Might Foldy-Wouthuysen Transformations Contain a Hidden Fermion Mass Generation Mechanism?

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1. Introduction

I have been studying the following three links for the Foldy-Wouthuysen transformation from the Dirac-Pauli to the Newton-Wigner representation of Dirac's equation. The first shows the calculation itself of this transformation:

I: http://www.physics.ucdavis.edu/~cheng/230A/RQM7.pdf.

The second, an excellent and lucid exposition of the physics (why this is of interest), is to be found at:

II: http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.27.3209&rep=rep1&type=pdf.

The third, dealing with Zitterbewegung motion and the velocity operator in the Dirac-Pauli representation, is at:

III: http://en.wikipedia.org/wiki/Zitterbewegung.

What I would like to discuss, is the possibility that a mechanism for generating fermion mass may be hidden in all of this, and to show a preliminary calculation.

I say this in particular because in the Dirac-Pauli representation, the velocity operator is given by:

$$v^k = \alpha^k \quad (1.1)$$

where $\alpha^k = \gamma^0\gamma^k$, see reference III. Further, the eigenvalues of this velocity operator constrain the velocity of the Fermion of be the speed of light, see reference II in the middle of page 3. This means that the fermion must be massless and luminous, in the Dirac-Pauli representation. Why this is so, has long been a mystery, and is thought not to make any sense, for obvious reasons. In particular, it is often asserted that the velocity operator in the Dirac-Pauli representation has "no physical meaning."

Now, transform into the Newton-Wigner representation via Foldy-Wouthuysen. The velocity operator in Newton-Wigner now takes the classical form:

$$v^k = dx^k / dt \quad (1.2)$$

where x^k is the position operator. But even more importantly, Newton-Wigner permits a range of eigenvalues less than the speed of light, and so, the fermions permitted by Newton-Wigner are massless and sub-luminous.

Following this to its logical conclusion, this seems to suggest that somewhere hidden in the Foldy-Wouthuysen transformation, we have gone from a fermion which is massless and luminous, to one which has a finite, non-zero rest mass and travels at sub-luminous velocity. It seems, then, that it would be important to specifically trace how the velocity operator (1) of the Dirac-Pauli representation with $\pm c$ eigenvalues transforms into the velocity operator (2) of Newton-Wigner which allows a continuous, sub-luminous velocity spectrum, and at the same time, to trace through how the rest mass goes from necessarily zero (with decoupled chiral components), to non-zero with chiral couplings. By doing so, perhaps one would find a mechanism for generating fermion masses.

One contrast to make here: think about how vector boson masses are generated. One starts with a Lagrangian in which the boson mass term is omitted entirely. Then, via a well-knows technique, one breaks the symmetry and reveals a boson mass. Perhaps the mystery of luminous velocity eigenvalues in the Dirac-Pauli representation is telling us a similar thing: Start out with a Dirac-Pauli Lagrangian in which the mass of the fermion is zero, i.e., without a mass term. After all, it is known that prior to symmetry breaking thought to entail a Higgs interaction, the Dirac Lagrangian needs in any event to omit a fermion mass term. Then, in this context, the +/- c velocity eigenvalues make sense. Transform that into the Newton-Wigner representation. Somewhere along the line, a mass must appear, because a subliminous velocity appears. After all, even though observed fermions has mass, nobody expects that the masses should be placed into the Dirac Lagrangian "by hand." Rather, one should expect to start with a zero mass fermion which then is, by definition, luminous, and by some process of symmetry breaking, to arrive at a non-zero rest mass.

Below, I lay out a preliminary calculation.

2. The Foldy-Wouthuysen Transformation for a Free Fermion

Let us first review the standard Foldy-Wouthuysen transformation for a free electron. Such an electron is specified by Dirac's equation, written in momentum space as:

$$m\psi = \gamma^\mu p_\mu \psi = \gamma^0 p_0 \psi + \gamma^k p_k \psi \quad (2.1)$$

In the usual way, one multiplies through by γ^0 and then rearranges terms to write:

$$H\psi \equiv p_0\psi = \gamma^0 m\psi - \alpha^k p_k\psi = \gamma^0 m\psi - \alpha^k p_k\psi = \beta m\psi + \boldsymbol{\alpha}\cdot\mathbf{p}\psi \quad (2.2)$$

where $\alpha^k \equiv \gamma^0\gamma^k$, $\beta \equiv \gamma^0$, and where we use $diag(\eta_{\mu\nu}) = (+1,-1,-1,-1)$ for the metric tensor so that $p_k = -p^k$ and so $\alpha^k p_k = -\boldsymbol{\alpha}\cdot\mathbf{p}$. It is clear from (2.2), that the p_0 component of the energy-momentum four-vector specifies the eigenvalues of the Hamiltonian operator matrix:

$$H \equiv \boldsymbol{\alpha}\cdot\mathbf{p} + \beta m. \quad (2.3)$$

This is the starting point for taking the Foldy-Wouthuysen transformation.

