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Massive Meson Phenomenology, Spontaneous Symmetry Breaking, and Propagator Matrices in Yang Mills Theory

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Abstract:

By giving careful attention to the formation of propagators in Yang Mills groups, particularly by understanding the term $p^\sigma p_\sigma - m^2$ not as a propagator denominator but rather as a matrix of internal symmetries the inverse of which multiplies the propagator spin sum, we carefully construct the matrix inverse $\left(p^\sigma p_\sigma - m^2\right)^{-1}$ in SU(2) as a "warm up" exercise, and then in SU(3). This includes a generalized approach to spontaneously break symmetry in Yang-Mills groups SU(N) with N>2. We then develop the framework for systematically characterizing the observed meson masses and lifetimes of QCD, while naturally overcoming the plague of infinite propagator poles without resort to any special measures.

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1. Introduction

The purpose of this paper is to develop an approach by which the massiveness, and the mass spectrum, of the massive vector mesons of QCD might be understood. That is, we seek to understand as simply as possible, the origin of experimentally-observed QCD vector meson masses such as can be found, for example, in the PDG table at [1]. Finding out how the vector mesons of QCD obtain their non-zero masses which make the strong QCD interaction short range despite its supposedly-massless gluons, is one aspect of the so-called "mass gap" problem, see [2] at page 3.

We start by reviewing how gauge boson mass is known to be generated in SU(2), as a template for considering SU(3) QCD. Since the SU(2) approach shown through equations (2.4) below is a well-known "warm-up" for developing SU(2)xU(1) electroweak interactions, we shall not discuss electroweak theory *per se*. Rather, we shall review how massive vector particles obtain mass in SU(2) via spontaneous symmetry breaking, and then seek to extend this to SU(3).

2. A Warm Up Exercise: Spontaneous Symmetry Breaking and Propagator Development in SU(2), and Non-Abelian Fourier Transforms

For the development throughout, we shall use the following Lagrangian density:

$$\begin{aligned}\mathfrak{L} &= \mathrm{Tr}\left(G_\mu\left(g^{\mu\nu}\partial^\sigma\partial_\sigma - \partial^\nu\partial^\mu\right)G_\nu\right) + g^2\varphi^\dagger G^\sigma G_\sigma\varphi + 2g\,\mathrm{Tr}\left(G_\mu J^\mu\right) - V\left(\varphi^\dagger\varphi\right) + \ldots \\ &= \tfrac{1}{2}G_{i\mu}\left(g^{\mu\nu}\partial^\sigma\partial_\sigma - \partial^\nu\partial^\mu\right)G^i{}_\nu + g^2\varphi^\dagger G^\sigma G_\sigma\varphi + gG_{i\mu}J^{i\mu} - V\left(\varphi^\dagger\varphi\right) + \ldots\end{aligned} \quad . \tag{2.1}$$

Above, $G^\mu \equiv T^i G_i^{\ \mu}$ and $J^\mu \equiv T^i J_i^{\ \mu}$ are Hermitian $N\times N$ matrices for $SU(N)$, g is the group coupling, and φ is a scalar with N complex components and $2N$ degrees of freedom. We also apply the normalization $\delta^i{}_j = 2\mathrm{Tr}\left(T^i T_j\right)$, which accounts for the factor of ½ between the top and bottom lines above. The reader will recognize that this is *only part of the standard model Lagrangian*. We have omitted terms, including those of higher than second order in G^μ and φ, which are not needed for this development below. The above $\mathfrak{L}$ may be applied to any non-Abelian, Yang Mills group SU(N).

Now, we turn specifically to SU(2), and focus on the term $g^2\varphi^\dagger G^\sigma G_\sigma\varphi$. We first define the scalar field in the usual manner:

$$\varphi = \frac{1}{\sqrt{2}}\begin{pmatrix} \varphi_1 + i\varphi_2 \\ \varphi_3 + i\varphi_4 \end{pmatrix}. \tag{2.2}$$

This contains 4(=2N) scalar degrees of freedom. Taking $V(\varphi^\dagger\varphi) \equiv \mu^2(\varphi^\dagger\varphi) + \lambda(\varphi^\dagger\varphi)^2 + \ldots$ as usual, we find the stationary points via $dV/\partial\varphi^\dagger = \mu^2\varphi + 2\lambda(\varphi^\dagger\varphi)\varphi = 0$ to define a non-trivial minimum at $(\varphi^\dagger\varphi)_0 = -\mu^2/2\lambda \equiv \frac{1}{2}v^2$,* where the $_0$ subscript specifies that this is the stationary point. Then we break the symmetry of $(\varphi^\dagger\varphi)_0 = \frac{1}{2}(\varphi_1{}^2 + \varphi_2{}^2 + \varphi_3{}^2 + \varphi_4{}^2) = \frac{1}{2}v^2$ by choosing $\varphi_3 = v$ and $\varphi_1 = \varphi_2 = \varphi_4 = 0$. Thus, showing a full expanded matrix calculation as a prelude to examining SU(3), and using the familiar Pauli spin matrices $\sigma^i = 2T^i$ of $O(3)$ rotations, we find:

$$\begin{aligned} g^2\varphi^\dagger G^\sigma G_\sigma\varphi &= g^2\varphi^\dagger T^i G_i{}^\sigma T^j G_{j\sigma}\varphi = \tfrac{1}{4}g^2\varphi^\dagger\sigma^i G_i{}^\sigma\sigma^j G_{j\sigma}\varphi \\ &= \tfrac{1}{8}\begin{pmatrix} v & 0 \end{pmatrix}\begin{pmatrix} G_3{}^\sigma & G_1{}^\sigma - iG_2{}^\sigma \\ G_1{}^\sigma + iG_2{}^\sigma & -G_3{}^\sigma \end{pmatrix}\begin{pmatrix} G_{3\sigma} & G_{1\sigma} - iG_{2\sigma} \\ G_{1\sigma} + iG_{2\sigma} & -G_{3\sigma} \end{pmatrix}\begin{pmatrix} 0 \\ v \end{pmatrix} \\ &= \tfrac{1}{8}v^2 g^2\left(G_1{}^\sigma G_{1\sigma} + G_2{}^\sigma G_{2\sigma} + G_3{}^\sigma G_{3\sigma}\right) = \tfrac{1}{2}M^2 G_i{}^\sigma G^i{}_\sigma = M^2\mathrm{Tr}(G^\sigma G_\sigma) \end{aligned} , \tag{2.3}$$

with $M = M_{(1)} = M_{(2)} = M_{(3)} = \frac{1}{2}vg$ "revealed" to be the masses of three gauge bosons $G_i{}^\sigma$.

At this point, we insert the results from (2.3) back into (2.1) to obtain:

$$\begin{aligned} \mathfrak{L} &= \mathrm{Tr}\left(G_\mu\left(g^{\mu\nu}\left(\partial^\sigma\partial_\sigma + M^2\right) - \partial^\nu\partial^\mu\right)G_\nu\right) + 2g\mathrm{Tr}\left(G_\mu J^\mu\right) - V(\varphi^\dagger\varphi) + \ldots \\ &= \tfrac{1}{2}G_{i\mu}\left(g^{\mu\nu}\left(\partial^\sigma\partial_\sigma + M^2\right) - \partial^\nu\partial^\mu\right)G^i{}_\nu + gG_{i\mu}J^{i\mu} - V(\varphi^\dagger\varphi) + \ldots \end{aligned} , \tag{2.4}$$

which takes the term $g^{\mu\nu}\partial^\sigma\partial_\sigma - \partial^\nu\partial^\mu$ and turns it into the Proca term $g^{\mu\nu}\left(\partial^\sigma\partial_\sigma + M^2\right) - \partial^\nu\partial^\mu$ for a massive particle. All of this is standard, known development. Now, we engage in a further exercise which is unnecessary overkill for SU(2), but which will be vital in considering SU(3).

