Kaluza-Klein Theory and Lorentz Force Geodesics

Jay R. Yablon*

910 Northumberland Drive

Schenectady, New York, 12309-2814

Abstract:

We examine a Kaluza-Klein-type theory of classical electrodynamics and gravitation in a five-dimensional Riemannian geometry. Based solely on the condition that the electrodynamic Lorentz force law must describe geodesic motion in this five-dimensional geometry, it appears possible to place all of Maxwell's electrodynamics, the theory of electrodynamic potentials, and the QED action on a solid geometrodynamic footing. We make no choice as between the fifth dimension being timelike or spacelike, but simply point out the impact in those places where this choice makes a difference. In the end, we deduce the Maxwell stress energy tensor, and in the process, learn that this fifth dimension must be spacelike.

PACS: 04.50.+h

* jyablon@nycap.rr.com

1. Introduction

The possibility of employing a fifth spacetime dimension to unite classical gravitation and electrodynamics has intrigued physicists for almost a century. [1], [2] Early theorists became perhaps overly-occupied with making assumptions about the scale or topology of the extra coordinate dimension. [3] Following the path of Wesson and other current-day theorists [4], we seek here to expose the main features of Kaluza- Klein theory irrespective of any particular model, and most importantly, to make the connection between Einstein's gravitation and Maxwell's electrodynamics which some have looked to 5-dimensional theories to provide, as clear and solid as possible, and as independent as possible of the detailed choice of model.

Most fundamentally, we adopt the view of the above-noted theorists that matter and electrodynamic charge are "induced" in the observed four dimensions of spacetime, from a vacuum in five dimensions, and so, in keeping with the spirit of Wheeler's program, [5] are of completely *geometrodynamic* origin. Particularly, we seek to show how classical electrodynamics emerges entirely from an Einstein-Hilbert Action of the general form $S = \frac{1}{2\kappa}\int R dV$ where R is a suitably-defined Ricci curvature scalar, integrated over a suitable multidimensional spacetime volume, and $\kappa = 8\pi G/c^4$ is the constant from Einstein's equation $-\kappa T^{\mu}{}_{\nu} = R^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}{}_{\nu} R$. The reader will observe that this omits any Lagrangian density $\mathfrak{L}_{\mathrm{Matter}}$ of matter, i.e., that it is *not* of the form $S = \int \left(\frac{1}{2\kappa} R + \mathfrak{L}_{\mathrm{Matter}}\right) dV$ and so is in the nature of action equation for the vacuum.[6] In different terms, we seek to induce the entirely of Maxwell's electrodynamics with sources, as well as the Maxwell stress-energy tensor, out of a gravitationally-based vacuum.

The main line of development will be deduced, based on a single proposition: we shall require that *the Lorentz force of electrodynamics,* $m\frac{d^2x^{\mu}}{d\tau^2} = qF^{\mu}{}_{\tau}\frac{dx^{\tau}}{d\tau}$, *must be represented as fully geodesic motion in the five-dimensional geometry.*

The foundation of this effort will be a five-dimensional Riemannian geometry, without any changes or enhancements, which merely extends the entire apparatus of gravitational theory into one more dimension. In five dimensions, we employ $g_{\mathrm{MN}} \equiv g_{\mathrm{NM}}$ with uppercase Greek indexes $\mathrm{M,N} = 0,1,2,3,5$ for the metric tensor, so $g_{\mu\nu}$ with lowercase $\mu,\nu = 0,1,2,3$ is the ordinary metric tensor in the spacetime subspace. Inverses are defined in the usual manner according to $g^{\mathrm{M\Sigma}} g_{\mathrm{\Sigma N}} = \delta^{\mathrm{M}}{}_{\mathrm{N}}$ and so $g^{\mathrm{M\Sigma}}$ and $g_{\mathrm{\Sigma N}}$ raise and lower indexes in the customary manner, but must be applied over all five dimensions to achieve proper five-covariance. The covariant derivative of the metric tensor $g_{\mathrm{MN;\Sigma}} = 0$, as always.

While most authors who still study Kaluza-Klein theories treat the fifth dimension as spacelike and a few have considered this to be timelike, e.g., [7], [8], [9], we shall approach the fifth dimension as independently of this choice as possible. Where this choice does make a difference, we shall point this out. If we define $g_{\mathrm{MN}} \equiv \eta_{\mathrm{MN}} + \overline{\kappa} h_{\mathrm{MN}}$ in the usual manner with $\overline{\kappa} = \sqrt{16\pi G/\hbar c^5}$, then for the weak-field limit $g_{\mathrm{MN}} \to \eta_{\mathrm{MN}}$. If the fifth dimension is timelike, $\mathrm{diag}(\eta_{\mathrm{MN}}) = (+1,-1,-1,-1,+1)$; if it is spacelike, then $\mathrm{diag}(\eta_{\mathrm{MN}}) = (+1,-1,-1,-1,-1)$. In either case, $\eta_{\mathrm{MN}} = 0$ for $\mathrm{M} \neq \mathrm{N}$. Note that the constant κ in Einstein's equation $-\kappa T^{\mu}{}_{\nu} = R^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}{}_{\nu} R$ is related to the foregoing $\overline{\kappa}$, with fundamental constants restored, by $\kappa = \frac{1}{2}\hbar c \overline{\kappa}^2 = 8\pi G/c^4$, with the overbar used to distinguish these two constants $\kappa, \overline{\kappa}$. The constant $\overline{\kappa}$ will appear frequently in the various equations herein.

At the end of section 10 see equations (10.14) and (10.15) infra, in the course of establishing the Maxwell stress-energy tensor, we will deduce that this fifth dimension must be spacelike.

2. Geodesic Motion in Five Dimensions, and the Lorentz Force

We start by maintaining the usual interval in the 4-dimensional spacetime subspace, using $d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu$, and define the five-space interval as:

$$\begin{aligned} d\mathrm{T}^2 \equiv g_{\mathrm{MN}} dx^{\mathrm{M}} dx^{\mathrm{N}} &= g_{\mu\nu} dx^\mu dx^\nu + g_{5\nu} dx^5 dx^\nu + g_{\mu 5} dx^\mu dx^5 + g_{55} dx^5 dx^5 \\ &= d\tau^2 + 2 g_{5\sigma} dx^5 dx^\sigma + g_{55} dx^5 dx^5 \end{aligned} . \tag{2.1}$$

The above is independent of whether the weak field $g_{55} \to \eta_{55} = \pm 1$, i.e., of whether the fifth dimension is timelike or spacelike, and is generally model-independent.

Like any metric equation, (2.1) can be algebraically-manipulated into:

$$1 = g_{\mathrm{MN}} \frac{dx^{\mathrm{M}}}{d\mathrm{T}} \frac{dx^{\mathrm{N}}}{d\mathrm{T}}, \tag{2.2}$$

which is the first integral of the equation of motion. In five dimensions, we specify the Christoffel connections in the usual manner, that is, $\Gamma^{\mathrm{M}}{}_{\Sigma\mathrm{T}} = \frac{1}{2} g^{\mathrm{MA}} \left(g_{\mathrm{A}\Sigma,\mathrm{T}} + g_{\mathrm{TA},\Sigma} - g_{\Sigma\mathrm{T},\mathrm{A}} \right)$, hence $\Gamma^{\mathrm{M}}{}_{\Sigma\mathrm{T}} = \Gamma^{\mathrm{M}}{}_{\mathrm{T}\Sigma}$. As noted, we employ $g_{\mathrm{MN};\Sigma} = 0$ as usual, with the usual first rank covariant derivative $A^{\mathrm{M}}{}_{;\Sigma} = A^{\mathrm{M}}{}_{,\Sigma} + \Gamma^{\mathrm{M}}{}_{\mathrm{A}\Sigma} A^{\mathrm{A}}$. We then take the covariant derivative of each side of (2.2) above, and after the usual reductions employed in four dimensions, and multiplying the result through by $d\mathrm{T}^2 / d\tau^2$, we arrive at a five-dimensional geodesic equation which bears an exact resemblance to the four-dimensional gravitational equation:

$$\frac{d^2 x^{\mathrm{M}}}{d\tau^2} + \Gamma^{\mathrm{M}}{}_{\Sigma\mathrm{T}} \frac{dx^{\Sigma}}{d\tau} \frac{dx^{\mathrm{T}}}{d\tau} = 0. \tag{2.3}$$

The above contains five independent equations. We are interested for now in the four equations for which $\mathrm{M} = \mu$, which specify motion in ordinary spacetime:

$$\frac{d^2 x^{\mu}}{d\tau^2} + \Gamma^{\mu}{}_{\Sigma\mathrm{T}} \frac{dx^{\Sigma}}{d\tau} \frac{dx^{\mathrm{T}}}{d\tau} = 0. \tag{2.4}$$

This expands, using the metric tensor symmetry $g_{\mathrm{MN}} = g_{\mathrm{NM}}$, to:

$$\frac{d^2 x^{\mu}}{d\tau^2} + \Gamma^{\mu}{}_{\sigma\tau} \frac{dx^{\sigma}}{d\tau} \frac{dx^{\tau}}{d\tau} + 2\Gamma^{\mu}{}_{5\sigma} \frac{dx^{5}}{d\tau} \frac{dx^{\sigma}}{d\tau} + \Gamma^{\mu}{}_{55} \frac{dx^{5}}{d\tau} \frac{dx^{5}}{d\tau} = 0. \tag{2.5}$$

Now, let us contrast (2.5) above to the gravitational geodesic equation which includes the Lorentz force law, namely, equation (20.41) of [10]:

$$\frac{d^2 x^{\mu}}{d\tau^2} + \Gamma^{\mu}{}_{\sigma\tau} \frac{dx^{\sigma}}{d\tau} \frac{dx^{\tau}}{d\tau} - \frac{q}{m} F^{\mu}{}_{\sigma} \frac{dx^{\sigma}}{d\tau} = 0. \tag{2.6}$$

We now take a critical step: *We require that the Lorentz force as expressed above, must be represented as nothing other than geodesic motion in the five-dimensional geometry.* The first two terms in (2.5) and (2.6) are identical, and they specify geodesic motion in an ordinary gravitational field absent any electrodynamic fields or sources. The absence of any mass or charge in the first two terms captures the Galilean principle of equivalence, and further expresses Newtonian inertial motion in a gravitational field via the Christoffel connections $\Gamma^{\mu}{}_{\sigma\tau}$.

If we require the Lorentz force to also be fashioned as geodesic motion through geometry, then we can do so by defining the third terms in (2.5) and (2.6) to be equivalent to one another, and the fourth term in (2.5) to be zero. Therefore, we now define:

$$2\Gamma^{\mu}{}_{5\sigma}\frac{dx^5}{d\tau}\frac{dx^\sigma}{d\tau} \equiv -\frac{q}{m}F^{\mu}{}_{\sigma}\frac{dx^\sigma}{d\tau}\text{, and} \tag{2.7}$$

$$\Gamma^{\mu}{}_{55} \equiv 0\,. \tag{2.8}$$

One might wish to consider $\Gamma^{\mu}{}_{55} \neq 0$, in which case $\Gamma^{\mu}{}_{55}\frac{dx^5}{d\tau}\frac{dx^5}{d\tau}$ in (2.5) would become an additional term in the Lorentz force law, but in the absence of experimental evidence for any deviations from the Lorentz force law, we shall proceed on the basis of (2.8).

The relationships (2.7) and (2.8), ensure that Lorentz force motion is in fact, no more and no less than geodesic motion in five dimensions. All else through section 7 will be deduced from (2.7) and (2.8).

3. Placing the Lorentz Force on a Geometrodynamic Footing as Geodesic Motion

Now, let us focus on equation (2.7). We can divide out $dx^\sigma/d\tau$ from (2.7), and then write the remaining terms as.

$$2\Gamma^{\mu}{}_{5\sigma}\frac{dx^5}{d\tau} \equiv -\sqrt{\frac{1}{\hbar c^5}}F^{\mu}{}_{\sigma}\frac{q}{m}\,, \tag{3.1}$$

where we have explicitly restored $\hbar = c = 1$. Now, we separate the proportionalities $dx^5/d\tau \propto q/m$ and $2\Gamma^{\mu}{}_{5\sigma} \propto -F^{\mu}{}_{\sigma}$, and turn the proportionalities $\propto$ into equalities by restoring their dimensional and numeric constants, starting with the former proportionality.

Irrespective of whether the fifth dimension is timelike or spacelike, we take dx^5 to be given in dimensions of time, so that $dx^5/d\tau$ is a dimensionless ratio. In the event that the fifth dimension is spacelike, one need merely divide through by c. In rationalized Heaviside-Lorentz units, the electric charge strength q (for a unit charge such as the electron, muon and tauon) is related to the dimensionless (running) coupling $\alpha = q^2/4\pi\hbar c$ which approaches $\alpha \to 1/137.036$ at low energy. The value of α is the same in all systems of units but the numerical value of q is different, so it is imperative that the exact expression for $dx^5/d\tau \propto q/m$ be based on α rather than q, and be independent of where the 4π factor appears. Further, to match dimensions with $\sqrt{\hbar c}$ the mass m needs to be multiplied by a factor of $\sqrt{G}$. Taking all of this into account, we now define:

$$\frac{dx^5}{d\tau} \equiv -\frac{1}{b}\frac{\sqrt{\hbar c \alpha}}{\sqrt{G}m} = -\frac{1}{b}\frac{1}{\sqrt{4\pi G}}\frac{q}{m} = -\frac{1}{\sqrt{\hbar c^5}}\frac{2}{b\overline{\kappa}}\frac{q}{m}. \tag{3.2}$$

where b is a dimensionless, numeric constant of proportionality that we are free at this moment to choose at will, which we will carry throughout the development, and which will ultimately be deduced to be $b^2 = 8$ when we obtain the Maxwell stress-energy tensor, see equations (10.14) and (10.15) infra. The equivalence between the first two terms is independent of the system of units but the terms containing q are in Heaviside-Lorentz units.

Then, we substitute (3.2) into (3.1) to obtain:

$$\Gamma^{\mu}{}_{5\sigma} \equiv \tfrac{1}{4} b\overline{\kappa} F^{\mu}{}_{\sigma}. \tag{3.3}$$

The definitions (3.2) and (3.3), together with $\Gamma^{\mu}{}_{55} \equiv 0$ from (2.8), when substituted into (2.5), turn the five-dimensional geodesic equation (2.5) into the Lorentz force law, and places this electrodynamic motion onto a totally-geometrodynamic footing. Of course, (3.3) is of further value, because it also relates the mixed field strength tensor $F^{\mu}{}_{\sigma}$ to the extra-dimensional connection components $\Gamma^{\mu}{}_{5\sigma}$, and this will lead to numerous other results. Although the $\Gamma^{\mathrm{M}}{}_{\Sigma\mathrm{T}}$ are not themselves tensors in general, (3.3) does suggest that that particular components $\Gamma^{\mu}{}_{5\sigma}$ do transform in the same way as the mixed tensor $F^{\mu}{}_{\sigma}$, multiplied by a the constant factor $\overline{\kappa}$. This "suggestion" is formally validated by the result (6.4), infra.

The question of whether the foregoing are fair suppositions, now rests on the correctness and sensibility of the deductions to which they lead.

4. Timelike versus Spacelike for the Fifth Dimension, and a Possible Connection to Intrinsic Spin

The results above are independent of whether the extra dimension is timelike or spacelike. In this section, we make a brief digression to examine each of these alternatives in a very basic way. This section can be safely skipped by the reader wishing to proceed straight into the main line of development.

Transforming into an "at rest" frame, $dx^1 = dx^2 = dx^3 = 0$, the spacetime metric equation $d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu$ reduces to $d\tau = \pm\sqrt{g_{00}} dx^0$, and (3.2) becomes:

$$\frac{dx^5}{dx^0} = \pm \frac{1}{b} \sqrt{\frac{g_{00}}{4\pi G}} \frac{q}{m} . \tag{4.1}$$

For a *timelike* fifth dimension, x^5 may be drawn as a second axis orthogonal to x^0, and the physics ratio q/m (which, by the way, results in the q/m material body in an electromagnetic field actually "feeling" a Newtonian force in the sense of $F = ma$ due to the *inequivalence* of electrical and inertial mass) measures the "angle" at which the material body moves through the x^5, x^0 "time plane."