Now, we subject the fermion wavefunction to the unitary transformation:

$$\psi \to \psi' = e^{iS}\psi, \text{ hence } \psi = e^{-iS}\psi' \quad (2.4)$$

where S is Hermitian. This means that $H\psi = He^{-iS}\psi'$. We then subject H to a bi-unitary transformation:

$$H' = e^{iS}He^{-iS}, \quad (2.5)$$

so that

$$e^{iS}H\psi = e^{iS}He^{-iS}\psi' = H'\psi' \quad (2.6)$$

The customary choice of the unitary operator e^{iS} is:

$$e^{iS} = e^{\beta\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}\theta} = \cos\theta + \beta\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}\sin\theta \quad (2.7)$$

where $\hat{\mathbf{p}} \equiv \mathbf{p}/|\mathbf{p}|$. That (2.7) is true, can be verified by taking the series expansion of $e^{\beta\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}\theta}$ and using the multiplication and commutation properties of α, β. This also means that the inverse unitary operator:

$$e^{-iS} = e^{-\beta\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}\theta} = \cos\theta - \beta\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}\sin\theta \quad (2.8)$$

Now, we use this together with (2.3) to expand and successively refine (2.5), as such:

$$\begin{aligned} H' = e^{iS}He^{-iS} &= (\cos\theta + \beta\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}\sin\theta)(\boldsymbol{\alpha}\cdot\mathbf{p} + \beta m)(\cos\theta - \beta\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}\sin\theta) \\ &= (\boldsymbol{\alpha}\cdot\mathbf{p} + \beta m)(\cos\theta - \beta\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}\sin\theta)^2 \\ &= (\boldsymbol{\alpha}\cdot\mathbf{p} + \beta m)e^{-2\beta\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}\theta} \\ &= (\boldsymbol{\alpha}\cdot\mathbf{p} + \beta m)(\cos 2\theta - \beta\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}\sin 2\theta) \end{aligned} \quad (2.9)$$

The second line makes use of $\beta\boldsymbol{\alpha} = -\boldsymbol{\alpha}\beta$, so that each of $\beta\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}(\boldsymbol{\alpha}\cdot\mathbf{p}) = -(\boldsymbol{\alpha}\cdot\mathbf{p})\beta\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}$ and $\beta\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}(\beta m) = -(\beta m)\beta\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}$. The third line uses (2.8) written as $(\cos\theta - \beta\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}\sin\theta)^2 = e^{-2\beta\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}\theta}$.

The fourth uses (2.8) written as $e^{-2\beta\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}\theta} = \cos 2\theta - \beta\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}\sin 2\theta$. Now let's expand the final line of (2.9) to write:

$$
\begin{aligned}
H' = e^{iS} H e^{-iS} &= (\boldsymbol{\alpha}\cdot\mathbf{p} + \beta m)(\cos 2\theta - \beta\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}\sin 2\theta) \\
&= +\boldsymbol{\alpha}\cdot\mathbf{p}\cos 2\theta - \beta m\beta\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}\sin 2\theta + \beta m\cos 2\theta - \boldsymbol{\alpha}\cdot\mathbf{p}\beta\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}\sin 2\theta \\
&= +\boldsymbol{\alpha}\cdot\mathbf{p}\left(\cos 2\theta - \frac{m}{|\mathbf{p}|}\sin 2\theta\right) + \beta\left(m\cos 2\theta + \frac{1}{|\mathbf{p}|}(\boldsymbol{\alpha}\cdot\mathbf{p})^2\sin 2\theta\right) \qquad (2.10) \\
&= +\boldsymbol{\alpha}\cdot\mathbf{p}\left(\cos 2\theta - \frac{m}{|\mathbf{p}|}\sin 2\theta\right) + \beta\left(m\cos 2\theta + |\mathbf{p}|\sin 2\theta\right)
\end{aligned}
$$

In the third line we $\beta^2 = 1$, and also $\beta\boldsymbol{\alpha} = -\boldsymbol{\alpha}\beta$ which flips the sign in the β term. In the final line, we use $\boldsymbol{\alpha}^2 = 1$ and $\mathbf{p}^2/|\mathbf{p}| = \left(p_x{}^2 + p_y{}^2 + p_z{}^2\right)/\left|\sqrt{p_x{}^2 + p_y{}^2 + p_z{}^2}\right| = |\mathbf{p}|$.