It is known that the propagator $D_{\nu\lambda}(p^\sigma)$ will be specified in momentum space, following a Fourier transform, by:

$$D_{\nu\lambda}(p^\sigma)\left(g^{\mu\nu}\left(\partial^\sigma\partial_\sigma + M^2\right) - \partial^\nu\partial^\mu\right)e^{ip^\sigma x_\sigma} = \delta^\mu{}_\lambda e^{ip^\sigma x_\sigma}. \tag{2.5}$$

However, because this is SU(2), the $p^\sigma = T^i p_i{}^\sigma$ in the Fourier factor $e^{ip^\sigma x_\sigma}$ needs to be a 2x2 Hermitian matrix. Why? If one considers $\left(g^{\mu\nu}\partial^\sigma\partial_\sigma - \partial^\nu\partial^\mu\right)G_\nu$ from (2.1), we know that the

* $v = 246.220\,\mathrm{GeV}$ is the vacuum expectation value (vev) based on the Fermi coupling constant $\sqrt{2}G_F = 1/v^2 = (1/246.220\,\mathrm{GeV})^2$ in units of $\hbar = c = 1$.

operator $\partial^{\mu} = \partial^{\mu} I_{(2\times 2)}$ is really a 2x2 unit matrix of four-gradients operating on the 2x2 Hermitian matrix G_{ν}. Once we perform the operation $\partial^{\mu} G_{\nu}$, this 2x2 Hermitian matrix structure will be "inherited" from G_{ν}, into the ∂^{μ} into the momentum space, and the way this is achieved in the calculation (2.5) is to employ the non-Abelian momentum $p^{\sigma} = T^{i} p_{i}^{\sigma}$ in the Fourier transform. Additionally, because there are three gauge bosons in SU(2), we in any event need three p_{i}^{σ}, which represent momenta associated with each of the corresponding A_{i}^{σ}, and from which the 2x2 matrix $p^{\sigma} = T^{i} p_{i}^{\sigma}$ is naturally formed. Finally, not only is this so-justified prospectively, it is also justified "retrospectively" by the problems which are resolved once we do employ $p^{\sigma} = T^{i} p_{i}^{\sigma}$ in the Fourier transform. As we shall now see, employing $p^{\sigma} = T^{i} p_{i}^{\sigma}$ in the Fourier transform resolves long-standing problems associated with infinite propagator "poles," and as we shall later see, this enables us to develop a framework to characterize the observed spectrum of QCD meson masses.

Using the non-Abelian $p^{\sigma} = T^{i} p_{i}^{\sigma}$, the term $\partial^{\nu}\partial^{\mu} e^{ip^{\sigma}x_{\sigma}} = i\partial^{\nu} p^{\mu} e^{ip^{\sigma}x_{\sigma}} = -p^{\nu} p^{\mu} e^{ip^{\sigma}x_{\sigma}}$ is straightforward to obtain, and so we use this to rewrite (2.5) above as:

$$D_{\nu\lambda}\left(p^{\sigma}\right)\left(-g^{\mu\nu}\left(p^{\sigma} p_{\sigma} - M^{2} I_{2\times 2}\right) - p^{\nu} p^{\mu}\right) = \delta^{\mu}{}_{\lambda} I_{2\times 2}. \tag{2.6}$$

It is *very important* to keep in mind that the above is a 2x2 matrix equation, because $p^{\sigma} = T^{i} p_{i}^{\sigma}$ is now a Yang-Mills (non-Abelian) momentum. Thus, $M^{2} = M^{2} I_{2\times 2}$ as well must be proportional to a 2x2 unit matrix. Consequently, without yet focusing on the known properties the *particular* 2x2 internal symmetry matrices $p^{\sigma} p_{\sigma}$ and $p^{\nu} p^{\mu}$ which happen to be based on the unitary generators of SU(2), which happen to be Hermitian, etc., the propagators $D_{\nu\lambda}\left(p^{\sigma}\right)$, to be completely accurately specified, *must be written using matrix inverses*, as:

$$D_{\nu\lambda}\left(p^{\sigma}\right) = \left(-g_{\nu\lambda} + \frac{p_{\lambda} p_{\nu}}{M^{2}}\right) \times \left\{p^{\sigma} p_{\sigma} - M^{2}\right\}^{-1}. \tag{2.7}$$

In particular, as a general rule, we simply cannot just put the 2x2 matrix $p^{\sigma} p_{\sigma} - M^{2}$ into a denominator because matrix mathematics does not work that way. In fact, because $p^{\sigma} p_{\sigma} - M^{2}$ is a matrix, if one were to write the "mass shell" relation as $\left(p^{\sigma} p_{\sigma} - M^{2}\right) x = 0$, where *x* is some two-component vector in the SU(2) internal symmetry space, one would have to say that M^2 gives the eigenvalues of vectors *x* for the matrix $p^\sigma p_\sigma$.

Of course, once we consider the particular 2x2 matrices which specify SU(2), which happen to be Hermitian, and all of which square to unity, this is substantial overkill, because:

$$\begin{aligned} p^\sigma p_\sigma - M^2 &= \begin{pmatrix} p_3{}^\sigma & p_1{}^\sigma - ip_2{}^\sigma \\ p_1{}^\sigma + ip_2{}^\sigma & -p_3{}^\sigma \end{pmatrix}\begin{pmatrix} p_{3\sigma} & p_{1\sigma} - ip_{2\sigma} \\ p_{1\sigma} + ip_{2\sigma} & -p_{3\sigma} \end{pmatrix} - \begin{pmatrix} M^2 & 0 \\ 0 & M^2 \end{pmatrix} \\ &= \begin{pmatrix} p_i{}^\sigma p^i{}_\sigma - M^2 & 0 \\ 0 & p_i{}^\sigma p^i{}_\sigma - M^2 \end{pmatrix} = \left(p_i{}^\sigma p^i{}_\sigma - M^2\right) I_{2\times 2} \end{aligned}, \tag{2.8}$$

that is, $p^\sigma p_\sigma - M^2$ is simply $p_i^\sigma p^i{}_\sigma - M^2$ times a unit matrix, and so can be inverted easily:

$$D_{\nu\lambda}\left(p^\sigma\right) = \left(-g_{\nu\lambda} + \frac{p_\lambda p_\nu}{M^2}\right) \times \begin{pmatrix} \dfrac{1}{p_i{}^\sigma p^i{}_\sigma - M^2} & 0 \\ 0 & \dfrac{1}{p_i{}^\sigma p^i{}_\sigma - M^2} \end{pmatrix} = \frac{-g_{\nu\lambda} + \dfrac{p_\lambda p_\nu}{M^2}}{p_i{}^\sigma p^i{}_\sigma - M^2} I_{2\times 2}. \tag{2.9}$$

This is *implicit* in known weak and electroweak interaction theory, though likely because of the ease of this inversion, this leading to calculation (2.9) is not ordinarily carried out.

We now take one final step: We take each of the three SU(2) gauge bosons G_i^μ for which the masses are $M = M_{(1)} = M_{(2)} = M_{(3)} = \frac{1}{2} vg$, and place them *on mass shell*. That is, we set $p_1{}^\sigma p^1{}_\sigma = p_2{}^\sigma p^2{}_\sigma = p_3{}^\sigma p^3{}_\sigma = M^2$, and we do not worry about, e.g., taking $p_3{}^\sigma p^3{}_\sigma - M^2 = 0$, because now, this is a matrix which is naturally written without difficulty as the eigenvalue equation $\left(p_3{}^\sigma p^3{}_\sigma - M^2\right)x = 0$, as noted in passing, above. Thus, with all the gauge bosons on mass shell, $p_i{}^\sigma p^i{}_\sigma = 3M^2$, and $p_i{}^\sigma p^i{}_\sigma - M^2 = 2M^2$. Then, we substitute this back into (2.9) to write:

$$D_{\nu\lambda}\left(p^\sigma\right) = \left(-g_{\nu\lambda} + \frac{p_\lambda p_\nu}{M^2}\right) \times \begin{pmatrix} \dfrac{1}{2M^2} & 0 \\ 0 & \dfrac{1}{2M^2} \end{pmatrix}. \tag{2.10}$$

which is manifestly finite, i.e., absent of infinite poles.

