \rightarrow {}^{*}F^{\mu\nu}$, one gets $-F_{\beta\sigma} P^{\sigma} = 0$. A second transformation $F^{\mu\nu} \rightarrow {}^{*}F^{\mu\nu}$ recovers the original equation, and combination of the two equations thereby yields ${}^{*}F_{\beta\sigma} J^{\sigma} - F_{\beta\sigma} P^{\sigma} = 0$, which is the proposed eq. (64). One cannot by this procedure, get ${}^{*}F_{\beta\sigma} J^{\sigma} + F_{\beta\sigma} P^{\sigma} = 0$, i.e., eq. (65). This is the same sort of procedure that was followed to develop electrodynamics with sources, for example, in combining the equations within (22), and within (35).

The specific miscalculation which takes place in <u>Geometrodynamics</u> therefore, occurs in the first paragraph of section 2.4.4. Particularly, one ends up combining (adding) the two equations:- 15

$$\begin{cases} (a) \quad {}^{*}\mathcal{E}_{B\mu} \mathcal{E}^{\mu\nu}{}_{;\mu} + {}^{*}\mathcal{E}_{B\mu} {}^{*}\mathcal{E}^{\mu\nu} a_{\nu} = 0 \\ (b) \quad \mathcal{E}_{B\mu} {}^{*}\mathcal{E}^{\mu\nu}{}_{;\mu} - \mathcal{E}_{B\mu} \mathcal{E}^{\mu\nu} a_{\nu} = 0 , \end{cases} \qquad (66)(A)$$

when one should really combine:

$$(a) \quad \begin{cases} {}^{*}\mathcal{E}_{B\mu} \mathcal{E}^{\mu\nu}{}_{;\mu} + {}^{*}\mathcal{E}_{B\mu} {}^{*}\mathcal{E}^{\mu\nu} a_{\nu} = 0 \\ \vdots \end{cases}$$

⋮

(66)(B)

(b) $\Big\{ - \varepsilon_{B\mu} {}^*\varepsilon^{\mu\nu}{}_{;\mu} + \varepsilon_{B\mu} \varepsilon^{\mu\nu} a_\nu = 0 .$

In particular, (66)(B)(a) and (b) really do differ from one another by a mere duality transformation $\varepsilon^{\mu\nu} \rightarrow {}^*\varepsilon^{\mu\nu}$. (66)(A)(a) and (b) do not however, as there is a sign reversal taking place under the transformation $\varepsilon^{\mu\nu} \rightarrow {}^*\varepsilon^{\mu\nu}$. The whole reason for combining the various equations on pp. 250-251 of Geometrodynamics, To begin with which was perhaps not made as explicit as it should have been, is to consolidate into a single equation, those sets of equations that differ from one another merely by the renaming of a duality rotation. The combination (66)(B) achieves This. (66)(A) does not.

Finally one should note, in general, that dualistic invariance under the second rank field transformation $F \rightarrow {}^*F$ is ~~normally~~ achieved, one the one hand, by using terms of the general form $F \cdot F + {}^*F \cdot {}^*F$; and on the other hand, by using terms of the form ${}^*F \cdot F - F \cdot {}^*F$. This is the fundamental source of the minus sign appearing in eq. (64) proposed here, rather than the corresponding plus sign in the known equation (62). Given the totally antisymmetric character of the Levi-Cevita formalism upon which these equations are based, this should not be surprising.

This completes the specialization of electrodynamics with sources to source-free electrodynamics in the extremal electric reference frame, via the source-free reduction $F^{\mu\nu} = A^{\nu;\mu} - A^{\mu;\nu}$ and the duality operator $F^{\mu\nu} = e^{*\alpha} \varepsilon^{\mu\nu}$; and the comparison of this specialization with the known equations of source-free already unified field theory.

We turn now to a number of other electrodynamical specializations of interest.

SECTION 9- Other Significant Specializations of Electrodynamics with Sources

Two further specializations of electrodynamics with sources, noted briefly in the introductory section, are the following:

$$\text{(a)} \quad T_{(max)}{}^{\mu}{}_{\nu} = -\frac{1}{8\pi}\left[F^{\mu\tau}F_{\nu\tau} + {}^{*}F^{\mu\tau}\,{}^{*}F_{\nu\tau}\right] = 0$$

and (67)

$$\text{(b)} \quad \tilde{J}^{\sigma} = J^{\sigma} + {}^{*}J^{\sigma} = 0.$$

These are of particular interest because of the following respective propositions, which are trivially deduced from the above:

$$\text{(a)} \quad * \equiv i = \sqrt{-1} \quad \underline{\text{iff}} \quad T_{(max)}{}^{\mu}{}_{\nu} = 0$$

and (68)

$$\text{(b)} \quad * \equiv -1 \quad \underline{\text{iff}} \quad \tilde{J}^{\sigma} = 0,$$

where the * in (a) is the one associated with second-second rank duality field transformations; and that in (b) is the one associated with first-third rank duality source transformations.