Now, we impose the condition that H' be a strictly diagonal operator. Therefore, we force the $\boldsymbol{\alpha}\cdot\mathbf{p}$ term to vanish, by imposing:

$\tan 2\theta \equiv |\mathbf{p}|/m$. (2.11)

By basic trigonometry, this also means that:

$\sin 2\theta = |\mathbf{p}|/\sqrt{m^2 + |\mathbf{p}|^2}$ and $\cos 2\theta = m/\sqrt{m^2 + |\mathbf{p}|^2}$. (2.12)

Substituting (2.12) into the remaining β term of (2.10) then yields:

$$
\boxed{H' = e^{iS} H e^{-iS} = \beta\left(\frac{m^2 + |\mathbf{p}|^2}{\sqrt{m^2 + |\mathbf{p}|^2}}\right) = \beta\sqrt{m^2 + |\mathbf{p}|^2}} \qquad (2.13)
$$

which is the Hamiltonian operator in the Newton-Wigner representation of the Dirac equation.

3. What Happens to the Rest Mass During a Foldy-Wouthuysen Transformation?

Now, let's specifically track what happens to the Fermion rest mass *m* during the free-fermion Foldy-Wouthuysen transformation reviewed above. Let us first return to (2.2), which we write as:

$H\psi = p_0\psi = \beta m\psi + \boldsymbol{\alpha}\cdot\mathbf{p}\psi$. (3.1)

We next use $\psi = e^{-iS}\psi'$ from (2.4) to write:

$He^{-iS}\psi' = p_0 e^{-iS}\psi' = \beta m e^{-iS}\psi' + \boldsymbol{\alpha}\cdot\mathbf{p}e^{-iS}\psi'$. (3.2)

Then, we multiply from the left by e^{iS} and use (2.5) to write:

$$H'\psi' = e^{iS}He^{-iS}\psi' = e^{iS}p_0e^{-iS}\psi' = p_0\psi' = e^{iS}\beta me^{-iS}\psi' + e^{iS}\boldsymbol{\alpha}\cdot\mathbf{p}e^{-iS}\psi' . \quad (3.3)$$

where $e^{iS}p_0e^{-iS}\psi' = p_0\psi'$, because, as noted above (2.3), the p_0 component of the energy-momentum four-vector specifies the eigenvalues of the Hamiltonian operator matrix and $p_0 = Ip_0$ is simply p_0 times a 4x4 unit matrix.

Then, we rearrange (3.3) to isolate the rest mass, thus:

$$e^{iS}\beta me^{-iS}\psi' = H'\psi' - e^{iS}\boldsymbol{\alpha}\cdot\mathbf{p}e^{-iS}\psi' = p_0\psi' - e^{iS}\boldsymbol{\alpha}\cdot\mathbf{p}e^{-iS}\psi' . \quad (3.4)$$

Note from (2.13), that $H' = \beta\sqrt{m^2 + |\mathbf{p}|^2}$. Now, in the above, the mass *m* is, of course, a scalar. But the "mass matrix" $M \equiv \beta m$ is *not* a scalar but rather, is a matrix which in general will share the same commutativity properties as $\beta \equiv \gamma^0$. If $M \equiv \beta m$ is *defined* as the mass matrix in the Dirac/Pauli representation, then we may define the matrix:

$$M' \equiv e^{iS}Me^{-iS} = e^{iS}\beta me^{-iS} \quad (3.5)$$

as the mass matrix in the Newton-Wigner representation. This is simply *M*, subject to a bi-unitary transformation. Therefore, we now rewrite (3.4) using (3.5) as:

$$M'\psi' = H'\psi' - e^{iS}\boldsymbol{\alpha}\cdot\mathbf{p}e^{-iS}\psi' = p_0\psi' - e^{iS}\boldsymbol{\alpha}\cdot\mathbf{p}e^{-iS}\psi' . \quad (3.6)$$

which also contains $H'\psi' = p_0\psi'$.