Finally, we return to the SU(3) current vector $J^\mu \equiv T^i J_i{}^\mu$, and use (2.10) above in the expression arrived at through the path integral:

$$W(J) = -\tfrac{1}{2}\int \frac{d^4 p}{(2\pi)^4}\mathfrak{M} = -\int \frac{d^4 p}{(2\pi)^4}\mathrm{Tr}\left[J^\nu(p^\sigma)D_{\nu\lambda}(p^\sigma)J^\lambda(p^\sigma)\right], \qquad (2.11)$$

where $\mathfrak{M}$ is an invariant amplitude, and where we need to use a trace on the right hand side, as is evident from the top line of (2.1), which trace also removes the ½ factor on the right hand side. We then use the fact that $J^\mu \equiv T^i J_i^{\ \mu}$ are also 2x2 Hermitian matrices, to rewrite the amplitude part of (2.11) as:

$$\begin{aligned}\tfrac{1}{2}\mathfrak{M} &= \mathrm{Tr}\left[J^\nu D_{\nu\lambda} J^\lambda\right] = J_i^{\ \nu} J_j^{\ \lambda}\mathrm{Tr}\left[T^i D_{\nu\lambda} T^j\right] \\ &= J_i^{\ \nu} J_j^{\ \lambda}\mathrm{Tr}\left[T^i\left(-g_{\nu\lambda} + \frac{p_\lambda p_\nu}{M^2}\right)\times\begin{pmatrix}\frac{1}{2M^2} & 0 \\ 0 & \frac{1}{2M^2}\end{pmatrix}T^j\right].\end{aligned} \qquad (2.12)$$

Where $p_\nu \cong (M,0,0,0)$ is comparatively small, and because $\mathrm{diag}(g_{\nu\lambda}) = (1,-1,-1,-1)$ absent a gravitational field of consequence, we may approximate $\mathrm{diag}\left(-g_{\nu\lambda} + \frac{p_\lambda p_\nu}{M^2}\right) \cong (0,-1,-1,-1)I_{2\times 2}$. So, we may further remove this "spin sum" term $-g_{\mu\nu} + p_\mu p_\nu / M^2 = \sum_{\lambda=-1,0,1}\varepsilon_\mu^{\ (\lambda)} * \varepsilon^{(\lambda)}{}_\nu$ from inside the trace, leaving:

$$\begin{aligned}\tfrac{1}{2}\mathfrak{M} &= J_i^{\ \nu} J_j^{\ \lambda}\mathrm{Tr}\left[T^i D_{\nu\lambda} T^j\right] = J_i^{\ \nu} J_j^{\ \lambda}\left(-g_{\nu\lambda} + \frac{p_\lambda p_\nu}{M^2}\right)\mathrm{Tr}\left[T^i\begin{pmatrix}\frac{1}{2M^2} & 0 \\ 0 & \frac{1}{2M^2}\end{pmatrix}T^j\right]. \\ &\equiv \tfrac{1}{4}J_i^{\ \nu} J_j^{\ \lambda}\left(-g_{\nu\lambda} + \frac{p_\lambda p_\nu}{M^2}\right)\left(\frac{1}{\mathrm{M}^2}\right)^{ij}\end{aligned} \qquad (2.13)$$

The heart of what we are after is *defined* above as the "inverse square mass matrix":

$$\tfrac{1}{4}\left(\frac{1}{\mathrm{M}^2}\right)^{ij} = \mathrm{Tr}\left[T^i\begin{pmatrix}\frac{1}{2M^2} & 0 \\ 0 & \frac{1}{2M^2}\end{pmatrix}T^j\right] = \tfrac{1}{4}\mathrm{Tr}\left[\sigma^i\begin{pmatrix}\frac{1}{2M^2} & 0 \\ 0 & \frac{1}{2M^2}\end{pmatrix}\sigma^j\right] = \mathrm{Tr}\left[T^i\left\{p^\sigma p_\sigma - M^2\right\}^{-1}T^j\right], \qquad (2.14)$$

which contains 9 components for SU(2). In general, for SU(N), the analogous matrix will contain $(N^2-1)^2$ components. Thus, for SU(3), this is a 64 component matrix. The factor of ¼ is to account for the normalization factor of ½ in $T^i = \tfrac{1}{2}\sigma^i$, given the two generators T^i, T^j which sandwich the matrix in (2.14). Now, how do we use this matrix?

For SU(2), using $T^i = \frac{1}{2}\sigma^i$, one may calculate that:

$$\left(\frac{1}{\mathrm{M}^2}\right)^{ij} = \mathrm{Tr}\left[\sigma^i \begin{pmatrix} \frac{1}{2M^2} & 0 \\ 0 & \frac{1}{2M^2} \end{pmatrix} \sigma^j\right] = \begin{pmatrix} \frac{1}{M^2} & 0 & 0 \\ 0 & \frac{1}{M^2} & 0 \\ 0 & 0 & \frac{1}{M^2} \end{pmatrix} = \begin{pmatrix} \left(\frac{2}{vg}\right)^2 & 0 & 0 \\ 0 & \left(\frac{2}{vg}\right)^2 & 0 \\ 0 & 0 & \left(\frac{2}{vg}\right)^2 \end{pmatrix}. \quad (2.15)$$

Therefore, (2.13) now reads:

$$\begin{aligned} \mathfrak{M} = J_i{}^{\nu} J_j{}^{\lambda} \mathrm{Tr}\left[T^i D_{\eta\nu} T^j\right] &= \tfrac{1}{2}\left(-g_{\mu\nu} + \frac{p_\mu p_\nu}{M^2}\right) J_i{}^{\nu} \begin{pmatrix} \frac{1}{M^2} & 0 & 0 \\ 0 & \frac{1}{M^2} & 0 \\ 0 & 0 & \frac{1}{M^2} \end{pmatrix} J_j{}^{\lambda} \\ &= \tfrac{1}{2}\left(-g_{\nu\lambda} + \frac{p_\lambda p_\nu}{M^2}\right)\left(\frac{J_1{}^{\nu} J_1{}^{\lambda}}{M^2} + \frac{J_2{}^{\nu} J_2{}^{\lambda}}{M^2} + \frac{J_3{}^{\nu} J_3{}^{\lambda}}{M^2}\right) \\ &= \tfrac{1}{2}\left(\varepsilon_\mu{}^{(-1)} * \varepsilon^{(-1)}{}_\nu + \varepsilon_\mu{}^{(0)} * \varepsilon^{(0)}{}_\nu + \varepsilon_\mu{}^{(1)} * \varepsilon^{(1)}{}_\nu\right)\left(\frac{J_1{}^{\nu} J_1{}^{\lambda}}{M^2} + \frac{J_2{}^{\nu} J_2{}^{\lambda}}{M^2} + \frac{J_3{}^{\nu} J_3{}^{\lambda}}{M^2}\right) \end{aligned}, \quad (2.16)$$

where we expressly show via the spin sum how there are actually 3x3=9 terms in the amplitude including for the transverse ($\lambda = \pm 1$) and longitudinal ($\lambda = 0$) polarizations. For the *on-diagonal* 1-1, 2-2 and 3-3 transitions, we simply read off the denominators in $\left(1/\mathrm{M}^2\right)^{ij}$, and find that the mass of the vector bosons which mediate each of these transitions is $M = \frac{1}{2}vg$. For the *off-diagonal* transitions, we read this to say that formally speaking, the mass of any vector boson which mediates these transitions is *infinite*, which simply means that this transition is forbidden because it contributes "zero" to the amplitude. There are only three masses observed mediating the 1-1, 2-2 and 3-3 transitions, and these masses all happen to be the same. Of course, this changes a bit in SU(2)xU(1) electroweak theory, but it is not our objective to review that here. The point is to be clear about how the masses which were first revealed in (2.3) following spontaneous symmetry breaking, make their way into the denominators of the terms in the amplitude where they come to specify the observed vector boson masses.

If, for contrast to the foregoing, we consider a propagator written in the usual form:

$$D_{\nu\lambda}\left(p^{\sigma}\right)=\frac{-g_{\nu\lambda}+\dfrac{p_{\lambda}p_{\nu}}{M^{2}}}{p^{\sigma}p_{\sigma}-M^{2}}, \tag{2.17}$$

we see that what is of fundamental importance about (2.16), and a solid "retrospective" justification for using the non-Abelian $p^{\sigma}=T^{i}p_{i}^{\ \sigma}$ in the Fourier transform (2.5), is that this completely inverts the usual problem with singular, propagator poles in propagator theory which require use of the "$+i\varepsilon$ prescription" or other means to dodge around the infinite poles. In (2.16), not only do we have a *finite* amplitude on the diagonal, but a *zero* amplitude off the diagonal. That is, all terms in the amplitude are finite.