The former specialization (67)(a) and its implied proposition (68)(a), when restated in terms of the duality operator $e^{*\alpha}$, and when connected with the Cartan spinor calculus, reveals that: -16

$$e^{*a}F^{\mu\nu}$$
$$\equiv e^{ia}F^{\mu\nu} = \tfrac{1}{4}\,\sigma^{\mu}{}_{A\dot{U}}\,\sigma^{\nu}{}_{B\dot{V}}\,e^{ia}F^{A\dot{U}B\dot{V}} \tag{69}$$
$$= \tfrac{1}{4}\sigma^{\mu}{}_{A\dot{U}}\sigma^{\nu}{}_{B\dot{V}}\,e^{ia}\left[\varepsilon^{A}\bar{\varepsilon}^{\dot{U}}(\varepsilon^{B}\bar{\pi}^{\dot{V}}+\pi^{B}\bar{\varepsilon}^{\dot{V}})-(\varepsilon^{A}\bar{\pi}^{\dot{U}}+\pi^{A}\bar{\varepsilon}^{\dot{U}})\varepsilon^{B}\bar{\varepsilon}^{\dot{V}}\right]$$

<u>if and only if</u> $T_{(max)}{}^{\mu}{}_{\nu}=0$. The $\sigma^{\mu}{}_{A\dot{U}}$ of course are the standard Hamilton/Pauli spin matrices: (in cartesian coordinates)

$$\sigma^{\mu A\dot{U}} = (\sigma_t{}^{A\dot{U}},\ \sigma_x{}^{A\dot{U}},\ \sigma_y{}^{A\dot{U}},\ \sigma_z{}^{A\dot{U}})$$
$$= \left(\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}0&1\\1&0\end{pmatrix},\begin{pmatrix}0&-i\\i&0\end{pmatrix},\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right). \tag{70}$$

In the context of the spinor formalism, as a result of the normalization condition $\varepsilon_A\pi^A = 1$,-17 the duality rotation $e^{*a}F^{\mu\nu}$ is precisely equivalent, when $T_{(max)}{}^{\mu}{}_{\nu}=0$, to a "phase" transformation in the basis spinor ε^A , of the form $e^{\frac{1}{2}ia}\varepsilon^A$. In other words:

$$F^{\mu\nu} \rightarrow e^{*a}F^{\mu\nu}$$
$$\equiv \tag{71}$$
$$\varepsilon^A \rightarrow e^{\frac{1}{2}ia}\varepsilon^A$$

iff $T_{(max)}{}^{\mu}{}_{\nu}=0$, where ε^A is the basis spinor for $F^{\mu\nu}$.

Therefore, if $T_{(max)}{}^{\mu}{}_{\nu}=0$, but only if this is so, then the duality rotation of $F^{\mu\nu}$ through a complexion angle a is precisely equivalent, in spinor language, with a phase rotation of ε^A through an angle exactly one-half as great as the angle of duality rotation, ie., through $\frac{1}{2}a$. The more general case where $T_{(max)}{}^{\mu}{}_{\nu}\neq 0$, and where i therefore reverts back to the usual second rank *, seems to suggest a corresponding generalization of the spinor formlism, whereby the second rank duality operator * is injected into the spinor calculus in place of i , at least for phase rotations, whenever the Maxwell tensor $T_{(em)\,\nu}^{\mu}$ is non-zero. In any case, this makes clear the very direct sort of connection that exists between second rank duality, and spinor phase rotations; and it shows how second rank dualistic invariance is just a generalized form of phase invariance. Thus, one particularly natural physical interpretation of second-second rank dualistic invariance, is that of invariance under arbitrary phase transformations.