For a *spacelike* fifth dimension, where one may wish to employ a compactified, hyper-cylindrical $x^5 \equiv R\phi$ (see [11], Figure 1) and R is a constant radius (distinguish from the Ricci scalar by context), $dx^5 \equiv Rd\phi$. Substituting this into (3.2), leaving in the $\pm$ ratio obtained in (4.1), and inserting c into the first term to maintain a dimensionless equation, then yields:

$$\frac{Rd\phi}{cd\tau} = \pm \frac{1}{b} \frac{\sqrt{\hbar c \alpha}}{\sqrt{G} m} = \pm \frac{1}{b} \frac{1}{\sqrt{4\pi G}} \frac{q}{m} . \tag{4.2}$$

We see that here, the physics ratio q/m measures an "angular frequency" of fifth-dimensional rotation. Interestingly, *this frequency runs inversely to the mass*, and by classical principles, this means that the angular momentum is independent of the mass, i.e., constant. If one doubles the mass, one halves the tangential velocity, and if the radius stays constant, then so too does the angular momentum. Together with the $\pm$ factor, one might suspect that this constant angular momentum is, by virtue of its constancy independently of mass, related to intrinsic spin. In fact, following this line of thought, one can arrive at an exact expression for the compactification radius R, in the following manner:

Assume that x^5 is spacelike, casting one's lot with the preponderance of those who study Kaluza-Klein theory. In (4.2), move the c away from the first term and move the m over to the first term. Then, multiply all terms by another R. Everything is now dimensioned as an angular momentum $m \cdot v \cdot R$, which we have just ascertained is constant irrespective of mass. So, set this all to $\pm\frac{1}{2}n\hbar$, which for $n=1$, represents intrinsic spin. The result is as follows:

$$m\frac{Rd\phi}{d\tau}R = \pm\frac{1}{b}\frac{\sqrt{\hbar c^3 \alpha}}{\sqrt{G}}R = \pm\frac{1}{b}\frac{c}{\sqrt{4\pi G}}qR = \pm\frac{1}{2}n\hbar . \tag{4.3}$$

Now, take the second and fourth terms, and solve for R with $n=1$, to yield:

$$R = \frac{b}{2\sqrt{\alpha}}\sqrt{\frac{G\hbar}{c^3}} = \frac{b}{2\sqrt{\alpha}}L_P , \tag{4.4}$$

where $L_P = \sqrt{G\hbar/c^3}$ is the Planck length. *This gives a definitive size for the compactification radius, and it is very close to the Planck length.* (Keep in mind that we will eventually find in (10.14) infra that $b^2 = 8$, so (4.4) will become $R = L_P\sqrt{2/\alpha}$.) What is of interest, is that α is a *running* coupling. At low probe energies, where $\alpha \to 1/137.036$, $R = 5.853 \cdot b \cdot L_P$. However, this is just the *apparent* radius relative to the low probe energy. If one were to probe to a regime where α becomes large, say, of order unity, $\alpha = 1$ then $R = \frac{b}{2}L_P$ is quite close to the Planck length of Wheeler's geometrodynamic vacuum "foam." [10] at §43.4, [12]* Since we have based the foregoing on a unit charge with spin ½, and since this is independent of the mass, the foregoing would appear to characterize the compactification radius R for all of the charged leptons, and to provide a geometric foundation for intrinsic spin. This suggests that for $\alpha = 1$ or on the order of unity, the compactification radius of the fifth dimension may become

---

* By way of review, the Planck mass, defined from the term atop Newton's law as a mass for which $G{M_P}^2 = \hbar c$, is thus $M_P = \sqrt{\hbar c/G}$. In the geometrodynamic vacuum, the negative gravitational energy between Planck masses separated by the Planck length $L_P = \sqrt{G\hbar/c^3}$ precisely counterbalances and cancels the positive energy of the Planck masses themselves. The Schwarzschild radius of a Plank mass $R_S = 2GM_P/c^2 = 2\sqrt{G\hbar/c^3} = 2L_P$.

synonymous with the Planck length itself, or the Schwarzschild radius of the vacuum, or something close to one of both of these.

While (4.2) applies generally for a compactified spacelike fifth dimension, before proceeding too far with this intrinsic spin interpretation (4.3), however, it is worth noting that for a neutral body, $q=0$, such as the neutrino, we have $d\phi / d\tau = 0$, and so there is no fifth-dimensional rotation. More generally, any electrically-neutral body must be considered to be non-moving through the x^5 dimension, $dx^5 = 0$. This would suggest that the neutrino has no intrinsic spin, which is, of course, contradicted by empirical knowledge. So, (4.3), while intriguing, does need to be studied further. Also, the intrinsic spin interpretation (4.3) suggests conversely, that any elementary scalar particle which has no intrinsic spin, must be electrically neutral. This is, in fact, true of the hypothesized Higgs boson. [13]

Despite the above puzzle regarding the neutrino, which we will return to in the hindsight of the conclusion, the use of the term "intrinsic" to describe an inherent quantized angular momentum of elementary particles, covers up what is actually a deep ignorance of what this really means. Why? For a material body to have an angular momentum, one must implicitly consider a radius R with which that body circles about an origin. At the same time, nobody believes that intrinsic spin represents an angular momentum about a radius R in the three usual spatial dimensions. By associating intrinsic spin with motion through a fourth, compactified, hyper-cylindrical spatial dimension, one simultaneously makes sense of intrinsic spin and of a compact fourth spatial dimension. The material body now has a spatial radius R outside of the usual three spatial dimensions to give meaning to its "intrinsic" spin, and the compactified fourth dimension now takes on meaning as something which is physically observed, via the phenomenon of intrinsic spin, and not merely a fictional idea that gives people pause about Klauza-Klein theories specifically, and dimensional compactification in general.

5. Symmetric Gravitation and Antisymmetric Electrodynamics

Now, following the brief digression in section 4, let us turn back to the association $\Gamma^{\mu}{}_{5\sigma} \equiv \frac{1}{4} b\overline{\kappa} F^{\mu}{}_{\sigma}$ in (3.3), which arises from the requirement that the Lorentz force be represented as geodesic motion in five dimensions. We know that $F^{\mu\nu} = -F^{\nu\mu}$ is an antisymmetric tensor. By virtue of (3.3), this will place certain constraints on the related Christoffel connections

$\Gamma^{M}{}_{5T} = \frac{1}{2} g^{MA}\left(g_{A5,T} + g_{TA,5} - g_{5T,A}\right)$, and it is important to find out what these are. These constraints, in the next section, will provide the basis for placing Maxwell's equations onto a purely geometrodynamic footing.

First, because we are working in five dimensions, we will find it desirable to generalize $F^{\mu\nu}$ to F^{MN}. We make no *a priori* supposition about the additional components in F^{MN}, other than to require that they be antisymmetric, $F^{MN} \equiv -F^{NM}$. *Any other information about these new components is to be deduced, not imposed.* Second, we generalize (3.3) into the full five dimensions, thus:

$$\Gamma^{M}{}_{5\Sigma} = \tfrac{1}{4} b\overline{\kappa} F^{M}{}_{\Sigma} . \tag{5.1}$$

By virtue of (2.8), $\Gamma^{\mu}{}_{55} \equiv 0$, we may immediately deduce that:

$$\Gamma^{\mu}{}_{55} = \tfrac{1}{4} b\overline{\kappa} F^{\mu}{}_{5} = 0 . \tag{5.2}$$

As it stands, $F^{M}{}_{\Sigma}$ is a mixed tensor, and it would be better to raise this into contravariant form where we can clearly examine the consequences of having an antisymmetric field strength $F^{MN} \equiv -F^{NM}$. Thus, let us now raise the lower index in (5.1), and at the same time equate this to the Christoffel connections, as such:

$$\tfrac{1}{4} b\overline{\kappa} F^{MN} = \tfrac{1}{4} b\overline{\kappa} g^{\Sigma N} F^{M}{}_{\Sigma} = g^{\Sigma N}\Gamma^{M}{}_{5\Sigma} = \tfrac{1}{2} g^{MA} g^{\Sigma N}\left(g_{A5,\Sigma} + g_{\Sigma A,5} - g_{5\Sigma,A}\right). \tag{5.3}$$

Now, we use (5.3) to write $F^{MN} = -F^{NM}$ completely in terms of the metric tensor g_{MN} and its first derivatives, as:

$$\tfrac{1}{4} b\overline{\kappa} F^{MN} = -\tfrac{1}{4} b\overline{\kappa} F^{NM} = g^{MA} g^{\Sigma N}\left(g_{A5,\Sigma} + g_{\Sigma A,5} - g_{5\Sigma,A}\right) = -g^{NA} g^{\Sigma M}\left(g_{A5,\Sigma} + g_{\Sigma A,5} - g_{5\Sigma,A}\right). \tag{5.4}$$

Renaming indexes, and using the symmetry of the metric tensor, this is readily reduced to::

$$g^{M\Sigma} g^{TN} g_{T\Sigma,5} = 0 . \tag{5.5}$$

This is an alternative, geometric way of saying that $F^{MN} = -F^{NM}$.

We can further simplify this using the inverse relationship $g^{TN} g_{T\Sigma} = \delta^{N}{}_{\Sigma}$, which we can differentiate to obtain $\left(g^{TN} g_{T\Sigma}\right)_{,A} = g^{TN}{}_{,A}\, g_{T\Sigma} + g^{TN} g_{T\Sigma,A} = 0$, i.e., $g^{TN} g_{T\Sigma,A} = -g^{TN}{}_{,A}\, g_{T\Sigma}$. This can then be used with $\mathrm{A}=5$ to reduce (5.4) to the very simple expressions, for both the covariant and contravariant metric tensor:

$$g^{\mathrm{MN}}{}_{,5}=0;\ g_{\mathrm{MN},5}=0. \tag{5.6}$$

This states that *all components of the metric tensor are constant when differentiated with respect to the fifth dimension*.

Now, we return to write out $\Gamma^{\mu}{}_{55}=\frac{1}{2}g^{\mu\mathrm{A}}\left(g_{\mathrm{A}5,5}+g_{5\mathrm{A},5}-g_{55,\mathrm{A}}\right)=0$ from (2.8), see also (5.2). Combined with $g_{\mathrm{MN},5}=0$ above and $g^{\mathrm{TN}}g_{\mathrm{T\Sigma,A}}=-g^{\mathrm{TN}}{}_{,\mathrm{A}}g_{\mathrm{T\Sigma}}$ we further deduce that:

$$g^{55}{}_{,\mathrm{A}}=0;\ g_{55,\mathrm{A}}=0 \tag{5.7}$$

This means, quite importantly, that $g_{55}=\text{constant}$ and $g^{55}=\text{constant}$, *everywhere in the five-dimensional geometry*.

To fix these constant values, consider the weak-field limit $g_{\mathrm{MN}}\rightarrow\eta_{\mathrm{MN}}$. If the fifth dimension is timelike, $\mathrm{diag}\left(\eta_{\mu\nu}\right)=\left(+1,-1,-1,-1,+1\right)$ and $g_{55}=g^{55}=+1$. If it is spacelike (briefly explored regarding intrinsic spin in section 4), then $\mathrm{diag}\left(\eta_{\mathrm{MN}}\right)=\left(+1,-1,-1,-1,-1\right)$ and $g_{55}=g^{55}=-1$. But, by (5.7), if the above expressions for g_{55} and g^{55} are true *anywhere*, then they are true *everywhere*. Therefore:

$$g_{55}=g^{55}=+1,\ \text{or}\ g_{55}=g^{55}=-1, \tag{5.8}$$

respectively, for a timelike or spacelike fifth dimension. In either case, timelike or spacelike, $g^{55}g_{55}=1$. The inverse $g^{\mathrm{T}5}g_{\mathrm{T}5}=g^{\tau 5}g_{\tau 5}+g^{55}g_{55}=g^{\tau 5}g_{\tau 5}+1=\delta^{5}{}_{5}=1$ then leads also to the null condition:

$$g^{\tau 5}g_{\tau 5}=0, \tag{5.9}$$

which applies *irrespective* of the timelike versus spacelike choice.

Finally, using (5.1) together with (5.6) and (5.7), we may deduce:

$$\tfrac{1}{4}b\overline{\kappa}F^{5}{}_{5}=\Gamma^{5}{}_{55}=\tfrac{1}{2}g^{5\mathrm{A}}\left(g_{\mathrm{A}5,5}+g_{5\mathrm{A},5}-g_{55,\mathrm{A}}\right)=0. \tag{5.10}$$

Taking this together with (5.2), $\Gamma^{\mu}{}_{55} = \frac{1}{4} b\overline{\kappa} F^{\mu}{}_{5} = 0$, we have now deduced that all of the newly-introduced fifth-dimensional components for the mixed field strength tensor are zero, i.e.,

$$\frac{1}{4} b\overline{\kappa} F^{M}{}_{5} = \Gamma^{M}{}_{55} = 0. \tag{5.11}$$

The free index in $F^{M}{}_{5} = 0$ above can easily be lowered to also find that the covariant:

$$F_{M5} = -F_{5M} = 0. \tag{5.12}$$

But, since the ordinary spacetime components of $F^{\mu}{}_{\nu}$ are non-zero, one should take care to ensure that the contravariant tensor components $F^{M5} = -F^{5M} = 0$ as well, that is, we want to make sure that the fixed index "5" in (5.11) can properly be raised. One can employ (5.1) together with the explicit components for $\Gamma^{M}{}_{5\Sigma}$ to write:

$$F^{MN} = g^{\Sigma N} F^{M}{}_{\Sigma} = g^{\Sigma N} \Gamma^{M}{}_{5\Sigma} = \frac{1}{2} g^{\Sigma N} g^{MA} \left(g_{A5,\Sigma} + g_{\Sigma A,5} - g_{5\Sigma,A} \right). \tag{5.13}$$

Expanding this to separate the μ from the 5 components, and applying (5.6), (5.7) and (5.9) as needed, together with $F^{MN} = -F^{NM}$ to eliminate the only term which (5.6), (5.7) and (5.9) cannot directly eliminate, one can indeed deduce that in addition to (5.11) and (5.12):

$$F^{M5} = -F^{5M} = 0. \tag{5.14}$$

Now, the free index can be easily lowered, referring also to (5.1), to find that:

$$\frac{1}{4} b\overline{\kappa} F^{5}{}_{M} = \Gamma^{5}{}_{5M} = \Gamma^{5}{}_{M5} = 0. \tag{5.15}$$

i.e., $F^{5}{}_{M} = 0$. So, we find that all of the newly-introduced fifth-dimensional components of the field strength tensor F^{MN}, whether in raised, lowered, or either mixed form, are equal to zero. Equations (5.11), $\Gamma^{M}{}_{55} = 0$, and (5.15), $\Gamma^{5}{}_{5M} = \Gamma^{5}{}_{M5} = 0$, taken together, tell us that as well, the "rule" that any Christoffel connection with "two or more fifth-dimension indexes," is also equal to zero.

Combining (5.1) with $F^{M5} = -F^{5M} = 0$ as well as $F_{M5} = -F_{5M} = 0$, we may deduce two further relationships:

$$g^{\Sigma M} \Gamma^{5}{}_{5\Sigma} = -g^{\Sigma 5} \Gamma^{M}{}_{5\Sigma} = 0 \text{ and } g_{TM} \Gamma^{M}{}_{55} = -g_{5M} \Gamma^{M}{}_{5T} = 0, \tag{5.16}$$

which are variations of the "two or more fifth dimension index" rule noted above.

It is also helpful as we shall soon see when we examine the Riemann tensor, to make note of the fact that:

$$\Gamma^{M}{}_{\Sigma T,5} = \tfrac{1}{2} g^{MA}{}_{,5}\left(g_{A\Sigma,T} + g_{TA,\Sigma} - g_{\Sigma T,A}\right) + \tfrac{1}{2} g^{MA}\left(g_{A\Sigma,T,5} + g_{TA,\Sigma,5} - g_{\Sigma T,A,5}\right) = 0 . \qquad (5.17)$$

This makes use of (5.6) and the fact that ordinary derivatives commute. A further variation of (5.17) employs (5.1) to also write, for the field strength tensor:

$$\Gamma^{M}{}_{5\Sigma,5} = \tfrac{1}{4} b\bar{\kappa} F^{M}{}_{\Sigma,5} = 0 . \qquad (5.18)$$

i.e., $F^{M}{}_{\Sigma,5} = 0$. Lust like the metric tensor, *all components of the field strength tensor are constant when differentiated with respect to the fifth dimension*.

Again, at bottom, every result in this section is a consequence of relationships (5.1) and (5.2), taken in combination with the antisymmetric field strength $F^{MN} \equiv -F^{NM}$. Now, we have the tools required to turn to the Riemann tensor, and to Maxwell's equations.