It should also be seen that if the *M* in (2.16) had been $M=0$ instead of finite, i.e., if we had not turned the term $g^{\mu\nu}\partial^{\sigma}\partial_{\sigma}-\partial^{\nu}\partial^{\mu}$ into $g^{\mu\nu}\left(\partial^{\sigma}\partial_{\sigma}+M^{2}\right)-\partial^{\nu}\partial^{\mu}$ in (2.4) by the spontaneous breaking of symmetry, then the diagonal would then have been infinite, and it would have become necessary to use methods such as that of Faddeev-Popov to fix the gauge for terms like $g^{\mu\nu}\partial^{\sigma}\partial_{\sigma}-\partial^{\nu}\partial^{\mu}$ which do not have an inverse. Fortunately, (2.16) averts this problem, because the *M* is not zero, and because the off-diagonal terms are zero. But most importantly, the need to fix the gauge when the mass is zero *only occurs in the special case of SU(2)*. As we shall later see, for SU(3) and higher, we can invert a matrix based even on the massless $g^{\mu\nu}\partial^{\sigma}\partial_{\sigma}-\partial^{\nu}\partial^{\mu}$ term, and may do so naturally, without having lift a finger to avert propagator poles by a variety of creative, but physically non-elegant, means.

In sum, what is already interesting about the result in (2.16) – even without yet moving on to SU(3) – is that it neatly solves the problem of propagator poles, simply because we take special care to form and carefully apply the matrix inverse $\left\{p^{\sigma}p_{\sigma}-M^{2}\right\}^{-1}$ in the propagator, even though this term is easily invertible to $1/\left(p_{i}^{\ \sigma}p^{i}{}_{\sigma}-M^{2}\right)$ and so tempts one to ignore these issues of properly developing the matrix inversion, as discussed in (2.8) and (2.9). Despite its seeming-obviousness based on a careful consideration of matrix algebra, this approach does not appear to heretofore have been discovered.

To simplify this result as much as possible for generality, and provide the simplest possible roadmap into SU(3), we note that because of the normalization $\mathrm{Tr}\left(T^{i}T_{j}\right)=\frac{1}{2}\delta^{i}{}_{j}$ which may be applied to *any* SU(N), the generators are reduced by a factor of 2, just as in SU(2) via

$T^i = \frac{1}{2}\sigma^1$. In SU(3), and generally for higher order groups, we will $T^i = \frac{1}{2}\lambda^i$ with a similar normalization, and we use Γ to denote λ^i in any Yang-Mills SU(N) group. Thus, we can bury the ¼ factor in (2.14) into Γ, and write out the mass **M** of the vector mesons, in general, to be determined in relation to the mass **m** of the gauge bosons, according to the simple and general:

$$\boxed{\frac{1}{\mathbf{M}^2} = \mathrm{Tr}\left[\Gamma\left\{\mathbf{p}^2 - \mathbf{m}^2\right\}^{-1}\Gamma\right]}. \tag{2.18}$$

We then conclude with one further point of interest. In this special-case context of SU(2) (and by extension electroweak SU(2)xU(1)), we find that for the three gauge bosons / vector mesons, $\mathbf{M} = \mathbf{m}$, so that *these masses observably manifest as one and the same*. That is, the gauge bosons of the SU(2) theory are synonymous with the vector bosons experimentally observed in the laboratory. *In SU(3), as we shall see, this is no longer the case*. The gauge bosons of SU(3) are typically taken to be 8 massless "gluons." The vector mesons of SU(3) are characterized in a plethora of particle data that reveals many more than 8 vector mesons, see, e.g., [1]. In SU(3), the **M** and **m** in (2.18) are decidedly not the same, $\mathbf{M} \neq \mathbf{m}$. The gluons are hidden from our experimental view, because the only *observable* in (2.18) is $\mathbf{M}$ and not $\mathbf{m}$.

Put in another way, referring to (2.3), the *observability* of a vector mass **M** comes *not* from the appearance of *M* in the term $g^2\varphi^\dagger G^\sigma G_\sigma \varphi = \frac{1}{2}M^2 G_i^{\ \sigma} G^i_{\ \sigma}$ in the $\mathfrak{L}$ of (2.4), but *rather from the appearance of M in the amplitude* $\mathfrak{M}$ *in (2.16)*. The only exception is SU(2) (and SU(2)xU(1)), because here $\mathbf{M} = \mathbf{m}$ and the gauge bosons manifest as free particles. Not so for the gluons of SU(3) or any larger group. In these larger groups, only **M** is observed but *not* **m**. The fact that in Quantum Chromodynamics, we only observe mesons and not gluons, may help "explain why we never see individual quarks," which is the second leg of the "mass gap" problem [2], and is fundamentally intertwined with the confinement issue.

3. Prelude to SU(3), and Spontaneous Symmetry Breaking in N>2 Gauge Groups

Beyond its (not negligible) utility in averting infinite propagator poles without resort to questionable means, the approach in section 2 is overkill for SU(2). But for any higher gauge group, this approach is vital if one is to understand the observed mass spectrum of, for example, the many vector mesons of SU(3) QCD. This is because for SU(3) and any larger groups, $\left\{p^\sigma p_\sigma - M^2\right\}^{-1}$ has on-diagonal elements which are not all the same, as well as off-diagonal elements which are non-zero, and so will not invert as in (2.8), (2.9) above, without a full, careful application of matrix inversion methods.

There are a few other things which change in SU(3) and larger groups of which it is also important to be aware, as we embark on detailed calculation. In SU(2), the masses determined via $\left(1/\mathrm{M}^2\right)^{ij}$ in (2.15) are the same as the masses determined in the expression $\frac{1}{2}M^2 G_i^{\ \sigma} G^i{}_\sigma$ in (2.3). Simply by way of nomenclature, let us now *define* the masses which appear in the term $g^2\varphi^\dagger G^\sigma G_\sigma \varphi$ after symmetry breaking as "gauge boson masses," and let us *define* the masses which appear in a $\left(1/\mathrm{M}^2\right)^{ij}$ matrix as in (2.15) as "vector meson masses," notwithstanding how these terms may be used elsewhere. This will give us the language to discuss these issues, whereby we carefully and deliberately distinguish the $g^2\varphi^\dagger G^\sigma G_\sigma \varphi$ "gauge boson" masses from the $\left(1/\mathrm{M}^2\right)^{ij}$ "vector meson" masses. In the special case of SU(2), *but only in this special case*, the "gauge boson masses" are identical with the "vector meson masses" because of the simple propagator inversion developed above in (2.9). *In SU(3) and higher, these two types of mass are not the same.*

As we shall see, in SU(3), there are only *three different gauge boson masses* after symmetry breaking: one mass >0 for the $G^{8\mu}$, one mass >0 for all of the $G^{7,6,5,4\mu}$, and finally, a *zero mass* for all of the $G^{3,2,1\mu}$. The 2N=6 degrees of freedom in the complex SU(3) scalar φ give mass > 0 to the *five new gauge bosons* that are added when one goes from SU(2) to SU(3), and the sixth degree of freedom is left over for the Higgs field. This is just as in SU(2) where there are four degrees of freedom, three of which are swallowed by the gauge bosons to acquire a longitudinal polarization and gain non-zero mass, and one of which stays with the Higgs.

In fact, this approach to breaking symmetry can be generalized to *any* SU(N). That is, for any SU(N), there are 2N degrees of freedom in the scalar φ, and the number of new gauge bosons introduced in going from any SU(N-1) to SU(N) is always equal to 2N-1. Thus, we can always give mass to only the 2N-1 new gauge bosons by setting $\varphi_1 = v$ in the scalar φ when we break symmetry, leave over a single degree of freedom for the Higgs, and can leave massless all the gauge bosons of the SU(N-1) subgroup. Thus, the vacuum remains invariant under transformations within the SU(N-1) subgroup. As a result, the $G^{\left(N^2-1\right)\mu}$ will always have one mass, the $G^{\left(N^2-2\right)\ldots(N-1)^2\mu}$ a second, different mass, and the remaining $G^{\left((N-1)^2-1\right)\ldots 1\mu}$ will be massless. This will, for any SU(N), put *three* masses (one of which is zero) rather than one into the $\left(\partial^\sigma \partial_\sigma + M^2\right)$ of (2.5), and thus yield *three* $\left(1/\mathrm{M}^2\right)^{ij}$ matrices rather than one by the time we arrive at the analog of (2.13). For example, in SU(3), of the three $\left(1/\mathrm{M}^2\right)^{ij}$ matrices, one is associated with the $G^{8\mu}$ gauge boson, one with the $G^{7,6,5,4\mu}$ gauge bosons, and one with the $G^{3,2,1\mu}$ gauge bosons.