Now, we apply (2.7) and (2.8) to write (3.6) as:

$$\begin{aligned} M'\psi' &= H'\psi' - (\cos\theta + \beta\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}\sin\theta)\boldsymbol{\alpha}\cdot\mathbf{p}(\cos\theta - \beta\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}\sin\theta)\psi' \\ &= p_0\psi' - (\cos\theta + \beta\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}\sin\theta)\boldsymbol{\alpha}\cdot\mathbf{p}(\cos\theta - \beta\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}\sin\theta)\psi' \end{aligned} \quad (3.7)$$

Now, let's calculate from the above. First, since $\beta\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}(\boldsymbol{\alpha}\cdot\mathbf{p}) = -(\boldsymbol{\alpha}\cdot\mathbf{p})\beta\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}$, this becomes:

$$\begin{aligned} M'\psi' &= H'\psi' - \boldsymbol{\alpha}\cdot\mathbf{p}(\cos\theta - \beta\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}\sin\theta)^2\psi' \\ &= p_0\psi' - \boldsymbol{\alpha}\cdot\mathbf{p}(\cos\theta - \beta\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}\sin\theta)^2\psi' \end{aligned} . \quad (3.8)$$

Via $(\cos\theta - \beta\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}\sin\theta)^2 = e^{-2\beta\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}\theta} = \cos 2\theta - \beta\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}\sin 2\theta$, this next becomes:

$$\begin{aligned} M'\psi' &= H'\psi' - \boldsymbol{\alpha}\cdot\mathbf{p}(\cos 2\theta - \beta\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}\sin 2\theta)\psi' \\ &= p_0\psi' - \boldsymbol{\alpha}\cdot\mathbf{p}(\cos 2\theta - \beta\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}\sin 2\theta)\psi' \end{aligned} . \quad (3.9)$$

Now, we make use of (2.12), and $H' = \beta\sqrt{m^2 + |\mathbf{p}|^2}$ from (2.13), to rewrite this as:

$$M'\psi' = H'\psi' - \boldsymbol{\alpha}\cdot\mathbf{p}\left(\frac{m}{\sqrt{m^2+|\mathbf{p}|^2}} - \beta\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}\frac{|\mathbf{p}|}{\sqrt{m^2+|\mathbf{p}|^2}}\right)\psi'$$
$$= p_0\psi' - \boldsymbol{\alpha}\cdot\mathbf{p}\left(\frac{m}{\sqrt{m^2+|\mathbf{p}|^2}} - \beta\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}\frac{|\mathbf{p}|}{\sqrt{m^2+|\mathbf{p}|^2}}\right)\psi' , \quad (3.10)$$

where, from (2.13), $H' = \beta\sqrt{m^2+|\mathbf{p}|^2}$. From here, we simply reduce using $\hat{\mathbf{p}} = \mathbf{p}/|\mathbf{p}|$,

$\boldsymbol{\alpha}\beta = -\beta\boldsymbol{\alpha}$, and $\boldsymbol{\alpha}^2 = 1$, thus (contrast $M\psi \equiv \beta m\psi$):

$$\boxed{M'\psi' = H'\psi' - \left(\frac{m\boldsymbol{\alpha}\cdot\mathbf{p} + \beta|\mathbf{p}|^2}{\sqrt{m^2+|\mathbf{p}|^2}}\right)\psi' = p_0\psi' - \left(\frac{m\boldsymbol{\alpha}\cdot\mathbf{p} + \beta|\mathbf{p}|^2}{\sqrt{m^2+|\mathbf{p}|^2}}\right)\psi'} . \quad (3.11)$$

Now, let's return to our starting point. The goal was to see if we could start with a zero rest mass in the Dirac-Pauli representation, which would then have a velocity operator which constrain the fermion to move at the speed of light, and end up the Newton-Wigner, via Foldy-Wouthuysen, at a non-zero rest mass in the new representation. Thus, in (3.11), we now set $m = 0$, to obtain:

$$M'\psi' = (H' - \beta|\mathbf{p}|)\psi' = (p_0 - \beta|\mathbf{p}|)\psi' . \quad (3.12)$$

Therefore, given a four-vector $p^\mu = (p^0, \mathbf{p})$, the mass matrix operator in the Newton-Wigner representation is given by:

$$M' = (p_0 - \beta|\mathbf{p}|) = \begin{pmatrix} p_0 - |\mathbf{p}| & 0 \\ 0 & p_0 + |\mathbf{p}| \end{pmatrix} \quad (3.13)$$

This non-zero mass matrix arises even though the mass *m* and the mass matrix $M \equiv \beta m$ in the Dirac representation is zero.