6. Maxwell's Equations as Pure Geometry

We have shown how Lorentz force motion might be described as simple geodesic motion in a five-dimensional Kaluza-Klein spacetime geometry. But equations of motion are only one part of a complete (classical) field theory. The other part is a specification of how the "sources" of that theory create the "fields" originating from those sources. In a complete theory, the equations of motion then describe motion through the fields originating from the sources. It is now time to place Maxwell's equations on a firm geometric footing.

In five dimensions, we specify the Riemann tensor in the usual way, albeit with an extra fifth-dimensional index. That is:

$$R^{A}{}_{BMN} = -\Gamma^{A}{}_{BM,N} + \Gamma^{A}{}_{BN,M} + \Gamma^{\Sigma}{}_{BN}\Gamma^{A}{}_{\Sigma M} - \Gamma^{\Sigma}{}_{BM}\Gamma^{A}{}_{\Sigma N} . \qquad (6.1)$$

Now, let's consider the $M = 5$ component of this equation, that is:

$$R^{A}{}_{B5N} = -\Gamma^{A}{}_{B5,N} + \Gamma^{A}{}_{BN,5} + \Gamma^{\Sigma}{}_{BN}\Gamma^{A}{}_{\Sigma 5} - \Gamma^{\Sigma}{}_{B5}\Gamma^{A}{}_{\Sigma N} . \qquad (6.2)$$

By virtue of $\Gamma^{M}{}_{\Sigma T,5} = 0$, equation (5.17), which is in turn a consequence of $g_{MN,5} = 0$, which is in turn a consequence of $F^{MN} \equiv -F^{NM}$, the second term zeros out, and (6.2) becomes:

$$R^{A}{}_{B5N} = -\Gamma^{A}{}_{B5,N} + \Gamma^{\Sigma}{}_{BN}\Gamma^{A}{}_{\Sigma 5} - \Gamma^{\Sigma}{}_{B5}\Gamma^{A}{}_{\Sigma N} . \qquad (6.3)$$

Substituting (5.1), i.e., $\Gamma^{\mathrm{M}}{}_{5\Sigma} = \frac{1}{4} b\overline{\kappa} F^{\mathrm{M}}{}_{\Sigma}$ into the above, and with some minor term rearrangement, we immediately arrive at the *very critical expression*:

$$R^{\mathrm{A}}{}_{\mathrm{B5N}} = -\tfrac{1}{4} b\overline{\kappa}\left(F^{\mathrm{A}}{}_{\mathrm{B,N}} + \Gamma^{\mathrm{A}}{}_{\Sigma\mathrm{N}} F^{\Sigma}{}_{\mathrm{B}} - \Gamma^{\Sigma}{}_{\mathrm{BN}} F^{\mathrm{A}}{}_{\Sigma}\right) = -\tfrac{1}{4} b\overline{\kappa} F^{\mathrm{A}}{}_{\mathrm{B;N}} . \tag{6.4}$$

In particular, these three remaining terms of $R^{\mathrm{A}}{}_{\mathrm{B5N}}$ *turn out to be identical with the expression for the gravitationally-covariant derivative* $F^{\mathrm{A}}{}_{\mathrm{B;N}}$ *of the mixed field strength tensor*, times the constant factor $-\frac{1}{4} b\overline{\kappa}$. This leads us immediately to a geometric foundation for Maxwell's equations in the following way:

As regards *Maxwell's electric charge equation*, we contract (6.4) down to its Ricci tensor component R_{B5} and define a five-current J_{B} with covariant 5-space index:

$$R_{\mathrm{B5}} = R^{\Sigma}{}_{\mathrm{B5}\Sigma} = -\tfrac{1}{4} b\overline{\kappa}\left(F^{\Sigma}{}_{\mathrm{B},\Sigma} + \Gamma^{\Sigma}{}_{\mathrm{T}\Sigma} F^{\mathrm{T}}{}_{\mathrm{B}} - \Gamma^{\mathrm{T}}{}_{\mathrm{B}\Sigma} F^{\Sigma}{}_{\mathrm{T}}\right) = -\tfrac{1}{4} b\overline{\kappa} F^{\Sigma}{}_{\mathrm{B};\Sigma} \equiv -\tfrac{1}{4} b\overline{\kappa} J_{\mathrm{B}} . \tag{6.5}$$

Now, we separate this into the two equations as such:

$$R_{\beta 5} = -\tfrac{1}{4} b\overline{\kappa}\left(F^{\sigma}{}_{\beta,\sigma} + \Gamma^{\sigma}{}_{\tau\sigma} F^{\tau}{}_{\beta} - \Gamma^{\tau}{}_{\beta\sigma} F^{\sigma}{}_{\tau}\right) = -\tfrac{1}{4} b\overline{\kappa} F^{\sigma}{}_{\beta;\sigma} \equiv -\tfrac{1}{4} b\overline{\kappa} J_{\beta}, \text{ and} \tag{6.6}$$

$$R_{55} = -\tfrac{1}{4} b\overline{\kappa}\left(F^{\Sigma}{}_{5,\Sigma} + \Gamma^{\Sigma}{}_{\mathrm{T}\Sigma} F^{\mathrm{T}}{}_{5} - \Gamma^{\mathrm{T}}{}_{5\Sigma} F^{\Sigma}{}_{\mathrm{T}}\right) = -\tfrac{1}{4} b\overline{\kappa} F^{\Sigma}{}_{5;\Sigma} \equiv -\tfrac{1}{4} b\overline{\kappa} J_{5} . \tag{6.7}$$

In (6.6), note that because $F^{5}{}_{\Sigma} = 0$ and $\Gamma^{5}{}_{\mathrm{T5}} = 0$ (see 5.15), we can easily drop the Σ, T indexes down to σ, τ. In (6.7), however, we leave $F^{\Sigma}{}_{5;\Sigma}$ as is because as we shall note in a moment, this term is not zero.

In (6.6), we discern the four-covariant derivative $F^{\sigma}{}_{\beta;\sigma} = F^{\sigma}{}_{\beta,\sigma} + \Gamma^{\sigma}{}_{\tau\sigma} F^{\tau}{}_{\beta} - \Gamma^{\tau}{}_{\beta\sigma} F^{\sigma}{}_{\tau}$, which is what allowed us to drop $F^{\Sigma}{}_{\beta;\Sigma}$ to $F^{\sigma}{}_{\beta;\sigma}$. This means that $J_{\beta} = F^{\sigma}{}_{\beta;\sigma}$ is the observed electromagnetic current source density, with covariant index. *This is Maxwell's electric charge equation, on a geometric foundation.*

For the fifth-dimensional component R_{55} in (6.7), we can use $F^{\mathrm{T}}{}_{5} = 0$ to eliminate the first two terms inside the parenthesis, but the third term is *not* zero. For the third term, we again employ the substitution $\Gamma^{\mathrm{M}}{}_{5\Sigma} = \frac{1}{4} b\overline{\kappa} F^{\mathrm{M}}{}_{\Sigma}$ from (5.1). Thus:

$$R_{55} = -\tfrac{1}{16} b^{2} \overline{\kappa}^{2} F^{\sigma\tau} F_{\sigma\tau} = -\tfrac{1}{4} b\overline{\kappa} F^{\Sigma}{}_{5;\Sigma} = -\tfrac{1}{4} b\overline{\kappa} J_{5} . \tag{6.8}$$

In the above, we have used $F^{\mathrm{T}}{}_{\Sigma}F^{\Sigma}{}_{\mathrm{T}} = F^{\mathrm{T}\Sigma}F_{\Sigma\mathrm{T}} = -F^{\Sigma\mathrm{T}}F_{\Sigma\mathrm{T}} = -F^{\sigma\tau}F_{\sigma\tau}$. Note, that we raise and lower indexes while they are five-dimensional, then we reduce to lowercase Greek indexes via $F^{\Sigma 5} = F_{\Sigma 5} = 0$.

Now, we begin to notice a significant result: Despite the $F^{\Sigma}{}_{5;\Sigma} = F^{\sigma}{}_{5;\sigma} + F^{5}{}_{5;5}$ term in (6.8) containing components of a mixed tensor which vanish in their own right, namely $F^{\Sigma}{}_{5} = 0$, this term for R_{55} is *not* equal to zero, and so, $F^{\Sigma}{}_{5;\Sigma} \neq 0$. Rather, we find that the covariant derivative term $F^{\Sigma}{}_{5;\Sigma} = F^{\sigma}{}_{5;\sigma} + F^{5}{}_{5;5} \neq 0$ *does not vanish* even though $F^{\Sigma}{}_{5} = 0$, and in fact, leaves a very central term $F^{\sigma\tau}F_{\sigma\tau}$ found in the QED free-field Lagrangian $\mathcal{L}_{QCD\,(Free)} = -\frac{1}{4}F^{\sigma\tau}F_{\sigma\tau}$ and in $T^{\mu}{}_{\nu\,Maxwell} = -\left(F^{\mu\sigma}F_{\nu\sigma} - \frac{1}{4}\delta^{\mu}{}_{\nu}F^{\sigma\tau}F_{\sigma\tau}\right)$, the Maxwell stress-energy tensor in Heaviside-Lorentz units. One may think of $F^{\Sigma}{}_{5;\Sigma} = \frac{1}{4}b\overline{\kappa}F^{\sigma\tau}F_{\sigma\tau} \neq 0$ as being "gravitationally induced" out of $F^{\Sigma}{}_{5} = 0$, solely as a *non-linear gravitational effect*, because in the absence of gravitation, covariant derivatives approach ordinary derivatives and so $F^{\Sigma}{}_{5;\Sigma} \rightarrow F^{\Sigma}{}_{5,\Sigma} = 0$. This induced term originates from the final term $-\Gamma^{\Sigma}{}_{\mathrm{BM}}\Gamma^{\mathrm{A}}{}_{\Sigma\mathrm{N}}$ of the Riemann tensor $R^{\mathrm{A}}{}_{\mathrm{BMN}}$, via the progression $\Gamma^{\Sigma}{}_{\mathrm{BN}}\Gamma^{\mathrm{A}}{}_{\Sigma\mathrm{M}} \rightarrow \Gamma^{\Sigma}{}_{5\mathrm{T}}\Gamma^{\mathrm{T}}{}_{\Sigma 5} = \frac{1}{16}b^2\overline{\kappa}^2F^{\Sigma}{}_{\mathrm{T}}F^{\mathrm{T}}{}_{\Sigma}$, starting from (6.1), and using $\Gamma^{\mathrm{M}}{}_{5\Sigma} = \frac{1}{4}b\overline{\kappa}F^{\mathrm{M}}{}_{\Sigma}$ from (5.1).

So, the upshot of (6.8), is that the fifth component of the five-covariant current source density in a five-dimensional spacetime, $J_5 = F^{\Sigma}{}_{5;\Sigma} = \frac{1}{4}b\overline{\kappa}F^{\sigma\tau}F_{\sigma\tau}$, is not zero despite $F^{\Sigma}{}_{5} = 0$, is gravitationally-induced from the term $\Gamma^{\Sigma}{}_{\mathrm{BN}}\Gamma^{\mathrm{A}}{}_{\Sigma\mathrm{M}}$ in the Riemann tensor, and carries the $F^{\sigma\tau}F_{\sigma\tau}$ scalar which is central to QED and the Maxwell stress-energy tensor and which, in the free-field Lagrangian density, represents the kinetic energy of a photon.

Turning now to Maxwell's magnetic equation, we first lower the A index in (6.4), $R_{5\mathrm{NMB}} = g_{\mathrm{MA}}R^{\mathrm{A}}{}_{\mathrm{B5N}}$, and use $R_{\mathrm{ABMN}} = R_{\mathrm{MNAB}}$ to write:

$$R_{5\mathrm{NMB}} = -\tfrac{1}{4}b\overline{\kappa}\left(g_{\mathrm{MA}}F^{\mathrm{A}}{}_{\mathrm{B,N}} + g_{\mathrm{MA}}\Gamma^{\mathrm{A}}{}_{\Sigma\mathrm{N}}F^{\Sigma}{}_{\mathrm{B}} - g_{\mathrm{MA}}\Gamma^{\Sigma}{}_{\mathrm{BN}}F^{\mathrm{A}}{}_{\Sigma}\right) = -\tfrac{1}{4}b\overline{\kappa}g_{\mathrm{MA}}F^{\mathrm{A}}{}_{\mathrm{B;N}} = -\tfrac{1}{4}b\overline{\kappa}F_{\mathrm{MB;N}}. \quad (6.9)$$

Maxwell's magnetic equation then arises straight from the 5-dimensional rendition of the "first" Bianchi identity:

$$R_{\mathrm{MNAB}} + R_{\mathrm{MABN}} + R_{\mathrm{MBNA}} = 0. \quad (6.10)$$

Making use of (6.9), the $\mathrm{M}=5$ component of this is:

$$R_{5\mathrm{NAB}}+R_{5\mathrm{ABN}}+R_{5\mathrm{BNA}}=-\tfrac{1}{4}b\bar{\kappa}\left(F_{\mathrm{AB;N}}+F_{\mathrm{BN;A}}+F_{\mathrm{NA;B}}\right)=-\tfrac{1}{4}b\bar{\kappa}\left(F_{\mathrm{AB,N}}+F_{\mathrm{BN,A}}+F_{\mathrm{NA,B}}\right)=0\,, \quad (6.11)$$

where we account for the well-known fact that in the cyclic combination of (6.11) with antisymmetric tensors, the Christoffel terms in the covariant derivatives cancel identically, so the covariant derivatives becomes ordinary derivatives. In the $\mathrm{NAB}=\nu\alpha\beta$ subset of this, we immediately obtain Maxwell's magnetic equation

$$F_{\alpha\beta,\nu}+F_{\beta\nu,\alpha}+F_{\nu\alpha,\beta}=0\,. \quad (6.12)$$

In light of our earlier having found some new terms in Maxwell's electric charge equation arising from the fifth dimension, see, e.g., the R_{55} equation in (6.8), one may ask whether there are any additional electrodynamic terms of interest in the (6.11) above, in the circumstance where more than a single fifth-dimensional index is employed. Because $R_{\mathrm{ABMN}}=R_{\mathrm{MNAB}}=-R_{\mathrm{BAMN}}$, it is clear that with more than two fifth-dimensional indexes, e.g., $R_{555\mu}$, (6.11) will identically reduce to zero. But we should explore whether there is any additional electrodynamic information to be gleaned when exactly two fifth-dimensional indexes are used in (6.11). Thus, we may examine, say:

$$R_{55\mathrm{AB}}+R_{5\mathrm{AB}5}+R_{5\mathrm{B}5\mathrm{A}}=-\tfrac{1}{4}b\bar{\kappa}\left(F_{\mathrm{AB;5}}+F_{\mathrm{B5;A}}+F_{\mathrm{5A;B}}\right)=-\tfrac{1}{4}b\bar{\kappa}\left(F_{\mathrm{AB,5}}+F_{\mathrm{B5,A}}+F_{\mathrm{5A,B}}\right)=0\,. \quad (6.13)$$

We learn from (6.8), especially $F^{\sigma}{}_{5;\sigma}=\tfrac{1}{4}b\bar{\kappa}F^{\sigma\tau}F_{\sigma\tau}\neq 0$, not to automatically eliminate a field strength term such as $F^{\sigma}{}_{5}$ when it appears in a *covariant* derivative, i.e., $F^{\sigma}{}_{5;\sigma}$. However, the migration of covariant to ordinary derivatives in the cyclic combination of (6.11) removes this complication. We know from (5.12) that $F_{\mathrm{B}5}=F_{5\mathrm{A}}=0$, so their the *ordinary* derivatives of these will vanish as well. The remaining $F_{\mathrm{AB},5}=\left(g_{\mathrm{A}\Sigma}F^{\Sigma}{}_{\mathrm{B}}\right)_{,5}=g_{\mathrm{A}\Sigma,5}F^{\Sigma}{}_{\mathrm{B}}+g_{\mathrm{A}\Sigma}F^{\Sigma}{}_{\mathrm{B},5}=0$ in (6.13), by virtue of (5.6), $g_{\mathrm{A}\Sigma,5}=0$, and (5.18), $F^{\Sigma}{}_{\mathrm{B},5}=0$. Thus, (6.13) is identically equal to zero, not only because of the Bianchi identity, but because of the inherent properties of the F_{AB} and g_{AB} developed in section 5. Thus, there is no additional electrodynamic information to be gleaned from (6.13).