Turning from the gauge boson masses to what we are calling the "vector meson masses," each of the three $\left(1/\mathrm{M}^2\right)^{ij}$ contains 8x8=64 components, a number of which are *independent* of one another. The 56 off-diagonal components are related (not independent) via transposition and so *at most* 56/2=28 can be mutually independent. Meanwhile, up to all 8 of the diagonal components can be independent, for a total of up to 28+8=36 *independent* "vector meson masses." With three of these $\left(1/\mathrm{M}^2\right)^{ij}$ for any SU(3), this makes available up to 36x3=108 *distinct and independent* "vector meson" masses, although some of these do turn out to be the same and others turn out to be zero (forbidden transitions), so the actual number is somewhat reduce as we shall see.

In general, adding the number of elements on the diagonal to half the number of elements off the diagonal and multiplying by three, the maximum number V_{Max} of distinct vector meson masses which can obtained is from any SU(N) by applying this symmetry breaking approach is specified by:

$$V_{Max} = 3\times\left[\left(N^2-1\right)+\left[\left(N^2-1\right)^2-\left(N^2-1\right)\right]/2\right]=3\times N^2\left(N^2-1\right)/2\,. \qquad (3.1)$$

In addition, as we shall see later, some of these masses are further "split" into more than one mass by various factors involving square roots and even fourth roots of $i \equiv \sqrt{-1}$.

Because there will always be three $\left(1/\mathrm{M}^2\right)^{ij}$ matrices no matter what the gauge group (which means that if SU(4) is a "leptoquark" gauge group there will still be only three $\left(1/\mathrm{M}^2\right)^{ij}$ matrices), one might look to this as a possible origin of *generation replication*, but it is too early to tell until a firm fit with experimental data is obtained. Certainly, no matter what, the vector meson masses contained in three $\left(1/\mathrm{M}^2\right)^{ij}$ will be classified most broadly by a) which of the precisely three $\left(1/\mathrm{M}^2\right)^{ij}$ they originate from, b) whether they originate from on or off the diagonal, and c) whether each of the two indexes $i, j = N^2 - 1$, or $N^2 - 2 \ge i, j \ge (N-1)^2$, or $(N-1)^2 - 1 \ge i, j \ge 1$, including each member of an index pair i, j being in the same range as one another, or being in different ranges, and d) whether they contain certain imaginary factors which further split the mass values. Section 11, infra, provides a more detailed look at all of this.

Finally, in SU(3), and indeed for any higher SU(N), some of the elements of the $(1/\mathrm{M}^2)^{ij}$ are *real, finite* numbers, some are *complex* numbers, some are *imaginary* numbers, and some are *zero*. These are suggestive, respectively, of *stable massive* particles, *massive* particles with specified *lifetimes*, *massless* particles with specified *lifetimes*, and zero-amplitude transitions.

Because this approach can be used to yield non-zero vector meson masses in QCD yielding the observed short range of strong interactions, this could resolve an important part of the so-called "mass gap" problem. And because the matrices $(1/\mathrm{M}^2)^{ij}$ can be used to generate numerical data in gory detail, again see section 11 infra, these predicted masses should be abundantly falsifiable using experimental particle data already in existence. We now turn to a detailed calculation along the foregoing lines, for SU(3).

4. SU(3) Symmetry Breaking, and Fourier Transformation to Momentum Space

We now return to Lagrangian (2.1), and start by considering the term $g^2\varphi^\dagger G^\sigma G_\sigma \varphi$ for the specific Yang Mills group SU(3). First, we form:

$$G_\sigma = \lambda^i G_{i\sigma} = \tfrac{1}{2}\begin{pmatrix} \frac{2}{\sqrt{3}}G_{8\sigma} & G_{6\sigma} - iG_{7\sigma} & G_{4\sigma} - iG_{5\sigma} \\ G_{6\sigma} + iG_{7\sigma} & -\frac{1}{\sqrt{3}}G_{8\sigma} + G_{3\sigma} & G_{2\sigma} - iG_{1\sigma} \\ G_{4\sigma} + iG_{5\sigma} & G_{2\sigma} + iG_{1\sigma} & -\frac{1}{\sqrt{3}}G_{8\sigma} - G_{3\sigma} \end{pmatrix} \qquad (4.1)$$

using the customary $T^i = \lambda^i$ SU(3) generators. We define a three-dimensional complex scalar field analogous to (2.2) with six degrees of freedom:

$$\varphi = \frac{1}{\sqrt{2}}\begin{pmatrix} \varphi_1 + i\varphi_2 \\ \varphi_3 + i\varphi_4 \\ \varphi_5 + i\varphi_6 \end{pmatrix}. \qquad (4.2)$$

With $V(\varphi^\dagger\varphi) \equiv \mu^2(\varphi^\dagger\varphi) + \lambda(\varphi^\dagger\varphi)^2 + \ldots$ we again use $(\varphi^\dagger\varphi)_0 = -\mu^2/2\lambda \equiv \tfrac{1}{2}v^2$ to find stationary points via $dV/\partial\varphi^\dagger = \mu^2\varphi + 2\lambda(\varphi^\dagger\varphi)\varphi = 0$, to define a $(\varphi^\dagger\varphi)_0 = -\mu^2/2\lambda \equiv \tfrac{1}{2}v^2$ non-trivial minimum.* We now we break symmetry of $\left(\varphi^{\dagger}\varphi\right)_0 = \frac{1}{2}\left(\varphi_1{}^2 + \varphi_2{}^2 + \varphi_3{}^2 + \varphi_4{}^2 + \varphi_5{}^2 + \varphi_6{}^2\right) = \frac{1}{2}v^2$ by choosing $\varphi_1 = v$ and $\varphi_2 = \varphi_3 = \varphi_4 = \varphi_5 = \varphi_6 = 0$. We then combine (4.1) and (4.2) to write:

$$G_\sigma \varphi = \frac{1}{2\sqrt{2}}\begin{pmatrix} \frac{2}{\sqrt{3}}G_{8\sigma} & G_{6\sigma} - iG_{7\sigma} & G_{4\sigma} - iG_{5\sigma} \\ G_{6\sigma} + iG_{7\sigma} & -\frac{1}{\sqrt{3}}G_{8\sigma} + G_{3\sigma} & G_{2\sigma} - iG_{1\sigma} \\ G_{4\sigma} + iG_{5\sigma} & G_{2\sigma} + iG_{1\sigma} & -\frac{1}{\sqrt{3}}G_{8\sigma} - G_{3\sigma} \end{pmatrix}\begin{pmatrix} v \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2\sqrt{2}}v\begin{pmatrix} \frac{2}{\sqrt{3}}G_{8\sigma} \\ G_{6\sigma} + iG_{7\sigma} \\ G_{4\sigma} + iG_{5\sigma} \end{pmatrix}, \tag{4.3}$$

therefore $\left(G^\sigma \varphi\right)^{\dagger} = \varphi^{\dagger} G^\sigma = \frac{1}{2\sqrt{2}} v\left(\frac{2}{\sqrt{3}}G^{8\sigma} \quad G^{6\sigma} - iG^{7\sigma} \quad G^{4\sigma} - iG^{5\sigma}\right)$, from which we deduce:

$$\begin{aligned} g^2\varphi^{\dagger}G^\sigma G_\sigma \varphi &= \frac{1}{8}v^2\left(\frac{2}{\sqrt{3}}G^{8\sigma} \quad G^{6\sigma} - iG^{7\sigma} \quad G^{4\sigma} - iG^{5\sigma}\right)\begin{pmatrix} \frac{2}{\sqrt{3}}G_{8\sigma} \\ G_{6\sigma} + iG_{7\sigma} \\ G_{4\sigma} + iG_{5\sigma} \end{pmatrix} \\ &= \frac{1}{8}v^2 g^2\left(\frac{4}{3}G_8{}^\sigma G_{8\sigma} + G^{7\sigma}G_{7\sigma} + G_6{}^\sigma G_{6\sigma} + G_5{}^\sigma G_{5\sigma} + G_4{}^\sigma G_{4\sigma}\right) \\ &= \begin{pmatrix} \frac{1}{6}v^2 g^2 G_8{}^\sigma G_{8\sigma} + \frac{1}{8}v^2 g^2 G_7{}^\sigma G_{7\sigma} + \frac{1}{8}v^2 g^2 G_6{}^\sigma G_{6\sigma} + \frac{1}{8}v^2 g^2 G_5{}^\sigma G_{5\sigma} + \frac{1}{8}v^2 g^2 G_4{}^\sigma G_{4\sigma} \\ + (0)G_3{}^\sigma G_{3\sigma} + (0)G_2{}^\sigma G_{2\sigma} + (0)G_1{}^\sigma G_{1\sigma} \end{pmatrix} \\ &\equiv \begin{pmatrix} \frac{1}{2}M_{(8)}{}^2 G_8{}^\sigma G_{8\sigma} + \frac{1}{2}M_{(7)}{}^2 G_7{}^\sigma G_{7\sigma} + \frac{1}{2}M_{(6)}{}^2 G_6{}^\sigma G_{6\sigma} + \frac{1}{2}M_{(5)}{}^2 G_5{}^\sigma G_{5\sigma} + \frac{1}{2}M_{(4)}{}^2 G_4{}^\sigma G_{4\sigma} \\ + \frac{1}{2}M_{(3)}{}^2 G_3{}^\sigma G_{3\sigma} + \frac{1}{2}M_{(2)}{}^2 G_2{}^\sigma G_{2\sigma} + \frac{1}{2}M_{(1)}{}^2 G_1{}^\sigma G_{1\sigma} \end{pmatrix} \end{aligned} \quad ,\text{(4.4)}$$

where the final line specifies the form expected in $\mathcal{L}$ for gauge boson masses. Consequently, we now extract the revealed gauge boson masses:

$$\begin{aligned} M_{(8)} &= \frac{1}{\sqrt{3}}vg \\ M_{(7)} = M_{(6)} = M_{(5)} = M_{(4)} &= \frac{1}{2}vg\ . \\ M_{(3)} = M_{(2)} = M_{(1)} &= 0 \end{aligned} \tag{4.5}$$

As described in section 3, five of the six scalar degrees of freedom in φ of (4.2) are swallowed to give mass > 0 to the five additional gauge bosons $G^{8\ldots4\mu}$ which arise when going from SU(2) to SU(3), the sixth degree of freedom goes over to Higgs as in SU(2) and SU(2)xU(1), and masses of the gauge bosons from the lower-rank embedded SU(2) subgroup are zero.** There are three distinct mass values revealed, namely, $\frac{1}{\sqrt{3}}vg$, $\frac{1}{2}vg$, and 0. And, the above approach may be generalized to larger gauge groups as outlined in section 3, though for now, we stick with SU(3).

---

* We shall regard *v* here, as a vev to be determined by experimental data, which as we will later see, in all likelihood does not turn out to coincide with the Fermi vev used in electroweak interactions.

** This leave open the possibility that unbroken SU(2) subgroup might be separately broken at the Fermi vev while crossed with U(1) to yield electroweak interactions, which could serve to unify the electroweak with the strong interaction.

Putting these revealed masses (4.5) back into (2.1) to arrive at the Proca terms is a little bit tricky, and must be done with care. Using the bottom line of (2.1), we expand out:

$$\begin{aligned}\mathfrak{L} &= \tfrac{1}{2}G_{i\mu}\left(g^{\mu\nu}\partial^\sigma\partial_\sigma - \partial^\nu\partial^\mu\right)G^i{}_\nu + g^2\varphi^\dagger G^\sigma G_\sigma\varphi \\ &= \tfrac{1}{2}G_{8\mu}\left(g^{\mu\nu}\left(\partial^\sigma\partial_\sigma + M_{(8)}{}^2\right) - \partial^\nu\partial^\mu\right)G^8{}_\nu \\ &+ \left\{\begin{array}{l}\tfrac{1}{2}G_{7\mu}\left(g^{\mu\nu}\left(\partial^\sigma\partial_\sigma + M_{(7)}{}^2\right) - \partial^\nu\partial^\mu\right)G^7{}_\nu + \tfrac{1}{2}G_{6\mu}\left(g^{\mu\nu}\left(\partial^\sigma\partial_\sigma + M_{(6)}{}^2\right) - \partial^\nu\partial^\mu\right)G^6{}_\nu \\ + \tfrac{1}{2}G_{5\mu}\left(g^{\mu\nu}\left(\partial^\sigma\partial_\sigma + M_{(5)}{}^2\right) - \partial^\nu\partial^\mu\right)G^5{}_\nu + \tfrac{1}{2}G_{4\mu}\left(g^{\mu\nu}\left(\partial^\sigma\partial_\sigma + M_{(4)}{}^2\right) - \partial^\nu\partial^\mu\right)G^4{}_\nu\end{array}\right\}, \\ &+ \left\{\begin{array}{l}\tfrac{1}{2}G_{3\mu}\left(g^{\mu\nu}\left(\partial^\sigma\partial_\sigma + M_{(3)}{}^2\right) - \partial^\nu\partial^\mu\right)G^3{}_\nu + \tfrac{1}{2}G_{2\mu}\left(g^{\mu\nu}\left(\partial^\sigma\partial_\sigma + M_{(2)}{}^2\right) - \partial^\nu\partial^\mu\right)G^2{}_\nu \\ + \tfrac{1}{2}G_{1\mu}\left(g^{\mu\nu}\left(\partial^\sigma\partial_\sigma + M_{(1)}{}^2\right) - \partial^\nu\partial^\mu\right)G^1{}_\nu\end{array}\right\}\end{aligned} \tag{4.6}$$

where we have grouped the terms in accordance with the three distinct masses in (4.5). To consolidate large expressions such as (4.6), we shall use $M_{(8)} = \frac{1}{\sqrt{3}}vg$ as is, but will define $M_{(7...4)} \equiv M_{(7)} = M_{(6)} = M_{(5)} = M_{(4)} = \frac{1}{2}vg$ to be representative of the $M_{(7...4)}$ mass values, and will define $M_{(3...1)} \equiv M_{(3)} = M_{(2)} = M_{(1)} = 0$ to be representative of this final set of mass values. We shall use, e.g., the notation $G_{7...4\mu}G^{7...4}{}_\nu \equiv G_{7\mu}G^7{}_\nu + G_{6\mu}G^6{}_\nu + G_{5\mu}G^5{}_\nu + G_{4\mu}G^4{}_\nu$ to represent an implied sum over the indicated subset of the Latin indexes. With this notational compaction, we rewrite (4.6) above as:

$$\begin{aligned}\mathfrak{L} &= \tfrac{1}{2}G_{i\mu}\left(g^{\mu\nu}\partial^\sigma\partial_\sigma - \partial^\nu\partial^\mu\right)G^i{}_\nu + g^2\varphi^\dagger G^\sigma G_\sigma\varphi \\ &= \tfrac{1}{2}G_{8\mu}\left(g^{\mu\nu}\left(\partial^\sigma\partial_\sigma + M_{(8)}{}^2\right) - \partial^\nu\partial^\mu\right)G^8{}_\nu + \tfrac{1}{2}G_{7...4\mu}\left(g^{\mu\nu}\left(\partial^\sigma\partial_\sigma + M_{(7...4)}{}^2\right) - \partial^\nu\partial^\mu\right)G^{7...4}{}_\nu\,, \\ &+ \tfrac{1}{2}G_{3...1\mu}\left(g^{\mu\nu}\partial^\sigma\partial_\sigma - \partial^\nu\partial^\mu\right)G^{3...1}{}_\nu\end{aligned} \tag{4.7}$$

including setting $M_{(3...1)} = 0$. This final point means that one of these factors, $g^{\mu\nu}\partial^\sigma\partial_\sigma - \partial^\nu\partial^\mu$, remains in the massless form usually thought to have no inverse and so giving rise to the need for Faddeev-Popov or similar gauge fixing. As we shall also see, now that we are using SU(3), the need for any such gauge fixing becomes obviated.