The latter specialization (67)(b) and associated proposition (68)(b), allows one to write:

$$\begin{aligned}\tilde{J}^{\sigma} &= 1 J^{\sigma} + {}^{*}J^{\sigma} \\ &= {}^{+}J^{\sigma} + {}^{-}J^{\sigma} = 0\,,\end{aligned} \tag{72}$$

in the special case where $\tilde{J}^{\sigma} = 0$. In this case, one sees that ${}^{+}J^{\sigma}$ may be interpreted as the four-vector of positive charges, and that ${}^{-}J^{\sigma}$ may be interpreted as that of negative charges. $\tilde{J}^{\sigma}$ therefore, is to be regarded as the four-vector of the net (positive combined with negative) charge. If one continues to interpret $\tilde{J}^{\sigma}$ generally as the net charge vector, then the above specialization where $\tilde{J}^{\sigma} = 0$ simply indicates an electrically neutral environment.

This leads to a subtle, but very important point about the convention, first introduced by Benjamin Franklin over 200 years ago, of denoting the two varieties of electric charge by the operators + and -. Particularly, from a mathematical standpoint, while this convention does apply precisely in an electrically neutral environment where $\tilde{J}^{\sigma} = 0$ and hence the first-third rank * goes to -1, it does not apply precisely in a non-neutral environment, where $\tilde{J}^{\sigma} \neq 0$. In this case, it is mathematically necessary to discard the convention which uses the operators + and -, and to replace it with a convention that uses the respective operators 1 and *. Again, this is a subtle, but very important distinction. Thus, the vector $1J^{\sigma}$ should be used in the general (non-neutral) context as that of positive charge; the vector $*J^{\sigma}$ as that of negative charge, and the vector $\tilde{J}^{\sigma}$ as that of net charge. The same sort of thing applies to monopoles P .

The existence of terms with the form $J + {}^{*}J$ and $P + {}^{*}P$ in already unified field theory with sources as developed here, in light of the above interpretations, appears to require that positive and negative charges shall enter symmetrically into the laws of physics -18 and this provides a natural physical intrpretation of invariance under first-third rank duality source transformations. It also makes the experimentally observed existence of positive and negative charges directly compatable with the given dualistically invariant theoretical formalism.

With The discussion of various electrodynamic specializations complete, we now turn to the broader issues of topology, and of quantization.

SECTION 10- On the Extension of Geometrodynamic Electrodynamics to Non-Euclidean Spacetime Topology: Toward the Natural Quantization of Classical Already Unified Field Theory

Classical electrodynamics is of course, a linear theory. Quantum particle solutions can only be obtained from non-linear theory. If we refrain from considering anything but classical Einstein spacetime, then the only way to quantize classical electrodynamics is to first incorporate it into a non-linear theory of electromagnetism and gravitation, and to then obtain quantum particle solutions to the resulting non-linear field equations. This is the route which Einstein himself pursued, in attempting to find a natural way to quantize classical field theory, though this ran counter to all of the other modes of quantization then being explored.-19

To this point, we have worked toward the development of a classical already unified field theory with sources, and have demonstrated how the entirety of Maxwell's electrodynamics, with sources, can be consolidated into the non-linear field equation:

$$R_{uv} = 0. \tag{73}$$

The question of how to quantize electrodynamics may therefore be boiled down to that of finding particle solutions for the above non-linear equation.

In developing the above, it has been assumed at all times that the topology of spacetime is simply connected, ("Euclidean topology") which in terms of the complexion gradient, is to say that:

$$\text{(a)} \qquad a^{\mu;\nu} - a^{\nu;\mu} = 0. \qquad (74)$$

It is natural at this point however, to generalize to above to account for multiple topological connections; ("Non-Euclidean topology") which is to say that (74)(a) above is now replaced by: (see eqs. (7))

$$\text{(b)} \qquad \oint a^{\mu} dx_{\mu} = 2\pi \cdot n \quad ; \quad n = 0,1,2,3 \ldots \qquad (74)$$

In the setting of non-Euclidean topology, one cannot help but be struck by the similarity between the above, and the Wilson-Bohr-Sommerfield rule (respective particle and wave formulations)

$$\text{(a)} \qquad \oint p^{\mu} dx_{\mu} = 2\pi \cdot n \cdot \hbar$$

$$\qquad ; \quad n = 0,1,2,3 \ldots \qquad (75)$$

$$\text{(b)} \qquad \oint k^{\mu} dx_{\mu} = 2\pi \cdot n$$

of the first quantization. (1916). While the applicability of this specific rule is limited to situations where particle wavelengths are small compared to the dimensions of particle confinement, the striking similarity between (75) and (74)(b) suggests strongly, that quantization may well be a fundamentally topological phenomemon. To the extent that quantum phenomena truly are the result of a non-Euclidean topology, one might expect that the entirety of physics could be idealized, in the main, simply as the superposition of quantized non-Euclidean topology, upon the already existing classical structures of non-Euclidean geometry. In short, it may well be possible to idealize quantum geometrodynamics as the admixture, simply, of non-Euclidean geometry plus topology.