We have now placed each of Maxwell's equations on a solely geometric footing. Maxwell's source equation in covariant (lower index) form is specified by (6.6), namely, $R_{\beta 5} = -\frac{1}{4} b\overline{\kappa} J_{\beta} = -\frac{1}{4} b\overline{\kappa} F^{\sigma}{}_{\beta;\sigma}$. The fifth component of this source equation, (6.8), contains the very central term $\mathfrak{L}_{QCD\,(Free)} = -\frac{1}{4} F^{\sigma\tau} F_{\sigma\tau}$, which is central to QED and to the Maxwell stress-energy tensor. Maxwell's magnetic equation is simply a fifth-dimensional component (6.11) of the first Bianchi identity $R_{\mathrm{MNAB}} + R_{\mathrm{MABN}} + R_{\mathrm{MBNA}} = 0$. And, the Lorentz force equation (2.6), upon which the foregoing geometrization of Maxwell's equations is based, is merely the equation for four-space geodesic motion in the five-dimensional geometry, $\frac{d^2 x^{\mu}}{d\tau^2} + \Gamma^{\mu}{}_{\Sigma\mathrm{T}} \frac{dx^{\Sigma}}{d\tau} \frac{dx^{\mathrm{T}}}{d\tau} = 0$, (2.4). With source equations producing fields and with material bodies in those fields moving over geodesics that are identical to and synonymous with the Lorentz force, Maxwell's classical electrodynamics with the Lorentz force law now rests on the firm geometrodynamic footing of a five-dimensional Kaluza-Klein geometry. Now, let's turn our efforts toward deriving the energy tensors and scalars associated with the foregoing.

7. Calculation of the Five-Dimensional Curvature Scalar

We begin discussion here by deriving the *five-dimensional* Ricci curvature scalar $R_{(5)} \equiv R^{\Sigma}{}_{\Sigma} = R + R^{5}{}_{5}$, where the ordinary *four-dimensional* curvature scalar $R = R^{\sigma}{}_{\sigma}$. We'll start with $R^{5}{}_{5}$.

In (6.5), we have already found $R_{\mathrm{B}5}$. So, all we need do is raise the index using $R^{\mathrm{M}}{}_{5} = g^{\mathrm{MB}} R_{\mathrm{B}5} = -\frac{1}{4} b\overline{\kappa} g^{\mathrm{MB}} F^{\Sigma}{}_{\mathrm{B};\Sigma} = -\frac{1}{4} b\overline{\kappa} F^{\Sigma\mathrm{M}}{}_{;\Sigma} = -\frac{1}{4} b\overline{\kappa} J^{\mathrm{M}}$, i.e.,

$$R^{\mathrm{M}}{}_{5} = -\tfrac{1}{4} b\overline{\kappa} F^{\Sigma\mathrm{M}}{}_{;\Sigma} = -\tfrac{1}{4} b\overline{\kappa}\left(F^{\Sigma\mathrm{M}}{}_{,\Sigma} + \Gamma^{\Sigma}{}_{\mathrm{T}\Sigma} F^{\mathrm{TM}} + \Gamma^{\mathrm{M}}{}_{\mathrm{T}\Sigma} F^{\Sigma\mathrm{T}}\right) = -\tfrac{1}{4} b\overline{\kappa} J^{\mathrm{M}}, \tag{7.1}$$

and then take the $\mathrm{M} = 5$ component. Above, we simply employ the definition of the covariant derivative of a second-rank contravariant tensor, particularly, of $F^{\Sigma\mathrm{M}}{}_{;\Sigma}$.

Now, we separate (7.1) into:

$$R^{\mu}{}_{5} = -\tfrac{1}{4} b\overline{\kappa} F^{\sigma\mu}{}_{;\sigma} = -\tfrac{1}{4} b\overline{\kappa}\left(F^{\sigma\mu}{}_{,\sigma} + \Gamma^{\sigma}{}_{\tau\sigma} F^{\tau\mu} + \Gamma^{\mu}{}_{\tau\sigma} F^{\tau\sigma}\right) = -\tfrac{1}{4} b\overline{\kappa} J^{\mu}, \text{ and} \tag{7.2}$$

$$R^{5}{}_{5} = -\tfrac{1}{4} b\overline{\kappa} F^{\Sigma 5}{}_{;\Sigma} = -\tfrac{1}{4} b\overline{\kappa}\left(F^{\Sigma 5}{}_{,\Sigma} + \Gamma^{\Sigma}{}_{\mathrm{T}\Sigma} F^{\mathrm{T}5} + \Gamma^{5}{}_{\mathrm{T}\Sigma} F^{\Sigma\mathrm{T}}\right) = -\tfrac{1}{4} b\overline{\kappa} J^{\mathrm{M}}. \tag{7.3}$$

In the former equation, (7.2), we employ the same set of reductions used in (6.6), and we see that $R^{\mu}{}_{5}$ contains the contravariant current source density $J^{\mu} \equiv \left(\rho, J_{(1)}, J_{(2)}, J_{(3)}\right)$. In (7.3), the first two terms can be eliminated because $F^{\mathrm{T}5} = 0$, so with suitable upper-to-lower-case reduction of Greek indexes also via $F^{\mathrm{T}5} = 0$, we have:

$$R^{5}{}_{5} = -\tfrac{1}{4} b\overline{\kappa} F^{\Sigma 5}{}_{;\Sigma} = -\tfrac{1}{4} b\overline{\kappa} \Gamma^{5}{}_{\tau\sigma} F^{\sigma\tau} = -\tfrac{1}{4} b\overline{\kappa} J^{5} . \tag{7.4}$$

While (5.1) tells us that $\Gamma^{\mathrm{M}}{}_{5\Sigma} = \frac{1}{4} b\overline{\kappa} F^{\mathrm{M}}{}_{\Sigma}$, this is the first time we have had to work with $\Gamma^{5}{}_{\tau\sigma}$, and because $\Gamma^{5}{}_{\tau\sigma} = \Gamma^{5}{}_{\sigma\tau}$, this cannot be related directly to $F_{\tau\sigma} = -F_{\sigma\tau}$. So, let's find out where the $F^{\sigma\tau} F_{\sigma\tau}$ term comes in.

Another way to arrive at (7.1) from (6.5) is to write:

$$R^{\mathrm{M}}{}_{5} = g^{\mathrm{MB}} R_{\mathrm{B}5} = -\tfrac{1}{4} b\overline{\kappa}\left(g^{\mathrm{MB}} F^{\Sigma}{}_{\mathrm{B},\Sigma} + g^{\mathrm{MB}} \Gamma^{\Sigma}{}_{\mathrm{T}\Sigma} F^{\mathrm{T}}{}_{\mathrm{B}} - g^{\mathrm{MB}} \Gamma^{\mathrm{T}}{}_{\mathrm{B}\Sigma} F^{\Sigma}{}_{\mathrm{T}}\right) = -\tfrac{1}{4} b\overline{\kappa} F^{\Sigma\mathrm{M}}{}_{;\Sigma} \equiv -\tfrac{1}{4} b\overline{\kappa} J^{\mathrm{M}} , \tag{7.5}$$

which merely entails using the g^{MB} to raise the indexes in a five-covariant manner. This equation is identical to (7.1), just in a different form. The $M = 5$ component is then:

$$R^{5}{}_{5} = -\tfrac{1}{4} b\overline{\kappa}\left(g^{5\beta}\left(F^{\sigma}{}_{\beta,\sigma} + \Gamma^{\sigma}{}_{\tau\sigma} F^{\tau}{}_{\beta} - \Gamma^{\tau}{}_{\beta\sigma} F^{\sigma}{}_{\tau}\right) - g^{55} \Gamma^{\tau}{}_{5\sigma} F^{\sigma}{}_{\tau}\right) = -\tfrac{1}{4} b\overline{\kappa} F^{\Sigma 5}{}_{;\Sigma} \equiv -\tfrac{1}{4} b\overline{\kappa} J^{5} , \tag{7.6}$$

where we again use suitable $F^{\mathrm{T}5} = 0$-based reductions, and have also expanded the final term $-g^{\mathrm{MB}} \Gamma^{\mathrm{T}}{}_{\mathrm{B}\Sigma} F^{\Sigma}{}_{\mathrm{T}}$ in (7.5) into its spacetime and fifth-dimensional parts. Contrasting with (6.6), we see that $F^{\sigma}{}_{\beta,\sigma} + \Gamma^{\sigma}{}_{\tau\sigma} F^{\tau}{}_{\beta} - \Gamma^{\tau}{}_{\beta\sigma} F^{\sigma}{}_{\tau} = J_{\beta}$ is simply the lower index current density J_{β}. And, in the remaining term, we may now employ the (5.1) substitution $\Gamma^{\mathrm{T}}{}_{5\Sigma} = \frac{1}{4} b\overline{\kappa} F^{\mathrm{T}}{}_{\Sigma}$. So, (7.6) now becomes:

$$R^{5}{}_{5} = -\tfrac{1}{4} b\overline{\kappa}\left(g^{5\beta} J_{\beta} + g^{55}\, \tfrac{1}{4} b\overline{\kappa} F^{\sigma\tau} F_{\sigma\tau}\right) = -\tfrac{1}{4} b\overline{\kappa}\left(g^{5\beta} J_{\beta} + g^{55} J_{5}\right) = -\tfrac{1}{4} b\overline{\kappa} g^{5\mathrm{B}} J_{\mathrm{B}} = -\tfrac{1}{4} b\overline{\kappa} J^{5} , \tag{7.7}$$

using $J_{5} = \frac{1}{4} b\overline{\kappa} F^{\sigma\tau} F_{\sigma\tau}$ from (6.8), and $F^{\mathrm{T}}{}_{\Sigma} F^{\Sigma}{}_{\mathrm{T}} = F^{\mathrm{T}\Sigma} F_{\Sigma\mathrm{T}} = -F^{\Sigma\mathrm{T}} F_{\Sigma\mathrm{T}} = -F^{\sigma\tau} F_{\sigma\tau}$ from following (6.8). So, simply put, $R^{5}{}_{5}$ also contains the $F^{\sigma\tau} F_{\sigma\tau}$ term, but it arises from the raising of the index in $g^{5\mathrm{B}} J_{\mathrm{B}} = J^{5}$, and so contains the term combination $g^{5\beta} J_{\beta} + g^{55}\, \frac{1}{4} b\overline{\kappa} F^{\sigma\tau} F_{\sigma\tau}$. It also helps to see J^{5} directly as:

$$J^{5} = g^{5\beta} J_{\beta} + g^{55}\, \tfrac{1}{4} b\overline{\kappa} F^{\sigma\tau} F_{\sigma\tau} , \tag{7.8}$$

This expression (7.8) will play a central role in the section 10 derivation of the Maxwell tensor.

Returning to compare (7.4) and (7.7), this also means that:

$$\Gamma^5{}_{\tau\sigma}F^{\sigma\tau} = g^{5\beta}J_\beta + g^{55}\tfrac{1}{4}b\overline{\kappa}F_{\sigma\tau}F^{\sigma\tau}. \tag{7.9}$$

So, now we have all the ingredients needed to write out the five-dimensional curvature scalar $R_{(5)} = R + R^5{}_5$, leaving R as a remaining unknown still to be deduced. Using (7.7), we simply write:

$$R_{(5)} = R + R^5{}_5 = R - \tfrac{1}{16}g^{55}b^2\overline{\kappa}^2F^{\sigma\tau}F_{\sigma\tau} - \tfrac{1}{4}b\overline{\kappa}g^{5\beta}J_\beta. \tag{7.10}$$

The four-dimensional Ricci scalar $R = R^\sigma{}_\sigma$ is still an unknown in (7.1). Now, let us see if there is a way to deduce R.

8. The Einstein Hilbert Action, and Derivation of the Energy Tensor and the Ricci Tensor, from Five-Dimensional Variation

At this phase of development, we are at a juncture: Up until this point, all of the development has been based on a single supposition introduced just after (2.6): the requirement that the Lorentz force must be represented as nothing other than geodesic motion in a five-dimensional geometry, as implemented through (2.7) and (2.8). Other than perhaps our imposing the requirement that $F^{\mathrm{MN}} \equiv -F^{\mathrm{NM}}$, every step taken since then has been fully deductive, with no other assumptions. We have even left open the question of whether the fifth dimension is timelike or spacelike, simply exploring the consequences in the alternative, as pertinent. This has enabled us to place Maxwell's equations, deductively, on a fully geometric footing, fully specify the fifth-dimensional components of the Ricci tensor $R^{\mathrm{M}}{}_5$, and obtain the five dimensional Ricci scalar $R_{(5)}$, *but only up to the four-dimensional scalar* $R = R^\sigma{}_\sigma$, which still stands out as undetermined. Determining R, would give us a window into $R^\mu{}_\nu$, and this in turn into the remaining $T^\mu{}_\nu$ components, among which, one would expect to find the Maxwell stress energy tensor, which would be a final check on the validity of this entire path of development. So, we need to find R. To deduce R, we must now, finally, make a new supposition beyond that of Lorentz force geodesics, which we do as follows:

Some theorists, particularly those who have adopted the so-called "Space-Time-Matter" view [4], seek the derivation of Einstein's equations out of a five-dimensional Riemannian geometry without the introduction of explicit matter source terms. There are perhaps several ways to frame this objective: the one we shall choose here, as set forth in the introduction, will be to employ an Einstein-Hilbert action of the general form $S = \frac{1}{2\kappa}\int R dV$, omitting any source term $\mathfrak{L}_{\text{Matter}}$, which is to say, *not* using an action $S = \int \left(\frac{1}{2\kappa} R + \mathfrak{L}_{\text{Matter}}\right) dV$. We do this is as follows:

Let us now posit that the action of the five-dimensional Riemannian geometry that we have been exploring herein, is to be *defined* over the four-dimensional spacetime of our common physical experience, in the form:

$$S(g_{\text{MN}}) \equiv \frac{1}{2\kappa}\int R_{(5)} dV = \int \left(\frac{1}{2\kappa} R + \frac{1}{2\kappa} R^5{}_5\right) dV . \tag{8.1}$$

This is a completely geometric definition of the action, without any explicit source term, of the general form $S = \frac{1}{2\kappa}\int R dV$, but in which R is replaced by the five-dimensional scalar $R_{(5)} = R^{\Sigma}{}_{\Sigma}$.

Now, although there is no *explicit* source term in (8.1), the $R^5{}_5$ component serves the role of an *implicit* source term, because if one contrasts (8.1) with $S = \int \left(\frac{1}{2\kappa} R + \mathfrak{L}_{\text{Matter}}\right) dV$, we see that one can associate:

$$S(g_{\text{MN}}) \equiv \frac{1}{2\kappa}\int R_{(5)} dV = \int \left(\frac{1}{2\kappa} R + \frac{1}{2\kappa} R^5{}_5\right) dV \equiv \int \left(\frac{1}{2\kappa} R + \mathfrak{L}_{Matter}\right) dV . \tag{8.2}$$

Then, employing $R^5{}_5 = -\frac{1}{4} b\overline{\kappa} g^{5\text{B}} J_{\text{B}} = -\frac{1}{4} b\overline{\kappa} J^5$ from (7.7), we have now effectively *defined*:

$$\begin{aligned}\mathfrak{L}_{Matter} &\equiv \frac{1}{2\kappa} R^5{}_5 = -\frac{1}{8\kappa} b\overline{\kappa} g^{5\text{B}} J_{\text{B}} = -\frac{1}{8\kappa} b\overline{\kappa} g^{\text{MN}} \delta^5{}_{\text{N}} J_{\text{M}} = -\frac{1}{8\kappa} b\overline{\kappa} J^5 \\ &= -\frac{1}{8\kappa} b\overline{\kappa}\left(g^{5\beta} J_{\beta} + \frac{1}{4} g^{55} b\overline{\kappa} F^{\sigma\tau} F_{\sigma\tau}\right)\end{aligned} . \tag{8.3}$$

Referring to the old adage that $R^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}{}_{\nu} R$ is made of "marble" but $T^{\mu}{}_{\nu}$ is made of "wood", the defining of $\mathfrak{L}_{Matter} \equiv \frac{1}{2\kappa} R^5{}_5$ allows us to fashion a $T^{\mu}{}_{\nu}$ or "marble" as well, because $R^5{}_5$ is a completely geometric object.