It is important to keep in mind, if one uses the same approach to symmetry breaking for higher order gauge groups SU(N>3), that there will still always be *only three distinct terms in the analog to (4.7)*, and that one of these will be of the massless form $g^{\mu\nu}\partial^\sigma\partial_\sigma - \partial^\nu\partial^\mu$. For example, for SU(4), the first term would be $\frac{1}{2}G_{15\mu}\left(g^{\mu\nu}\left(\partial^\sigma\partial_\sigma + M_{(15)}{}^2\right) - \partial^\nu\partial^\mu\right)G^{15}{}_\nu$, the second $\frac{1}{2}G_{14...9\mu}\left(g^{\mu\nu}\left(\partial^\sigma\partial_\sigma + M_{(14...9)}{}^2\right) - \partial^\nu\partial^\mu\right)G^{14...9}{}_\nu$, and the third $G_{8...1\mu}\left(g^{\mu\nu}\partial^\sigma\partial_\sigma - \partial^\nu\partial^\mu\right)G^{8...1}{}_\nu$, but there would still be only three terms. This is what was meant in section 3 when we spoke of always having exactly three gauge boson masses, two of which are >0 and one of which =0, no matter what rank of the gauge group SU(N). And, this is why one might suspect that the replication of nature into three generations emanates from the existence of three terms in (4.7) *and from the fact that this generalizes to any gauge group of any size*.

Now, contrasting with (2.5), we use *each* of the three main terms in (4.7) to carry out *three distinct Fourier transforms*:

$$\begin{cases} D_{\nu\lambda}(p^\sigma)\left(g^{\mu\nu}\left(\partial^\sigma\partial_\sigma + M_{(8)}{}^2\right) - \partial^\nu\partial^\mu\right)e^{ip^\sigma x_\sigma} = \delta^\mu{}_\lambda e^{ip^\sigma x_\sigma} \\ D_{\nu\lambda}(p^\sigma)\left(g^{\mu\nu}\left(\partial^\sigma\partial_\sigma + M_{(7\ldots4)}{}^2\right) - \partial^\nu\partial^\mu\right)e^{ip^\sigma x_\sigma} = \delta^\mu{}_\lambda e^{ip^\sigma x_\sigma} \\ D_{\nu\lambda}(p^\sigma)\left(g^{\mu\nu}\partial^\sigma\partial_\sigma - \partial^\nu\partial^\mu\right)e^{ip^\sigma x_\sigma} = \delta^\mu{}_\lambda e^{ip^\sigma x_\sigma} \end{cases}. \tag{4.8}$$

However, we can economize on mathematical calculation by representing all of the above via:

$$D_{\nu\lambda}(p^\sigma)\left(g^{\mu\nu}\left(\partial^\sigma\partial_\sigma + M^2\right) - \partial^\nu\partial^\mu\right)e^{ip^\sigma x_\sigma} = \delta^\mu{}_\lambda e^{ip^\sigma x_\sigma}, \tag{4.9}$$

and thereafter substituting $M = M_{(8)} = \frac{1}{\sqrt{3}}vg$, $M = M_{(7\ldots4)} = \frac{1}{2}vg$, and $M = M_{(3\ldots1)} = 0$, in essence, as what will shortly become a three-value quantum number. So, from (4.9), we again calculate $\partial^\nu\partial^\mu e^{ip^\sigma x_\sigma} = i\partial^\nu p^\mu e^{ip^\sigma x_\sigma} = -p^\nu p^\mu e^{ip^\sigma x_\sigma}$, and thus obtain:

$$D_{\nu\lambda}(p^\sigma)\left(-g^{\mu\nu}\left(p^\sigma p_\sigma - M^2 I_{3\times3}\right) + p^\nu p^\mu\right) = \delta^\mu{}_\lambda I_{3\times3}, \tag{4.10}$$

just as in (2.6), but now, with a 3x3 rather than 2x2 matrix equation, and with the ability to plug in any one of the mass values $M = M_{(8)} = \frac{1}{\sqrt{3}}vg$, $M = M_{(7\ldots4)} = \frac{1}{2}vg$, and $M = M_{(3\ldots1)} = 0$.

The 3x3 propagator is then specified as in (2.7) by the matrix inverse relation:

$$D_{\nu\lambda}(p^\sigma) = \left(-g_{\nu\lambda} + \frac{p_\lambda p_\nu}{M^2}\right) \times \left\{p^\sigma p_\sigma - M^2\right\}^{-1}. \tag{4.11}$$

Now, all we need to do is calculate the inverse $\left\{p^\sigma p_\sigma - M^2\right\}^{-1}$, and we can then start to crank out *vector meson* masses, using $\frac{1}{4}\left(1/\mathrm{M}^2\right)^{ij} = \mathrm{Tr}\left[T^i\left\{p^\sigma p_\sigma - M^2\right\}^{-1}T^j\right]$ from (2.14). However, this is an involved, tedious calculation that requires very careful double and triple checking.

To being this calculation, patterned on (4.1), we first write:

$$p_\sigma = \lambda^i p_{i\sigma} = \frac{1}{2}\begin{pmatrix} \frac{2}{\sqrt{3}}p_{8\sigma} & p_{6\sigma} - ip_{7\sigma} & p_{4\sigma} - ip_{5\sigma} \\ p_{6\sigma} + ip_{7\sigma} & -\frac{1}{\sqrt{3}}p_{8\sigma} + p_{3\sigma} & p_{2\sigma} - ip_{1\sigma} \\ p_{4\sigma} + ip_{5\sigma} & p_{2\sigma} + ip_{1\sigma} & -\frac{1}{\sqrt{3}}p_{8\sigma} - p_{3\sigma} \end{pmatrix}, \tag{4.12}$$

so that:

$$
\begin{aligned}
&p^{\sigma} p_{\sigma} - M^2 \\
&= \tfrac{1}{4}\begin{pmatrix} \frac{2}{\sqrt{3}} p^{8\sigma} & p^{6\sigma} - ip^{7\sigma} & p^{4\sigma} - ip^{5\sigma} \\ p^{6\sigma} + ip^{7\sigma} & -\frac{1}{\sqrt{3}} p^{8\sigma} + p^{3\sigma} & p^{2\sigma} - ip^{1\sigma} \\ p^{4\sigma} + ip^{5\sigma} & p^{2\sigma} + ip^{1\sigma} & -\frac{1}{\sqrt{3}} p^{8\sigma} - p^{3\sigma} \end{pmatrix}
\begin{pmatrix} \frac{2}{\sqrt{3}} p_{8\sigma} & p_{6\sigma} - ip_{7\sigma} & p_{4\sigma} - ip_{5\sigma} \\ p_{6\sigma} + ip_{7\sigma} & -\frac{1}{\sqrt{3}} p_{8\sigma} + p_{3\sigma} & p_{2\sigma} - ip_{1\sigma} \\ p_{4\sigma} + ip_{5\sigma} & p_{2\sigma} + ip_{1\sigma} & -\frac{1}{\sqrt{3}} p_{8\sigma} - p_{3\sigma} \end{pmatrix} - M^2
\end{aligned}
$$