There is little doubt that Einstein himself favored a topological approach toward quantization; and equally little doubt that such an approach necessarily presupposes a non-linear theory. Thus, to quantize electrodynamics, one would at the very least have to unify linear electrodynamics with sources, into non-linear geometrodynamics.[20] The absence of such a unification however, even to this day, has rendered next to impossible, the realistic consideration of topological quantization.

With the development of a classical unified field theory of electromagnetism and gravitation completed, one looks quite naturally toward the quantization of that theory. The advantage to be gained, once a non-linear theory of this sort has been produced, is that a topological approach to quantization realy does become feasable, because one now has available a non-linear electrodynamic theory.

While the sorts of topological structures that one is forced by natural experiment to consider may be quite rich and varied, one nevertheless seems bound to ask how much of today's elementary particle phenomenology, and various other aspects of quantum theory, (such as atomic theory, small scale vacuum fluctuation theory, strong and weak interaction theory, etc.) may ultimately be deduced from a topological approach. One also wonders how a non-Euclidean topology would affect such macroscopic domains of physics as cosmological and stellar theory. Particularly, as quantization eliminated theoretical atomic collapse in the early part of this century, one wonders whether the quantization of classical geometrodynamics through topology, could eliminate in some global way, the nagging macroscopic singularities of classical theory. -21

Answers to these and similar questions may not be forthcoming for some time yet. Nevertheless, the development of a classical unified field theory of electromagnetism and gravitation, independent of quantization, and independent of all the other interactions, is a long standing problem that has confronted physical theorists for the better part of a century. With the satisfactory development of such a theory, it finally becomes possible to address these broader, and highly fascinating questions of fundamental physics.

REFERENCES

1. Ann. of Phys., 2, 525-603, (1957); reprinted in Wheeler, J.A., Geometrodynamics, Academic Press, (1962), pp. 225-253.

2.

3. See Einstein, A., Relativistic Theory of the Non-Symmetric Field, in The Meaning of Relativity, Fifth Edition, Princeton University Press, (1956), Appendix II, pp. 133-139, for a parallel discussion.

4. Op. Cit. note 1, pg. 250, eqs (60), in a slightly different representation; and pp. 238-239, eqs. (13) and (15).

5. ibid, pp. 228, eqs. (4),(5); pg. 250, eq. (59); pg. 230, eq. (8).

6. ibid, pg. 251, note 22.

7. ibid, pg. 250, eqs. (60).

8. ibid, derivable from pg. (239), eq. (15), or the well known fact that the Maxwell tensor is traceless.

9. Misner, C.W.; Thorne, K.S.; Wheeler, J.A.; Gravitation, W.H. Freeman and Co., (1973), pg. 88.

10. ibid, pp. 87-88.

11. Op. Cit. note 3.

12. ibid, pg. 139.

13. Op. Cit. note 1, pg. 250, eq (64).

14. ibid, pg. 240.

15. ibid, pg. 250,251, eqs. (62),(63).

16. Op. Cit. note 9, eqs. (41.78), (41.89), (41.95).

17. ibid, eq. (41.81).

18. Op. Cit., note 3, contrast with a similar discussion on pg. 150.

19. Gursey, F., Geometrization of Unified Fields, in Ne'eman, Y., ed., To Fulfill A Vision..., Addison-Wesley, (1981), pg. 24, discussion of "Wave-Particle Duality."

20. ibid.

21. Op. Cit., note 1, see generally pp. 66-87, specifically pp. 66-67. Also, Op. Cit., note 9, see generally Chapter 44, specifically, section 44.1.

BIBLIOGRAPHY

Einstein, A.; The Meaning of Relativity, Fifth Edition; Princeton University Press; (1956).

Einstein, A.; The Principle of Relativity; Dover Publications; (1952).

Misner, C.W.; Thorne, K.S.; Wheeler, J.A.; Gravitation; W.H. Freeman and Co.; (1973).

Ne'eman Y., ed.; To Fulfill A Vision: Jerusalem Einstein Centennial Symposium on Gauge Theories and the Unification of Physical Forces; Addison-Wesley; (1981).

Ohanian, H.C.; Gravitation and Spacetime; W.W. Norton; (1976).

Rainich, G.Y.; Electrodynamics in the General Relativity Theory; Trans. Am. Math. Soc., Vol. 27, pp. 106-136; (1925).

Wheeler, J.A.; Geometrodynamics, Academic Press, (1962).