Now, we can use variational principles to immediate calculate the energy tensor. Specifically, the variation of the 5-dimensional metric tensor determinant $g_{(5)}$ is specified by $\frac{1}{\sqrt{-g_{(5)}}}\frac{\delta\sqrt{-g_{(5)}}}{\delta g^{\mathrm{MN}}}=-\frac{1}{2}g_{\mathrm{MN}}$. The 5-dimensional energy tensor may be defined from the matter term $\mathcal{L}_{Matter}$ according to: (See [6]):

$$T_{\mathrm{MN}}\equiv-\frac{2}{\sqrt{-g}}\frac{\partial\left(\sqrt{-g}\,\mathcal{L}_{Matter}\right)}{\delta g^{\mathrm{MN}}}=-2\frac{\delta\mathcal{L}_{Matter}}{\delta g^{\mathrm{MN}}}+g_{\mathrm{MN}}\mathcal{L}_{Matter}. \tag{8.4}$$

Then, we simply substitute the five-geometry-based $\mathcal{L}_{Matter}$ from (8.3) into the above, thus:

$$\kappa T_{\mathrm{MN}}=\left(\tfrac{1}{4}b\overline{\kappa}\delta^5{}_{\mathrm{N}}J_{\mathrm{M}}\right)-\tfrac{1}{2}g_{\mathrm{MN}}\left(\tfrac{1}{4}b\overline{\kappa}g^{5\mathrm{B}}J_{\mathrm{B}}\right), \tag{8.5}$$

We note from (8.5) that the four-dimensional energy tensor $\kappa T_{\mu\nu}=-\frac{1}{8}g_{\mu\nu}b\overline{\kappa}g^{5\mathrm{B}}J_{\mathrm{B}}$ is symmetric, $T_{\mu\nu}=-T_{\nu\mu}$ with $\delta^5{}_\mu=0$, but that the fifth-dimensional components T_{MN} appear to be non-symmetric, because $\delta^5{}_{\mathrm{N}}J_{\mathrm{M}}\neq\delta^5{}_{\mathrm{M}}J_{\mathrm{N}}$. Keep in mind, $J^5=g^{5\mathrm{B}}J_{\mathrm{B}}$. The transposed (8.5) is:

$$\kappa T_{\mathrm{NM}}=\left(\tfrac{1}{4}b\overline{\kappa}\delta^5{}_{\mathrm{M}}J_{\mathrm{N}}\right)-\tfrac{1}{2}g_{\mathrm{MN}}\left(\tfrac{1}{4}b\overline{\kappa}g^{5\mathrm{B}}J_{\mathrm{B}}\right), \tag{8.6}$$

The mixed tensors formed from the above by raising M, respectively, are:

$$-\kappa T^{\mathrm{M}}{}_{\mathrm{N}}=-\left(\tfrac{1}{4}b\overline{\kappa}\delta^5{}_{\mathrm{N}}J^{\mathrm{M}}\right)+\tfrac{1}{2}\delta^{\mathrm{M}}{}_{\mathrm{N}}\left(\tfrac{1}{4}b\overline{\kappa}g^{5\mathrm{B}}J_{\mathrm{B}}\right),\ \text{and} \tag{8.7}$$

$$-\kappa T_{\mathrm{N}}{}^{\mathrm{M}}=-\left(\tfrac{1}{4}b\overline{\kappa}g^{5\mathrm{M}}J_{\mathrm{N}}\right)+\tfrac{1}{2}\delta^{\mathrm{M}}{}_{\mathrm{N}}\left(\tfrac{1}{4}b\overline{\kappa}g^{5\mathrm{B}}J_{\mathrm{B}}\right). \tag{8.8}$$

This non-symmetry is further emphasized by the two different mixed tensors (8.7) and (8.8). There are two possibilities: either the fifth-dimensional components $T_{5\mathrm{N}}\neq T_{\mathrm{N}5}$ really are and ought to be non-symmetric, or we will need to take steps to make this tensor symmetric. We defer this for the moment pending a bit more development.

First, despite the non-symmetry, the five-dimensional trace energy from (8.7) and (8.8) turn out to be identical:

$$\kappa T_{(5)}=-\tfrac{3}{2}\cdot\tfrac{1}{4}b\overline{\kappa}J^5. \tag{8.9}$$

Note that the $\frac{3}{2}$ factors arises because with an extra dimension, $\delta^{\Sigma}{}_{\Sigma} = 5$. If we now consider the Einstein equation in five dimensions as $-\kappa T^{\mathrm{M}}{}_{\mathrm{N}} = R^{\mathrm{M}}{}_{\mathrm{N}} - \frac{1}{2}\delta^{\mathrm{M}}{}_{\mathrm{N}} R_{(5)}$, then this contracts down to $\kappa T_{(5)} = \frac{3}{2} R_{(5)}$. Therefore, from (8.9) we deduce to for either (8.7) or (8.8):

$$R_{(5)} = -\tfrac{1}{4} b\overline{\kappa} J^{5}, \tag{8.10}$$

Finally, from the inverse Einstein equation, we deduce from (8.7) and (8.8) respectively, also using the common (8.9), that the $\frac{3}{2} \cdot \frac{2}{3} = 1$ factors cancel, all of the $\delta^{\mathrm{M}}{}_{\mathrm{N}}$ terms cancel, and we are left with:

$$R^{\mathrm{M}}{}_{\mathrm{N}} = -\kappa T^{\mathrm{M}}{}_{\mathrm{N}} + \tfrac{2}{3} \cdot \tfrac{1}{2} \delta^{\mathrm{M}}{}_{\mathrm{N}} \kappa T_{(5)} = -\tfrac{1}{4} b\overline{\kappa} \delta^{5}{}_{\mathrm{N}} J^{\mathrm{M}}, \tag{8.11}$$

$$R_{\mathrm{N}}{}^{\mathrm{M}} = -\kappa T_{\mathrm{N}}{}^{\mathrm{M}} + \tfrac{2}{3} \cdot \tfrac{1}{2} \delta^{\mathrm{M}}{}_{\mathrm{N}} \kappa T_{(5)} = -\tfrac{1}{4} b\overline{\kappa} g^{5\mathrm{M}} J_{\mathrm{N}}, \tag{8.12}$$

In retrospect, (8.11) and (8.12) could have been gleaned directly from (8.7) and (8.8), which were written suggestively for that very reason. However, it is useful to confirm that this works via the use of the inverse field equation, even with the extra dimension. Lowering the upper indexes in the above, we obtain the respective covariant:

$$R_{\mathrm{MN}} = -\tfrac{1}{4} b\overline{\kappa} \delta^{5}{}_{\mathrm{N}} J_{\mathrm{M}}, \tag{8.13}$$

$$R_{\mathrm{NM}} = -\tfrac{1}{4} b\overline{\kappa} \delta^{5}{}_{\mathrm{M}} J_{\mathrm{N}}, \tag{8.14}$$

which also in non-symmetric in the fifth-dimensional components $R_{5\mathrm{N}} \neq R_{\mathrm{N}5}$, just like the energy tensor, contrast (8.5). However, what we also deduce from either (8.13) or (8.14) that the covariant curvature tensor *in four spacetime dimensions* is:

$$R_{\mu\nu} = R_{\nu\mu} = -\tfrac{1}{4} b\overline{\kappa} \delta^{5}{}_{\nu} J_{\mu} = -\tfrac{1}{4} b\overline{\kappa} \delta^{5}{}_{\mu} J_{\nu} = 0, \tag{8.10}$$

that is: $R_{\mu\nu} = 0$. The $\delta^{5}{}_{\mathrm{N}}$ which first made its appearance in (8.3) and (8.5), is effectively a "screen factor" which shuts all four-dimensional components of the covariant Ricci tensor $R_{\mu\nu}$ down to zero, and leaves a four-dimensional vacuum under the five-dimensional variation (8.4).

Although $R_{\mu\nu} = 0$, this is not so for $T_{\mu\nu}$, because by (8.5) or (8.6) and (7.8):

$$\kappa T_{\mu\nu} = \kappa T_{\nu\mu} = -\tfrac{1}{8} b\overline{\kappa} g_{\mu\nu} J^5 = -\tfrac{1}{8} b\overline{\kappa} g_{\mu\nu} \left(g^{5\beta} J_\beta + g^{55} \tfrac{1}{4} b\overline{\kappa} F^{\sigma\tau} F_{\sigma\tau} \right), \tag{8.11}$$

Although non-zero, this tensor *is* symmetric in four dimensions.

Finally, we set out at the beginning of this section to deduce the four-dimensional Ricci scalar R. Combining (7.7) with (8.8) yields $R_{(5)} = R + R^5{}_5 = R - \tfrac{1}{4} b\overline{\kappa} g^{5\Sigma} J_\Sigma = -\tfrac{1}{4} b\overline{\kappa} g^{5\Sigma} J_\Sigma$, i.e.:

$$R = 0. \tag{8.12}$$

More directly, this also comes from $R_{\mu\nu} = 0$, (8.10). However, the ordinary, four-dimensional trace energy is not zero, but from (8.11), is:

$$\kappa T = -\tfrac{1}{2} b\overline{\kappa} J^5 = -\tfrac{1}{2} b\overline{\kappa} \left(g^{5\beta} J_\beta + g^{55} \tfrac{1}{4} b\overline{\kappa} F^{\sigma\tau} F_{\sigma\tau} \right). \tag{8.13}$$

The derivation in this section made use of a five-dimensional variation, i.e., a variation using δg^{MN}. In section 10, we shall see how a four-dimensional variation $\delta g^{\mu\nu}$ leads to the Maxwell stress energy tensor. But first, we pause to examine the non-symmetry of the fifth-dimensional component of the Ricci tensor, $R_{5\mathrm{N}} \neq R_{\mathrm{N}5}$, and the energy tensor $T_{5\mathrm{N}} \neq T_{\mathrm{N}5}$.

9. A Non-Symmetry Ricci Tensor for the Fifth-Dimensional Components?

What are we to make of the fact that $R_{5\mathrm{N}} \neq R_{\mathrm{N}5}$ and $T_{5\mathrm{N}} \neq T_{\mathrm{N}5}$ in section 8 above? It is helpful to directly examine the definition of the Riemann tensor (6.1):

$$R_{\mathrm{BM}} = R^{\mathrm{A}}{}_{\mathrm{BMA}} = -\Gamma^{\mathrm{A}}{}_{\mathrm{BM,A}} + \Gamma^{\mathrm{A}}{}_{\mathrm{BA,M}} + \Gamma^{\Sigma}{}_{\mathrm{BA}} \Gamma^{\mathrm{A}}{}_{\Sigma\mathrm{M}} - \Gamma^{\Sigma}{}_{\mathrm{BM}} \Gamma^{\mathrm{A}}{}_{\Sigma\mathrm{A}}, \tag{9.1}$$

and the reverse-indexed:

$$R_{\mathrm{MB}} = R^{\mathrm{A}}{}_{\mathrm{MBA}} = -\Gamma^{\mathrm{A}}{}_{\mathrm{MB,A}} + \Gamma^{\mathrm{A}}{}_{\mathrm{MA,B}} + \Gamma^{\Sigma}{}_{\mathrm{MA}} \Gamma^{\mathrm{A}}{}_{\Sigma\mathrm{B}} - \Gamma^{\Sigma}{}_{\mathrm{MB}} \Gamma^{\mathrm{A}}{}_{\Sigma\mathrm{A}}, \tag{9.2}$$

The first and fourth terms are clearly identical, because $\Gamma^{\Sigma}{}_{\mathrm{B}5} = \Gamma^{\Sigma}{}_{5\mathrm{B}}$. The third terms are also identical if one renames indexes. However, the second terms are *not necessarily* the same, $\Gamma^{\mathrm{A}}{}_{\mathrm{BA,M}} \neq \Gamma^{\mathrm{A}}{}_{\mathrm{MA,B}}$, and specifically:

$$\Gamma^{\mathrm{A}}{}_{\mathrm{BA,M}} = \tfrac{1}{2} g^{\mathrm{A}\Delta}{}_{,\mathrm{M}} \left(g_{\Delta\mathrm{B,A}} + g_{\mathrm{A}\Delta,\mathrm{B}} - g_{\mathrm{BA},\Delta} \right) + \tfrac{1}{2} g^{\mathrm{A}\Delta} \left(g_{\Delta\mathrm{B,A,M}} + g_{\mathrm{A}\Delta,\mathrm{B,M}} - g_{\mathrm{BA},\Delta,\mathrm{M}} \right), \text{ and} \tag{9.3}$$

$$\Gamma^{\mathrm{A}}{}_{\mathrm{MA,B}} = \tfrac{1}{2} g^{\mathrm{A}\Delta}{}_{,\mathrm{B}} \left(g_{\Delta\mathrm{M,A}} + g_{\mathrm{A}\Delta,\mathrm{M}} - g_{\mathrm{MA},\Delta} \right) + \tfrac{1}{2} g^{\mathrm{A}\Delta} \left(g_{\Delta\mathrm{M,A,B}} + g_{\mathrm{A}\Delta,\mathrm{M,B}} - g_{\mathrm{MA},\Delta,\mathrm{B}} \right), \tag{9.4}$$

which, as a general rule, are not by identity, the same. If they are the same, it has to be because of particular constraints on the g_{MN}. But as a general rule, it is possible to entertain field equations, i.e., Ricci tensors and energy tensors which are not transposition symmetric, even when the $\Gamma^{\Sigma}{}_{\mathrm{AB}} = \Gamma^{\Sigma}{}_{\mathrm{BA}}$ and $g_{\mathrm{AB}} = g_{\mathrm{BA}}$.

This possibility has long been known, and is the precise problem that Einstein pointed out in [14], see his contrast of equations (4a) and (4b). It has also been noted that "starting with a general (though still symmetric) connection allowed Eddington – and Einstein following him in 1923 – to obtain a non-symmetric Ricci tensor, the antisymmetric part of which could then be taken as a representation of the (antisymmetric) electromagnetic field tensor." [15] Given the foregoing, as well as the fact that although non-symmetric in five dimensions, the four-dimensional energy tensor and Ricci tensor (8.11) and (8.10) retain their $T_{\mu\nu} = T_{\nu\mu}$ and $R_{\mu\nu} = R_{\nu\mu}$ transposition symmetry, we shall accept the non-symmetric $R_{5\mathrm{N}} \neq R_{\mathrm{N}5}$ and $T_{5\mathrm{N}} \neq T_{\mathrm{N}5}$ as is, and not attempt to make these symmetric in the $\mathrm{N}5$ indexes. That is, we shall take $R_{5\mathrm{N}} \neq R_{\mathrm{N}5}$ and $T_{5\mathrm{N}} \neq T_{\mathrm{N}5}$ uncovered in the previous section as an indication that in nature, wherein Maxwell's electric charge source equation is effectively represented along those fifth-dimensional components, (see sections 6 and 7) the fifth-dimensional components of R_{MN} and T_{MN} are non-symmetric.

Therefore, we return to (8.9), which we redefine in the opposite manner as before, reversing M and N, as follows:

$$R_{\mathrm{MN}} \equiv -\tfrac{1}{4} b\overline{\kappa}\delta^{5}{}_{\mathrm{N}} J_{\mathrm{M}} \tag{9.5}$$

We do this so as to be consistent with the results in section 6. Thus, from (9.5) we find, just as in (6.6) and (6.7), respectively, that:

$$R_{\beta 5} = -\tfrac{1}{4} b\overline{\kappa}\delta^{5}{}_{5} J_{\beta} = -\tfrac{1}{4} b\overline{\kappa} J_{\beta} \tag{9.6}$$

$$R_{55} = -\tfrac{1}{4} b\overline{\kappa}\delta^{5}{}_{5} J_{5} = -\tfrac{1}{4} b\overline{\kappa} J_{5}\,. \tag{9.7}$$

However:

$$R_{5\beta} = -\tfrac{1}{4} b\overline{\kappa}\delta^{5}{}_{\beta} J_{5} = 0\,, \tag{9.8}$$

which demonstrates explicitly the non-symmetric character of $R_{5\mathrm{N}} \neq R_{\mathrm{N}5}$. The entire fifth "column" $R_{\mathrm{B}5} = -\frac{1}{4} b\overline{\kappa}\left(J_\beta, J_5\right) = -\frac{1}{4} b\overline{\kappa}\left(J_\beta, \frac{1}{4} b\overline{\kappa} F^{\sigma\tau} F_{\sigma\tau}\right)$ of the covariant Ricci tensor contains the Maxwell source charge current density J_β transforming as part of a five-vector with a $F^{\sigma\tau} F_{\sigma\tau}$ term, while the fifth "row" $R_{5\beta}$, except for R_{55}, is zero. Combined with $R_{\mu\nu} = 0$ from (8.10), we may summarize that $R_{\mathrm{M}5} = -\frac{1}{4} b\overline{\kappa} J_{\mathrm{M}}$, $R_{\mathrm{M}\nu} = 0$.

As we shall now see, allowing the Ricci and energy tensors to stay non-symmetric will validate itself by leading to the Maxwell stress-energy tensor, which we take to be a point of contact between theory and settled empirical observation.

10. Derivation of the Maxwell Stress-Energy Tensor, using a Four-Dimensional Variation

In section 8, we derived the energy tensor based on the variational calculation (8.4), in five dimensions, i.e., by the variation δg^{MN}. Let us repeat this same calculation, but in a slightly different way.

In section 8, we used (8.3) in the form of $\mathfrak{L}_{Matter} = -\frac{1}{8\kappa} b\overline{\kappa} g^{5\mathrm{B}} J_{\mathrm{B}} = -\frac{1}{8\kappa} b\overline{\kappa} g^{\mathrm{MN}} \delta^5{}_{\mathrm{M}} J_{\mathrm{N}}$, because that gave us a contravariant g^{MN} against which to obtain the five-dimensional variation $\delta\mathfrak{L}_{Matter} / \delta g^{\mathrm{MN}}$. Let us instead, here, use the very last term in (8.3) as $\mathfrak{L}_{Matter}$, writing this as:

$$\mathfrak{L}_{Matter} \equiv \frac{1}{2\kappa} R^5{}_5 = -\frac{1}{8\kappa} b\overline{\kappa}\left(g^{5\beta} J_\beta + \frac{1}{4} g^{55} b\overline{\kappa} F^{\sigma\tau} F_{\sigma\tau}\right) = -\frac{1}{8\kappa} b\overline{\kappa}\left(g^{\mu\nu} \delta^5{}_\nu J_\mu + \frac{1}{4} g^{55} g^{\mu\nu} b\overline{\kappa} F_\mu{}^\tau F_{\nu\tau}\right). \quad (10.1)$$

It is important to observe that the term $g^{5\beta} J_\beta$ is only summed over four spacetime indexes. The fifth term, $g^{55} J_5 = \frac{1}{4} g^{55} b\overline{\kappa} F^{\sigma\tau} F_{\sigma\tau}$, see, e.g., (6.8). For consistency with the non-symmetric (9.5), we employ $g^{5\beta} J_\beta = g^{\mu\nu} \delta^5{}_\nu J_\mu$ rather than $g^{5\beta} J_\beta = g^{\mu\nu} \delta^5{}_\mu J_\nu$. By virtue of this separation, in which we can only introduce $g^{\mu\nu}$ and not g^{MN} as in section 8, we can only take a four-dimensional variation $\delta\mathfrak{L}_{Matter} / \delta g^{\mu\nu}$, which, in contrast to (8.4), is now given by:

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\partial\left(\sqrt{-g}\,\mathfrak{L}_{Matter}\right)}{\delta g^{\mu\nu}} = -2 \frac{\delta\mathfrak{L}_{Matter}}{\delta g^{\mu\nu}} + g_{\mu\nu} \mathfrak{L}_{Matter}. \quad (10.2)$$

Substituting from (10.1) then yields:

$$T_{\mu\nu} = \tfrac{1}{4\kappa} b\overline{\kappa}\left(\delta^5{}_\nu J_\mu + \tfrac{1}{4} g^{55} b\overline{\kappa} F_\mu{}^\tau F_{\nu\tau}\right) - \tfrac{1}{2} g_{\mu\nu} \tfrac{1}{4\kappa} b\overline{\kappa}\left(g^{5\beta} J_\beta + \tfrac{1}{4} g^{55} b\overline{\kappa} F^{\sigma\tau} F_{\sigma\tau}\right). \tag{10.3}$$

Now, the non-symmetry of sections 8 and 9 comes into play, and this will yield the Maxwell tensor. Because $\delta^5{}_\nu = 0$, the first term drops out and the above reduces to:

$$\kappa T_{\mu\nu} = \tfrac{1}{4} b\overline{\kappa}\left(\tfrac{1}{4} g^{55} b\overline{\kappa} F_\mu{}^\tau F_{\nu\tau}\right) - \tfrac{1}{2} g_{\mu\nu} \tfrac{1}{4} b\overline{\kappa}\left(g^{5\beta} J_\beta + \tfrac{1}{4} g^{55} b\overline{\kappa} F^{\sigma\tau} F_{\sigma\tau}\right). \tag{10.4}$$

Note that this four-dimensional tensor *is* symmetric, and that we would arrive at an energy tensor which is identical if (10.3) contained a $\delta^5{}_\mu J_\nu$ rather than $\delta^5{}_\nu J_\mu$. One again, the screen factor $\delta^5{}_\nu = 0$ is at work.

In mixed form, starting from (10.3), there are two energy tensors to be found. If we raise the μ index in (10.3), the first term becomes $\delta^5{}_\nu J^\mu = 0$ and we obtain:

$$-\kappa T^\mu{}_\nu = -\tfrac{1}{4} b\overline{\kappa}\left(\tfrac{1}{4} g^{55} b\overline{\kappa} F^{\mu\tau} F_{\nu\tau}\right) + \tfrac{1}{2} \delta^\mu{}_\nu \tfrac{1}{4} b\overline{\kappa}\left(g^{5\beta} J_\beta + \tfrac{1}{4} g^{55} b\overline{\kappa} F^{\sigma\tau} F_{\sigma\tau}\right). \tag{10.5}$$

with this first term still screened out. However, if we transpose (10.3) and then raise the μ index, the first term becomes $g^{5\mu} J_\nu$ and this term does *not* drop out, i.e.,

$$-\kappa T_\nu{}^\mu = -\tfrac{1}{4} b\overline{\kappa}\left(g^{5\mu} J_\nu + \tfrac{1}{4} g^{55} b\overline{\kappa} F^{\mu\tau} F_{\nu\tau}\right) + \tfrac{1}{2} \delta^\mu{}_\nu \tfrac{1}{4} b\overline{\kappa}\left(g^{5\beta} J_\beta + \tfrac{1}{4} g^{55} b\overline{\kappa} F^{\sigma\tau} F_{\sigma\tau}\right). \tag{10.6}$$

So, there are two mixed tensors to consider, and this time, unlike in section 8, these each yield different four-dimensional energy tensors. Contrasting (10.5) and (10.6), we see that $\delta^5{}_\nu = 0$ has effectively "broken" a symmetry that is apparent in (10.6), but "hidden" in (10.5). At this time, we focus on (10.5), because, as we shall now see, this is the Maxwell stress-energy tensor $T^\mu{}_\nu = -\left(F^{\mu\tau} F_{\nu\tau} - \tfrac{1}{4} \delta^\mu{}_\nu F^{\sigma\tau} F_{\sigma\tau}\right)$, before reduction into this more-recognizable form.

Purposely leaving constant factors separated, the trace equation of (10.5) is then:

$$\kappa T = R = -2 \tfrac{1}{4} b\overline{\kappa} g^{5\beta} J_\beta - \tfrac{1}{4} b\overline{\kappa}\left(\tfrac{1}{4} g^{55} b\overline{\kappa} F^{\sigma\tau} F_{\sigma\tau}\right). \tag{10.7}$$

and so, via the inverse equation $R^\mu{}_\nu = -\kappa T^\mu{}_\nu + \delta^\mu{}_\nu \kappa T$, from (10.5) and (10.7):

$$R^\mu{}_\nu = -\tfrac{1}{4} b\overline{\kappa}\left(\tfrac{1}{4} g^{55} b\overline{\kappa} F^{\mu\tau} F_{\nu\tau}\right) + \tfrac{1}{2} \delta^\mu{}_\nu \tfrac{1}{4} b\overline{\kappa}\left(-3 g^{5\beta} J_\beta - \tfrac{1}{4} g^{55} b\overline{\kappa} F^{\sigma\tau} F_{\sigma\tau}\right). \tag{10.8}$$

Note that here, traceable to the screened term, lost via $\delta^5{}_\nu = 0$, that one cannot simply glean $R^\mu{}_\nu$ from (10.5) as we were able to for (8.9). It was necessary to use the full inverse field equation $R^\mu{}_\nu = -\kappa T^\mu{}_\nu + \delta^\mu{}_\nu \kappa T$. Now, we take the trace of (10.8) to obtain:

$$\kappa T = R = -6\tfrac{1}{4} b\overline{\kappa} g^{5\beta} J_\beta - 3\tfrac{1}{4} b\overline{\kappa}\left(\tfrac{1}{4} g^{55} b\overline{\kappa} F^{\sigma\tau} F_{\sigma\tau}\right). \tag{10.9}$$

Interestingly, this does not look to be the same as the trace in (10.7), yet these are the same. This means that a further relationship must subsist, and if we look closely, (10.9) is the same as (10.7), multiplied by a factor of 3. If $x = 3x$, then $x = 0$, so this is an indication that the trace $\kappa T = R = 0$, which is characteristic of Maxwell's tensor.

So, setting (10.7) equal to (10.9), we obtain:

$$R = -2\tfrac{1}{4} b\overline{\kappa} g^{5\beta} J_\beta - \tfrac{1}{4} b\overline{\kappa}\left(\tfrac{1}{4} g^{55} b\overline{\kappa} F^{\sigma\tau} F_{\sigma\tau}\right) = -6\tfrac{1}{4} b\overline{\kappa} g^{5\beta} J_\beta - 3\tfrac{1}{4} b\overline{\kappa}\left(\tfrac{1}{4} g^{55} b\overline{\kappa} F^{\sigma\tau} F_{\sigma\tau}\right), \tag{10.10}$$

and we find after reducing, that:

$$g^{5\beta} J_\beta = -\tfrac{1}{2}\left(\tfrac{1}{4} g^{55} b\overline{\kappa} F^{\sigma\tau} F_{\sigma\tau}\right). \tag{10.11}$$

Now, we return to the energy tensor (10.5) and shift some terms to rewrite this as:

$$4\kappa T^\mu{}_\nu = b\overline{\kappa}\left(\tfrac{1}{4} g^{55} b\overline{\kappa} F^{\mu\tau} F_{\nu\tau}\right) - \tfrac{1}{2}\delta^\mu{}_\nu b\overline{\kappa}\left(\tfrac{1}{4} g^{55} b\overline{\kappa} F^{\sigma\tau} F_{\sigma\tau}\right) - \tfrac{1}{2}\delta^\mu{}_\nu b\overline{\kappa} g^{5\beta} J_\beta . \tag{10.12}$$

Then, we substitute $g^{5\beta} J_\beta$ from (10.11) into (10.12), and do some further rearranging, including making use of $\overline{\kappa}^2 = 2\kappa/\hbar c$, to obtain:

$$\frac{16\kappa}{b^2 \overline{\kappa}^2} T^\mu{}_\nu = \frac{8}{b^2}\hbar c T^\mu{}_\nu = g^{55}\left(F^{\mu\tau} F_{\nu\tau} - \frac{1}{4}\delta^\mu{}_\nu F^{\sigma\tau} F_{\sigma\tau}\right). \tag{10.13}$$

If we now set $\hbar = c = 1$ as well as:

$$b^2 = 8 \text{ and } g^{55} = -1, \tag{10.14}$$

then (10.13) now reduces, rather fortuitously, to the Maxwell stress-energy tensor:

$$T^\mu{}_\nu = -\left(F^{\mu\tau} F_{\nu\tau} - \tfrac{1}{4}\delta^\mu{}_\nu F^{\sigma\tau} F_{\sigma\tau}\right), \tag{10.15}$$

in the Heaviside-Lorentz units that we have been employing from the outset. The factor b which we have employed all along is now determined to be $b^2 = 8$. Further, because we have deduced that $g^{55} = -1$ we no longer need to straddle between a timelike and a spacelike fifth dimension: *we have deduced that the fifth dimension must be spacelike*. Also, despite the five-dimensional non-symmetry that we started with, the net result is still a symmetric tensor in four-dimensions. The stress-energy tensor is an important result, because this tensor is underpinned by extensive empirical evidence.

We can then also derive the mixed Ricci tensor corresponding to the stress-energy (10.15). We start with (10.8), substitute (10.11), and reduce, to obtain:

$$16R^{\mu}{}_{\nu} = -b^2\overline{\kappa}^2\left(g^{55}F^{\mu\tau}F_{\nu\tau}\right) + \tfrac{1}{4}\delta^{\mu}{}_{\nu}b^2\overline{\kappa}^2\left(g^{55}F^{\sigma\tau}F_{\sigma\tau}\right). \tag{10.16}$$

Clearly, this is also traceless, as it should be. Further use of $\overline{\kappa}^2 = 2\kappa/\hbar c$ with $\hbar = c = 1$, and $b^2 = 8$ and $g^{55} = -1$ from (10.14), then reduces to:

$$R^{\mu}{}_{\nu} = \kappa\left(F^{\mu\tau}F_{\nu\tau} - \tfrac{1}{4}\delta^{\mu}{}_{\nu}F^{\sigma\tau}F_{\sigma\tau}\right), \tag{10.17}$$

which is summarized by the traceless field equation $-\kappa T^{\mu}{}_{\nu} = R^{\mu}{}_{\nu}$, as expected.

Finally, in being able to derive the traceless equation (10.15) which among many things tells us that electromagnetic energy propagates at the speed of light, we have solved the essential riddle which concerned Einstein in [16], see equations (1) versus (1a) and (3) therein, which was to find a compatibility between the field equation $-\kappa T^{\mu}{}_{\nu} = R^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}{}_{\nu}R$ which contains a non-zero scalar trace, and (10.15) and (10.17) above which are scalar-free. More fundamentally, since (10.15) was derived by rigorously applying the field equation $-\kappa T^{\mu}{}_{\nu} = R^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}{}_{\nu}R$, we have demonstrated that Einstein's equation, which one ordinarily applies to trace matter which can be placed at rest, is also fully compatible with, and is indeed the foundation for, the energy tensor of traceless, luminous electromagnetic radiation.

11. Relation between the Electrodynamic Vector and Gravitational Tensor Potentials

We now formally introduce the four-vector potential $A^{\mu} \equiv (\phi, A_1, A_2, A_3)$, related to the field strength tensor according to $F^{\mu\nu} = A^{\mu;\nu} - A^{\nu;\mu} = A^{\mu,\nu} - A^{\nu,\mu}$, where, as is well-known, the covariant derivatives become ordinary derivatives in the particular combination used to form $F^{\mu\nu}$. Introducing A^{μ} is desirable and indeed required, for as Witten points out, ([17] at pg. 28) the vector potential A^{μ} is essential to the quantum mechanical treatment of electromagnetism.

So far, we have restricted ourselves strictly to classical electrodynamics and classical gravitation, based on five-dimensional Riemannian geometry. We now venture a tentative, introductory foray, from here, into the quantum world.

Once again, we start with (5.1), written out using with $g_{\Sigma\mathrm{T},5}=0$ from (5.6), as:

$$\tfrac{1}{4}b\bar{\kappa}F^{\mathrm{M}}{}_{\mathrm{T}}=\Gamma^{\mathrm{M}}{}_{\mathrm{T}5}=\tfrac{1}{2}g^{\mathrm{MA}}\left(g_{\mathrm{AT},5}+g_{5\mathrm{A},\mathrm{T}}-g_{\mathrm{T}5,\mathrm{A}}\right)=\tfrac{1}{2}g^{\mathrm{MA}}\left(g_{5\mathrm{A},\mathrm{T}}-g_{5\mathrm{T},\mathrm{A}}\right). \tag{11.1}$$

It is helpful to lower the indexes in field strength tensor and connect this to the covariant vector potentials A_μ, generalized into 5-dimensions as A_{M} via $F_{\Sigma\mathrm{T}}\equiv A_{\Sigma;\mathrm{T}}-A_{\mathrm{T};\Sigma}=A_{\Sigma,\mathrm{T}}-A_{\mathrm{T},\Sigma}$, as such:

$$\tfrac{1}{4}b\bar{\kappa}\left(A_{\Sigma;\mathrm{T}}-A_{\mathrm{T};\Sigma}\right)=\tfrac{1}{4}b\bar{\kappa}F_{\Sigma\mathrm{T}}=\tfrac{1}{4}b\bar{\kappa}g_{\Sigma\mathrm{M}}F^{\mathrm{M}}{}_{\mathrm{T}}=\tfrac{1}{2}g_{\Sigma\mathrm{M}}g^{\mathrm{MA}}\left(g_{5\mathrm{A},\mathrm{T}}-g_{5\mathrm{T},\mathrm{A}}\right)=\tfrac{1}{2}\left(g_{5\Sigma,\mathrm{T}}-g_{5\mathrm{T},\Sigma}\right). \tag{11.2}$$

The relationship $\tfrac{1}{4}b\bar{\kappa}F_{\Sigma\mathrm{T}}=\tfrac{1}{4}b\bar{\kappa}\left(A_{\Sigma;\mathrm{T}}-A_{\mathrm{T};\Sigma}\right)=\tfrac{1}{2}\left(g_{5\Sigma,\mathrm{T}}-g_{5\mathrm{T},\Sigma}\right)$ expresses clearly, the antisymmetry of $F_{\Sigma\mathrm{T}}$ in terms of the non-zero connection terms $\tfrac{1}{2}\left(g_{5\Sigma,\mathrm{T}}-g_{5\mathrm{T},\Sigma}\right)$ involving the gravitational potential. Of particular interest, is that we may extract from (11.2), the relation:

$$\tfrac{1}{4}b\bar{\kappa}A_{\Sigma;\mathrm{T}}=\tfrac{1}{2}g_{5\Sigma,\mathrm{T}}=\tfrac{1}{2}\bar{\kappa}h_{5\Sigma,\mathrm{T}}, \tag{11.3}$$

using also $g_{\mathrm{MN}}=\eta_{\mathrm{MN}}+\bar{\kappa}h_{\mathrm{MN}}$ for the gravitational potential energy h_{MN}. If one forms $A_{\Sigma;\mathrm{T}}-A_{\mathrm{T};\Sigma}$ from (11.3) and then renames indexes and uses $g_{\mathrm{MN}}=g_{\mathrm{NM}}$, one arrives back at (11.2). So (11.3) is just the simplest form of equation (11.2). The reason we did not remove the covariant derivative via $F_{\Sigma\mathrm{T}}\equiv A_{\Sigma;\mathrm{T}}-A_{\mathrm{T};\Sigma}=A_{\Sigma,\mathrm{T}}-A_{\mathrm{T},\Sigma}$, is that in (11.3), $A_{\Sigma;\mathrm{T}}$ is considered separated from $-A_{\mathrm{T};\Sigma}$, and the covariant derivatives do not become ordinary unless and until one forms the combination $F_{\Sigma\mathrm{T}}\equiv A_{\Sigma;\mathrm{T}}-A_{\mathrm{T};\Sigma}=A_{\Sigma,\mathrm{T}}-A_{\mathrm{T},\Sigma}$. When the terms are separated as in (11.3), the covariant derivatives must be left intact.

Equation (11.3) makes perfect sense classically: after all, the oft-employed $\Gamma^{\mathrm{M}}{}_{5\Sigma}=\tfrac{1}{4}b\bar{\kappa}F^{\mathrm{M}}{}_{\Sigma}$ of (5.1) is simply a first order differential equation between the vector potential A^μ and the gravitational field $h^{\mu\nu}$, and each is a dynamical field. Equation (11.3) above merely states that differential equation explicitly. But quantum mechanically, (11.3) raises questions, because we are talking about a relationship between a spin-1 photon and a spin-2 graviton, and so need now to come to better terms with what (11.3) implies quantum mechanically.

Equation (11.3) is a first order differential equations which tells us up to a constant factor, that the *covariant* derivative of the electrodynamic potential A_Σ is equal to the *ordinary* derivative of the gravitational potential $h_{5\Sigma}$. In the weak field limit / linear approximation, where covariant derivatives become *approximately* equal to ordinary derivatives, we have $\frac{1}{2} g_{5\Sigma,\mathrm{T}} = \frac{1}{4} b\overline{\kappa} A_{\Sigma;\mathrm{T}} \approx \frac{1}{4} b\overline{\kappa} A_{\Sigma,\mathrm{T}}$, and so, integrating based on this *linear approximation*, we obtain:

$$g_{5\Sigma} \approx \tfrac{1}{2} b\overline{\kappa} A_\Sigma . \tag{11.4}$$

Keep in mind, (11.3) is exact and non-linear; (11.4) only applies to the weak-field, linear approximation $A_{\Sigma;\mathrm{T}} \approx A_{\Sigma,\mathrm{T}}$.

Now, the reader will recall that the term $g^{5\beta} J_\beta$ and $J^5 = g^{5\mathrm{B}} J_\mathrm{B}$ has shown up repeatedly throughout many of the prior equations, going all the way back to (7.7), and most recently, in the energy tensor (10.3) which later turned into the Maxwell stress-energy tensor, via the "keystone" relationship (10.11) which enabled us to uncover the Maxwell tensor.

Equation (11.4) is in lower-index (covariant) form. We now wish to obtain a suitable contravariant expression for $g^{5\mathrm{B}}$ which is akin to (11.4), so that this can be employed in the various equations where $g^{5\mathrm{B}}$ or $g^{5\beta}$ appear. To properly raise (11.4), let us first raise the free index in (11.4) to write $\delta_5{}^\Sigma \approx \frac{1}{2} b\overline{\kappa} A^\Sigma$. Next, we write $g^{\mathrm{MN}} = g^{\mathrm{M}\Sigma} g^{\mathrm{NT}} g_{\Sigma\mathrm{T}}$, then take the $M = 5$ component, and use $\delta^\mathrm{N}{}_5 \approx \frac{1}{2} b\overline{\kappa} A^\mathrm{N}$ to obtain:

$$g^{5\mathrm{N}} = g^{5\Sigma} g^{\mathrm{NT}} g_{\Sigma\mathrm{T}} = g^{5\sigma} g^{\mathrm{NT}} g_{\sigma\mathrm{T}} + g^{55} g^{\mathrm{NT}} g_{5\mathrm{T}} = g^{5\sigma} \delta^\mathrm{N}{}_\sigma + g^{55} \delta_5{}^\mathrm{N} \approx g^{5\sigma} \delta^\mathrm{N}{}_\sigma + g^{55} \tfrac{1}{2} b\overline{\kappa} A^\mathrm{N} . \tag{11.5}$$

We now separate this out into:

$$g^{5\nu} \approx g^{5\sigma} \delta^\nu{}_\sigma + g^{55} \tfrac{1}{2} b\overline{\kappa} A^\nu = g^{5\nu} + g^{55} \tfrac{1}{2} b\overline{\kappa} A^\nu \text{, and} \tag{11.6}$$

$$g^{55} \approx g^{5\sigma} \delta^5{}_\sigma + g^{55} \tfrac{1}{2} b\overline{\kappa} A^5 = g^{55} \tfrac{1}{2} b\overline{\kappa} A^5 . \tag{11.7}$$

The latter (11.7) reduces to $1 \approx \frac{1}{2} b\overline{\kappa} A^5$. Since $\overline{\kappa} = \sqrt{16\pi G/\hbar c^5}$ and the Planck energy $E_P = \sqrt{\hbar c^5/G}$, and also using $b^2 = 8$ from (10.14), we may restate (11.7) as:

$$A^5 \approx \tfrac{1}{8} \sqrt{\tfrac{2}{\pi}} E_P . \tag{11.8}$$

Apparently, the fifth component of A^{N} has a huge energy, on the scale of the Planck vacuum.

It is a little trickier to reduce (7.6), because $g^{5\nu} \approx g^{5\nu} + g^{55}\frac{1}{2}b\overline{\kappa}A^{\nu}$ reduces to $g^{55}\frac{1}{2}b\overline{\kappa}A^{\nu} \approx 0$, which is another way of saying that $A^{\nu} <<< E_P$, i.e., that in the linear approximation, the spacetime part of the electromagnetic vector potential, A^{ν}, has an energy much less than the Planck energy. That is obvious, by definition. Let's instead make additional use of $g^{\mathrm{MN}} = \eta^{\mathrm{MN}} + \overline{\kappa}h^{\mathrm{MN}}$, and especially, $g^{5\nu} = \eta^{5\nu} + \overline{\kappa}h^{5\nu} = \overline{\kappa}h^{5\nu}$, and also $g^{55} = -1$ from (10.14) to rewrite (11.6) as:

$$g^{5\nu} \approx \overline{\kappa}\left(h^{5\nu} - \tfrac{1}{2}bA^{\nu}\right). \tag{11.9}$$

This yields a workable expression, and we find that the fifth component $h^{5\nu}$ of the gravitational potential is added to A^{ν} in this linear approximation.

Now, we make further use of $h^{\mu\nu} = \phi^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}\phi$ which in the linear approximation has the gravitational field equation $-\kappa T^{\mu\nu} = \partial_{\sigma}\partial^{\sigma}\phi^{\mu\nu}$ with gauge condition $\partial_{\mu}\phi^{\mu\nu} = 0$. Quantum-mechanically, $\phi^{\mu\nu}$ is of course representative of spin-2 gravitons, and we know that A^{ν} is representative of a spin-1 photon. Also, $\phi = \phi^{\mu}{}_{\mu}$ is the scalar (spin-0) trace of $\phi^{\mu\nu}$. We generalize to five dimensions, i.e., $h^{\mathrm{MN}} \equiv \phi^{\mathrm{MN}} - \frac{1}{2}\eta^{\mathrm{MN}}\phi$. Because $g^{55} = -1$, $h^{55} = \phi^{55} = 0$. But $h^{5\nu} = \phi^{5\nu} - \frac{1}{2}\eta^{5\nu}\phi = \phi^{5\nu}$, and so (11.9) can be turned into:

$$g^{5\nu} \approx \overline{\kappa}\left(\phi^{5\nu} - \tfrac{1}{2}bA^{\nu}\right). \tag{11.10}$$

This suggests that $g^{5\nu}$ decomposes into a vector potential, interpreted quantum mechanically as a spin-1 photon, together with the $\phi^{5\nu}$ components of a gravitational wave, interpreted quantum mechanically as a spin-2 graviton. If (11.10) tells us about the decomposition of $g^{5\nu}$ in the linear approximation, and we already know that $g^{55} = -1$ so $h^{55} = \phi^{55} = 0$, what about the ordinary spacetime components of the contravariant $g^{\mu\nu}$? How do these decompose?

Going back to (11.5), we still start with $g^{\mathrm{MN}} = g^{\mathrm{M\Sigma}}g^{\mathrm{NT}}g_{\Sigma\mathrm{T}}$, but now we write:

$$g^{\mathrm{MN}} = g^{\mathrm{M\Sigma}}g^{\mathrm{NT}}g_{\Sigma\mathrm{T}} = g^{\mathrm{M}\sigma}g^{\mathrm{NT}}g_{\sigma\mathrm{T}} + g^{\mathrm{M5}}g^{\mathrm{NT}}g_{5\mathrm{T}} = g^{\mathrm{M}\sigma}\delta^{\mathrm{N}}{}_{\sigma} + g^{\mathrm{M5}}\delta^{\mathrm{N}}{}_{5} \approx g^{\mathrm{M}\sigma}\delta^{\mathrm{N}}{}_{\sigma} + \tfrac{1}{2}b\overline{\kappa}g^{\mathrm{M5}}A^{\mathrm{N}}, \tag{11.11}$$

where we again use $\delta^{\mathrm{N}}{}_5 \approx \frac{1}{2} b\overline{\kappa} A^{\mathrm{N}}$ which we earlier used for (11.5). The four-dimensional spacetime components are:

$$g^{\mu\nu} \approx g^{\mu\sigma}\delta^{\nu}{}_{\sigma} + \tfrac{1}{2} b\overline{\kappa} g^{\mu 5} A^{\nu} = g^{\mu\nu} + \tfrac{1}{2} b\overline{\kappa} g^{\mu 5} A^{\nu} \approx g^{\mu\nu} + b\kappa\left(\phi^{\mu 5} - \tfrac{1}{2} b A^{\mu}\right) A^{\nu} . \tag{11.12}$$

where we have also used $g^{\mu 5} \approx \overline{\kappa}\left(\phi^{\mu 5} - \frac{1}{2} b A^{\mu}\right)$ from (11.10) and $2\kappa = \overline{\kappa}^2$ with $\hbar = c = 1$. Proceeding forward, we then employ $g^{\mu\nu} = \eta^{\mu\nu} + \overline{\kappa} h^{\mu\nu} = \eta^{\mu\nu} + \overline{\kappa}\left(\phi^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}\phi\right)$ to write:

$$g^{\mu\nu} \approx g^{\mu\nu} + b\kappa\left(\phi^{\mu 5} - \tfrac{1}{2} b A^{\mu}\right) A^{\nu} = \eta^{\mu\nu} + \overline{\kappa}\left(\phi^{\mu\nu} - \tfrac{1}{2}\eta^{\mu\nu}\phi\right) + b\kappa\left(\phi^{\mu 5} - \tfrac{1}{2} b A^{\mu}\right) A^{\nu} . \tag{11.13}$$

Restructuring a bit, and using $b^2 = 8$ from (10.14), we finally rewrite (11.13) as:

$$g^{\mu\nu} \approx \eta^{\mu\nu}\left(1 - \tfrac{1}{2}\overline{\kappa}\phi\right) + \overline{\kappa}\phi^{\mu\nu} + 2\sqrt{2}\kappa\phi^{\mu 5} A^{\nu} - 4\kappa A^{\mu} A^{\nu} . \tag{11.14}$$

Thus, in the *linear approximation* $A_{\Sigma;\mathrm{T}} \approx A_{\Sigma,\mathrm{T}}$, we find that $g^{\mu\nu}$ decomposes into the Minkowski tensor $\eta^{\mu\nu}$, a term $-\frac{1}{2}\overline{\kappa}\phi\eta^{\mu\nu}$ which effectively multiplies the Minkowski tensor by the factor $\left(1 - \frac{1}{2}\overline{\kappa}\phi\right)$ which includes a spin-0 scalar ϕ, a gravitational wave $\overline{\kappa}\phi^{\mu\nu}$ with a quantum mechanical interpretation as a graviton, a $2\sqrt{2}\kappa\phi^{\mu 5} A^{\nu}$ term which may be thought of as containing both a photon A^{ν} and the $\phi^{\mu 5}$ components of a graviton (remember, gravitation affects light propagation), and finally, the term $-4\kappa A^{\mu} A^{\nu}$ which contains two photons, and which is provocatively reminiscent of the $\left[G^{\mu}, G^{\nu}\right]$ terms which appear in the field strength tensor $F^{\mu\nu} = \partial^{[\mu} G^{\nu]} - ig\left[G^{\mu}, G^{\nu}\right]$ of Yang Mills theory, which in forms language is simply $F = dG + igG^2$.

So, we have now obtained contravariant expression for $g^{5\beta}$ which is akin to (11.4), namely (11.10), $g^{5\nu} \approx \overline{\kappa}\left(\phi^{5\nu} - \frac{1}{2} b A^{\nu}\right)$, so that this can be employed in the various equations where $g^{5\beta}$ appear. Along the way, we have seen the contravariant g^{MN} decompose into spin-2 gravitons, spin-1 photons, and spin-0 scalars. Now, let's see how (11.9) comes into play, particularly in those equations which contain the term $g^{5\beta} J_{\beta}$.

To sample (11.10) further, let's for example, substitute this into the "keystone" relationship (10.11), which we rewrite as $g^{5\beta} J_{\beta} - \frac{1}{8} b\overline{\kappa} F^{\sigma\tau} F_{\sigma\tau} = 0$ with $g^{55} = -1$. Because we have written (10.11) so as to be equal to zero, we can multiply through by any constant we choose. From the traceless Maxwell tensor $T^{\mu}{}_{\nu} = -\left(F^{\mu\tau}F_{\nu\tau} - \frac{1}{4}\delta^{\mu}{}_{\nu}F^{\sigma\tau}F_{\sigma\tau}\right)$ of (10.15), and (10.17), we know that $R = 0$. Let's choose to represent the "0" of (10.11) in terms of the Ricci scalar R of (10.7). We also use $g^{55} = -1$, $b^2 = 8$ and $2\kappa = \overline{\kappa}^2$ with $\hbar = c = 1$. So, by employing (11.10), and rearranging terms a bit, we may write (10.11) as:

$$0 = \frac{1}{4\kappa}R = \left(\frac{1}{2}b\overline{\kappa}g^{5\beta}J_{\beta} - \frac{1}{16}b^2\overline{\kappa}^2F^{\sigma\tau}F_{\sigma\tau}\right) \approx \frac{\sqrt{2}}{2}\phi^{5\beta}J_{\beta} - A^{\beta}J_{\beta} - \frac{1}{4}F^{\sigma\tau}F_{\sigma\tau} = \frac{\sqrt{2}}{2}\phi^{5\beta}J_{\beta} + \mathcal{L}_{QED}. \quad (11.15)$$

The fact that $\mathcal{L}_{QCD} = -A^{\beta}J_{\beta} - \frac{1}{4}F^{\sigma\tau}F_{\sigma\tau}$ now also appears in this way, is very much of interest, because it connects $\mathcal{L}_{QCD}$, and thereby the action $S(A^{\mu}) = \int \mathcal{L}_{QCD}dV$, to geometry. Once we have a geometric foundation for the action, we are but a single theoretical step away from employing that action in a path integral $Z = \int DAe^{iS(A)} \equiv e^{iW(J)}$, and thereby establishing a geometric foundation for the quantum field theory of electrodynamics. Of course, the calculations are far from trivial, and the connection in (11.15) is only in the linear approximation and is far-more complex when one must employ the exact $\frac{1}{4}b\overline{\kappa}A_{\Sigma;\mathrm{T}} = \frac{1}{2}g_{5\Sigma,\mathrm{T}} = \frac{1}{2}\overline{\kappa}h_{5\Sigma,\mathrm{T}}$ of (11.3), rather than the approximate $g_{5\Sigma} \approx \frac{1}{2}b\overline{\kappa}A_{\Sigma}$ of (11.4). But, in theory, (11.15) does establish a connection between geometrodynamics and quantum field theory.

Beyond the connection to $\mathcal{L}_{QCD}$, (11.15) is also very much of interest, because it demonstrates how any term:

$$g^{5\beta}J_{\beta} \approx \overline{\kappa}\left(\phi^{5\beta} - \frac{1}{2}bA^{\beta}\right)J_{\beta}, \quad (11.16)$$

taken in the linear approximation $A_{\Sigma;\mathrm{T}} \approx A_{\Sigma,\mathrm{T}}$, contains a source current J_{β} coupled to both a photon, $A^{\beta}J_{\beta}$, and a graviton, $\phi^{5\beta}J_{\beta}$.

This brings us full circle back to the early results of section 3, where in (3.2) we associated the ratio q/m of a material body with both "electrical mass" q and "inertial / gravitational" mass m with motion in the fifth dimension, which, with $\hbar = c = 1$, and knowing now that $b^2 = 8$, we may write as $\overline{\kappa}\frac{dx^5}{d\tau} = -\frac{\sqrt{2}}{2}\frac{q}{m}$. The whole purpose of this was to represent the Lorentz force as geodesic motion in five-dimensions, and provide a geometrodynamic foundation for the Newtonian "forces," in the sense of $F = ma$, experienced by any material body for which the electrical (interaction) mass is *inequivalent* to the gravitational (inertial) mass. Such a material body with both an electrical mass q and an inertial mass m will necessarily interact both electromagnetically, and gravitationally. In quantum mechanical terms, such a body will emit and absorb both photons and gravitons. Equation (11.16), derived from the linear approximation, represents the interaction via photons and gravitons, of a material body with both an electrical mass q, and an inequivalent inertial mass m.

12. Summary and Conclusion

We have attempted throughout the discussion to specify all of classical electrodynamics on the basis of a five-dimensional Kaluza-Klein type theory, built on the bedrock of Riemannian geometry, and independently of whether this fifth dimension is timelike or spacelike. The fact that it turns out that $g_{55} = g^{55} = \pm 1 = \text{constant}$ for respective timelike and spacelike extensions is very helpful in this endeavor, because one can capture the $\pm$ character of this contrast in fully covariant fashion. Only by deduction, in connection with deducing the Maxwell stress energy tensor in (10.15), do we find that this fifth dimension must be spacelike.

The assumptions used in this exposition were minimal, conservative, and based on what we observe to be true in the natural world. Our mathematical foundation was a five-dimensional Riemannian geometry, without any changes or enhancements, but merely extending the entire apparatus of gravitational theory into one more dimension. The key assumption, which drove almost all of the mathematical development, was the requirement effectively imposed by equations (2.7) and (2.8), that the Lorentz force as it is experimentally observed, must be represented as nothing other than geodesic motion in the five-dimensional Riemannian geometry. In section 3, where we started to implement this requirement, there was a freedom with respect to choosing the constant labeled b throughout. We carried this constant throughout the development, and only finally fixed its value in (10.14), when obtaining the Maxwell tensor.

In section 4, which is an "optional" digression from the main course of development, we showed that, *if* the fifth dimension is taken to be spacelike and compactified, *then* angular frequency movements (4.2) through this dimension, may well be the foundation of intrinsic spin, and that if this is so, then one can actually calculate the compactification radius with precision, and that this radius is quite close to the Planck length, as finally given in (8.12). In light of our subsequent finding in (10.14) that the fifth dimension is indeed spacelike, the precise nature of the spacelike fifth dimension does warrant deeper consideration.

We noted in section 4 that the neutrino may appear to present a problem for such an intrinsic spin interpretation, because it does not have electric charge. In hindsight, we now see that the theory developed herein is a U(1) theory of electromagnetism and gravitation. Specifically, the q in the q/m ratio upon which the intrinsic spin interpretation is based, is a U(1) charge generator. Therefore, the only particles one can talk about in this context are electrons, photons, and gravitons. Strictly speaking, one cannot even talk about neutrinos, unless and until the development here is extended to Yang-Mills theory, and specifically, the SU(2)xU(1) theory of electroweak interactions. When SU(2)xU(1) is considered, the (left-chiral) neutrino, though having $q = 0$, does obtain a non-zero weak isospin $I^3 = \frac{1}{2}$. This isospin charge, one would suspect, may provide the basis for understanding the intrinsic spin of the neutrino through a compactified fifth spatial dimension.

Section 5 introduced but one further assumption, as conservative as can be, that the field strength tensor, when extended to five dimensions, must continue to be fully antisymmetric, $F^{\mathrm{MN}} \equiv -F^{\mathrm{NM}}$. All else was deductive, and in the end, we found that all of these new components $F^{\mathrm{M5}} \equiv -F^{\mathrm{5M}}$ were equal to zero. We also found other helpful relations involving the metric tensor, the most important of which is that $g_{55} = g^{55} = \pm 1 = \text{constant}$, for a timelike and spacelike fifth dimension, respectively.

In section 6, we used the tools developed in section 5 to examine the Riemann tensor. Almost immediately, in (6.4), we deduced a critical, central relation which laid the foundation for casting both of Maxwell's equations, including sources, on a totally geometric foundation, see (6.6) and (6.11). We also, in the course of this development, demonstrated how one of the central terms of QED, $F^{\sigma\tau}F_{\sigma\tau} = \mathbf{B}^2 - \mathbf{E}^2$ in Minkowski space, is induced out of the five-dimensional gravitational interaction, see (6.8). This laid the groundwork for the later deduction of the Maxwell tensor in (10.15), and the uncovering of the QED Lagrangian in (11.15).

Section 7 laid some further deductive groundwork, but most importantly, enabled us to determine the exact expression for the five-dimensional Ricci scalar $R_{(5)} = R^{\sigma}{}_{\sigma} + R^{5}{}_{5}$, up to the four dimensional curvature scalar $R = R^{\sigma}{}_{\sigma}$, which was still left unknown.

In section 8, we reached a juncture. The Lorentz force geodesic requirement had done as much as it can, and we needed a new foundation to deduce the four dimensional scalar $R = R^{\sigma}{}_{\sigma}$. To do so, we adopted the space-time-matter view [3] that the action over five dimensions is to be expressed as the vacuum-type equation $S(g_{\mathrm{MN}}) \equiv \frac{1}{2\kappa}\int R_{(5)} dV = \int \left(\frac{1}{2\kappa} R + \frac{1}{2\kappa} R^{5}{}_{5}\right) dV$ of (8.1). A five-dimensional variation δg^{MN} enabled us to deduce that in the four dimensions of spacetime, $R_{\mu\nu} = 0$, equation (8.10). We also uncovered a possible non-symmetry in the fifth-dimensional components of R_{MN}, and in section 9, decided to accept this fifth-dimensional non-symmetry, summarized by $R_{\mathrm{M}5} = -\frac{1}{4} b\bar{\kappa} J_{\mathrm{M}}$, $R_{\mathrm{M}\nu} = 0$. In effect, the covariant (lower index) Ricci tensor contains a single column five-vector $R_{\mathrm{M}5} = -\frac{1}{4} b\bar{\kappa} J_{\mathrm{M}}$, and all other components $R_{\mathrm{M}\nu} = 0$ zero.

In section 10, based on this fifth-dimensional non-symmetry, we followed the same variational procedure as in section 8, but using only a four-dimensional variation $\delta g^{\mu\nu}$. This led to the single most important result of this entire effort, wherein we were able to deduce the Maxwell tensor $T^{\mu}{}_{\nu} = -\left(F^{\mu\tau} F_{\nu\tau} - \frac{1}{4}\delta^{\mu}{}_{\nu} F^{\sigma\tau} F_{\sigma\tau}\right)$ (10.15), free of the scalar trace, directly from the full Einstein equation $-\kappa T^{\mu}{}_{\nu} = R^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}{}_{\nu} R$ with the scalar trace. This empirically-supported result, more than any other, underscores that the five-dimensional theory developed here may in fact be a plausible candidate for describing the real, material universe in which we live.

Finally, in section 11, we made a foray into quantum theory, by developing the relationship between the electrodynamic vector and gravitational tensor potentials, and in the linear approximation, decomposing the contravariant g^{MN} into its physical constituents, see (11.10) and (11.14), plus $g^{55} = -1$. We also uncovered in the traceless Ricci scalar R of the Maxwell stress energy, the QED Lagrangian density $\mathfrak{L}_{QCD} = -A^{\beta} J_{\beta} - \frac{1}{4} F^{\sigma\tau} F_{\sigma\tau}$.

What we have *not* yet done here, is to select a Clifford Algebra so as to reproduce η^{MN}, which we have learned from (10.14) is given by $\eta^{\mathrm{MN}} = (1,-1,-1,-1,-1)$. This suggests that we choose $\Gamma^{\mathrm{M}} = (\gamma^{\mu}, -i\gamma^{5})$, which it will be noted, will yield the correct metric signature when employed in $\frac{1}{2}\left\{\Gamma^{\mathrm{M}}\Gamma^{\mathrm{N}} + \Gamma^{\mathrm{N}}\Gamma^{\mathrm{M}}\right\} \equiv \eta^{\mathrm{MN}}$. (See [11], section 3). Nor have we proceeded to develop all of the foregoing in a formal manner, via the Dirac equation. Nor have we at all explored the so-called "chirality problem" which is often seen to plague five-dimensional Kaluza-Klein theories, again, see, e.g., [11], section 3. It is, however, of interest that we appear to have uncovered a non-symmetry in the fifth dimension summarized by $R_{\mathrm{M}5} = -\frac{1}{4} b\overline{\kappa} J_{\mathrm{M}}$, $R_{\mathrm{M}\nu} = 0$, so, that, for example, $R_{\mu 5} \neq R_{5\mu}$. This is at least reminiscent of the fact that weak interactions are not symmetric under transformations between the projection operators $\gamma^\mu P_R \equiv \frac{1}{2}\left(\gamma^\mu + \gamma^\mu \gamma^5\right)$ and $\gamma^\mu P_L \equiv \frac{1}{2}\left(\gamma^\mu - \gamma^\mu \gamma^5\right) = \frac{1}{2}\left(\gamma^\mu - \gamma^5 \gamma^\mu\right)$, i.e., under transposition of the μ and 5 indexes. Thus, there may be within these results, a plausible foundation for understanding the breaking of chiral symmetry. These are all important avenues for exploration, which will be left to a future endeavor in the hope that the connections which have been established here – despite those which have not – will nevertheless lend some plausibility to the development presented here.

Now, let's step back and look at the larger picture.

Ever since Galileo's refutation of Aristotle which legend situates at the Leaning Tower of Pisa, it has been understood that heavier and lighter gravitational masses similarly-disposed in a gravitational field will accelerate at the same rate and reach the ground after identical times have elapsed, because of the so-called "weak equivalence" of gravitational and inertial mass. As a material body becomes more massive and so more-susceptible to the pull of a gravitational field (back when gravitation was viewed as action at a distance), so too this increase in massiveness causes the material body in equal measure to resist the gravitational pull. Along his path to developing the General Theory of Relativity (GTR), Albert Einstein made a brief stop in 1911 in an imaginary elevator, to conduct a *gedanken* in which he concluded that the physical experience of an observer falling freely in a gravitational field before terminally hitting the ground is no different from what was commonly thought of as Newton's inertial motion in which a body in motion remained in motion unless acted upon by a "force." [18] With the exception of the tidal forces later found in the General Theory of Relativity, this *gedanken* remains an accurate guide to the present day.

But electrical masses have long presented a dilemma, because the electrical mass of a material body, say, an electron, is *not* equal to its inertial mass, and this inequivalence is the mainspring of the forces we feel which clearly, as a physical sensation, differentiate the acceleration of Newton's a=F/m from that of the gravitational a=9.8 meters/sec^2 near the surface of the earth. The General Theory of Relativity, in the end, captured inertial motion and its close cousin of free-fall motion in a gravitational field, in the most elegant way, as simple geodesic motion in a curved Riemannian geometry along geodesic paths which coincide precisely with the paths one observes for bodies moving under gravitational influences. But the electrical motion of the Lorentz force has long been the "odd man out," because it was something distinct from gravitation: it did not appear to follow a geodesic path, and it did not "feel" like inertial or gravitational free fall motions because it created the sensation of a force which we can measure when we place a scale between ourselves and the ground on which we stand or the elevator which accelerates us upward, because of the collective electrical repulsion between billions of electrons in our bodies and billions more in the surface against which we are pressing.

The key to unlocking this mystery, and ultimately, to placing gravitation and electromagnetism within the same framework, is to understand the motion of electrical masses – as governed by the experimentally-grounded Lorentz force law – to be geodesic motion no less that that of gravitation, but in a spacetime that is extended to contain a single additional fifth dimension: the dimension first proposed long ago by Kaluza and Klein. By placing electrical masses onto their own geodesics in a five-dimensional Riemannian geometry which embeds the spacetime of our daily experience into its seamless fabric, we find that the long-standing quest to unite gravitation and electrodynamics may finally arrive at a safe haven on a firm foundation.

Because the remaining interactions of nature are but carbon copies of electrodynamics, based on a group theory pioneered by Yang and Mills, the five-dimensional unity of electro-gravitational phenomenon exposited here, may presage the development of a non-Abelian Kaluza-Klein spacetime geometry which could make further strides toward uncovering nature's underlying unity.

References

[1] Kaluza, T., On the problem of unity in physics, *Sitzungsber. Preuss. Akad. Wiss. Berlin.* (*Math. Phys.*) 966-972 (1921).

[2] Klein, O., Quantum theory and five dimensional theory of relativity, *Z. Phys.* 37 895-906 (1926).

[3] See the Introduction at http://astro.uwaterloo.ca/~wesson/.

[4] See the Members page at http://astro.uwaterloo.ca/~wesson/.

[5] Wheeler, J. A., *Geometrodynamics*, Academic Press, pp. 225-253 (1962).

[6] A good discussion of the Einstein-Hilbert action is found at http://en.wikipedia.org/wiki/Einstein-Hilbert_action.

[7] Billyard, A., Wesson, P.S. 1996. Gen. Rel. Grav. 28, 129.

[8] Bars, I., *Survey of Two-Time Physics*, http://arxiv.org/abs/hep-th/0008164 (2000). See also http://www.physorg.com:80/news98468776.html.

[9] Matsuda, S., and Seki, S., *Gravitational Stability and Screening Effect from D Extra Timelike Dimensions*, http://arxiv.org/abs/hep-th/0008216 (2000).

[10] Misner, C. W., Wheeler, J. A., and Thorne, K. S., *Gravitation*, Freeman (1973).

[11] Sundrum, R., *TASI 2004 Lectures: To the Fifth Dimension and Back*, http://arxiv.org/abs/hep-th/0508134 (2005).

[12] Wheeler, J. A., *On the Nature of Quantum Geometrodynamics*, Annals of Physics: 2, 604-614 (1957).

[13] The author thanks and acknowledges Daryl McCullough of Ithaca, New York for pointing out some of these implications of this intrinsic spin interpretation in an online discussion.

[14] Einstein, A., *Relativistic Theory of the Non-Symmetry Field*, in "The Meaning of Relativity," Fifth Edition, (1956), pp. 133-166.

[15] Ashtekar, A. et al., *Revisiting the Foundations of Relativistic Physics*, Springer (2003).

[16] Einstein. A., *Do Gravitational Fields Play an Essential Part in the Structure of the Elementary Particles of Matter*, in "The Principle of Relativity," Dover (1952).

[17] Witten, E., *Duality, Spacetime and Quantum Mechanics*, Physics Today, (May 1997)

[18] Einstein, A., *On the Influence of Gravitation on the Propagation of Light*, Annalen der Physik, 35 (1911).