$$
= \begin{pmatrix}
\begin{matrix} \frac{4}{3} p^{8\sigma} p_{8\sigma} + p^{7\ldots4\sigma} p_{7\ldots4\sigma} \\ - M^2 \end{matrix}
&
\begin{matrix} \frac{2}{\sqrt{3}} p^{8\sigma}\left(p_{6\sigma} - ip_{7\sigma}\right) \\ + \left(p^{6\sigma} - ip^{7\sigma}\right)\left(-\frac{1}{\sqrt{3}} p_{8\sigma} + p_{3\sigma}\right) \\ + \left(p^{4\sigma} - ip^{5\sigma}\right)\left(p_{2\sigma} + ip_{1\sigma}\right) \end{matrix}
&
\begin{matrix} \left(\frac{2}{\sqrt{3}} p^{8\sigma}\right)\left(p_{4\sigma} - ip_{5\sigma}\right) \\ + \left(p^{6\sigma} - ip^{7\sigma}\right)\left(p_{2\sigma} - ip_{1\sigma}\right) \\ + \left(p^{4\sigma} - ip^{5\sigma}\right)\left(-\frac{1}{\sqrt{3}} p_{8\sigma} - p_{3\sigma}\right) \end{matrix}
\\[2ex]
\begin{matrix} \left(p^{6\sigma} + ip^{7\sigma}\right)\left(\frac{2}{\sqrt{3}} p_{8\sigma}\right) \\ + \left(-\frac{1}{\sqrt{3}} p^{8\sigma} + p^{3\sigma}\right)\left(p_{6\sigma} + ip_{7\sigma}\right) \\ + \left(p^{2\sigma} - ip^{1\sigma}\right)\left(p_{4\sigma} + ip_{5\sigma}\right) \end{matrix}
&
\begin{matrix} \left(-\frac{1}{\sqrt{3}} p^{8\sigma} + p^{3\sigma}\right)\left(-\frac{1}{\sqrt{3}} p_{8\sigma} + p_{3\sigma}\right) \\ + p^{7,6\sigma} p_{7,6\sigma} + p^{2,1\sigma} p_{2,1\sigma} \\ - M^2 \end{matrix}
&
\begin{matrix} \left(p^{6\sigma} + ip^{7\sigma}\right)\left(p_{4\sigma} - ip_{5\sigma}\right) \\ + \left(-\frac{1}{\sqrt{3}} p^{8\sigma} + p^{3\sigma}\right)\left(p_{2\sigma} - ip_{1\sigma}\right) \\ + \left(p^{2\sigma} - ip^{1\sigma}\right)\left(-\frac{1}{\sqrt{3}} p_{8\sigma} - p_{3\sigma}\right) \end{matrix}
\\[2ex]
\begin{matrix} \left(p^{4\sigma} + ip^{5\sigma}\right)\left(\frac{2}{\sqrt{3}} p_{8\sigma}\right) \\ + \left(p^{2\sigma} + ip^{1\sigma}\right)\left(p_{6\sigma} + ip_{7\sigma}\right) \\ + \left(-\frac{1}{\sqrt{3}} p^{8\sigma} - p^{3\sigma}\right)\left(p_{4\sigma} + ip_{5\sigma}\right) \end{matrix}
&
\begin{matrix} \left(p^{4\sigma} + ip^{5\sigma}\right)\left(p_{6\sigma} - ip_{7\sigma}\right) \\ + \left(p^{2\sigma} + ip^{1\sigma}\right)\left(-\frac{1}{\sqrt{3}} p_{8\sigma} + p_{3\sigma}\right) \\ + \left(-\frac{1}{\sqrt{3}} p^{8\sigma} - p^{3\sigma}\right)\left(p_{2\sigma} + ip_{1\sigma}\right) \end{matrix}
&
\begin{matrix} \left(-\frac{1}{\sqrt{3}} p^{8\sigma} - p^{3\sigma}\right)\left(-\frac{1}{\sqrt{3}} p_{8\sigma} - p_{3\sigma}\right) \\ + p^{5,4\sigma} p_{5,4\sigma} + p^{2,1\sigma} p_{2,1\sigma} \\ - M^2 \end{matrix}
\end{pmatrix} . \quad (4.13)
$$

Immediately we see why this inversion is not at all trivial, because (4.13) is the SU(3) equivalent of (2.8) in SU(2). Equation (2.8) was easily invertible, because the diagonal was proportional to the unit matrix, and the off-diagonal elements were zero. Equation affords us no such luxury: The diagonal is not proportional to the unit matrix, the off diagonal elements are not zero, and in general there is a complicated mix of both real and imaginary terms. That is why it was so important to work through SU(2) meticulously in preparation for SU(3). But (4.13), nonetheless, is the matrix that we must invert to obtain the SU(3) propagator (2.7), (4.11), as well as the inverse square mass matrices $\frac{1}{4}\left(1/\mathrm{M}^2\right)^{ij} = \mathrm{Tr}\left[T^i \left\{p^{\sigma} p_{\sigma} - M^2\right\}^{-1} T^j\right]$ using each of the three mass values $M = M_{(8)} = \frac{1}{\sqrt{3}} vg$, $M = M_{(7\ldots4)} = \frac{1}{2} vg$, and $M = M_{(3\ldots1)} = 0$ revealed in (4.5).

The symmetry is broken; we now turn to the formidable task of inverting this matrix.

5. Simplification and Reduction of the SU(3) Momentum Space Matrix

Following the same path as we did following (2.9), we will want to put all the gauge bosons on mass shell, and them make this on-shell substitution into (4.13). Thus, referring to (4.5), we set:

$$
\begin{aligned}
&{M_{(8)}}^2 = \tfrac{1}{3}v^2 g^2 = p^{8\sigma} p_{8\sigma} \\
&{M_{(7\ldots4)}}^2 = \tfrac{1}{4}v^2 g^2 = p^{7\sigma} p_{7\sigma} = p^{6\sigma} p_{6\sigma} = p^{5\sigma} p_{5\sigma} = p^{4\sigma} p_{4\sigma} \; . \\
&{M_{(3\ldots1)}}^2 = 0 = p^{3\sigma} p_{3\sigma} = p^{2\sigma} p_{2\sigma} = p^{1\sigma} p_{1\sigma}
\end{aligned}
\tag{5.1}
$$

This immediately enables us to simplify the diagonal in (4.13). Keeping in mind that $p^{7\ldots4\sigma} p_{7\ldots4\sigma} \equiv p^{7\sigma} p_{7\sigma} + p^{6\sigma} p_{6\sigma} + p^{5\sigma} p_{5\sigma} + p^{4\sigma} p_{4\sigma}$, the upper left element in (4.13), for example, $\tfrac{4}{3} p^{8\sigma} p_{8\sigma} + p^{7\ldots4\sigma} p_{7\ldots4\sigma} - M^2$, becomes $\tfrac{4}{9}v^2 g^2 + v^2 g^2 - M^2 = \tfrac{13}{9}v^2 g^2 - M^2$, which is either $\tfrac{10}{9}v^2 g^2$ if we insert $M^2 = {M_{(8)}}^2 = \tfrac{1}{3}v^2 g^2$, $\tfrac{43}{36}v^2 g^2$ for $M^2 = {M_{(7\ldots4)}}^2 = \tfrac{1}{4}v^2 g^2$, or $\tfrac{13}{9}v^2 g^2$ for $M^2 = {M_{(3\ldots1)}}^2 = 0$. The same sort of reduction applies to other terms.

We also note a proliferation of cross terms, such as $p^{8\sigma} p_{3\sigma}$, and inspection of the off-diagonal elements of (4.13) makes clear that these cross terms are not the exception, but the rule. Thus, we need to supplement (5.1) to deal with these cross terms.

Each term $p^{i\sigma} p_{j\sigma}$ in (4.13) is a scalar Lorentz invariant, describing the square of the total energy when an $A^{i\sigma}$ collides with an $A_{j\sigma}$. Some of the cross terms are simple. For scalar products within the same "sector" of SU(3) wherein each gauge bosons has identical mass – analogous classically to two billiard balls colliding – we may easily use (5.1) to write:

$$
\begin{aligned}
&p^{7\sigma} p_{6\sigma} = p^{7\sigma} p_{5\sigma} = p^{7\sigma} p_{4\sigma} = p^{6\sigma} p_{5\sigma} = p^{6\sigma} p_{4\sigma} = p^{5\sigma} p_{4\sigma} = \tfrac{1}{4}v^2 g^2 \\
&p^{3\sigma} p_{2\sigma} = p^{3\sigma} p_{1\sigma} = p^{2\sigma} p_{1\sigma} = 0
\end{aligned}
\; .
\tag{5.2}
$$

The product between the $M_{(8)}$ and the $M_{(7\ldots4)}$ sectors is a little more complex, but still straightforward. From the ${M_{(8)}}^2 = \tfrac{1}{3}v^2 g^2 = p^{8\sigma} p_{8\sigma}$ sector, each of the $p^{8\sigma}$ contributes a $\tfrac{1}{\sqrt{3}}vg$ factor, and from ${M_{(7\ldots4)}}^2 = \tfrac{1}{4}v^2 g^2 = p^{7\sigma} p_{7\sigma} = p^{6\sigma} p_{6\sigma} = p^{5\sigma} p_{5\sigma} = p^{4\sigma} p_{4\sigma}$, each of the $p^{7\ldots4\sigma}$ contributes a $\tfrac{1}{2}vg$ factor. (We just multiply square roots here, similarly to what one would do classically if the two billiard balls had different masses.) Thus:

$$
p^{8\sigma} p_{7\sigma} = p^{8\sigma} p_{6\sigma} = p^{8\sigma} p_{5\sigma} = p^{8\sigma} p_{4\sigma} = \tfrac{1}{2\sqrt{3}}v^2 g^2 \; .
\tag{5.3}
$$

The somewhat puzzling terms, are those in which a massive gauge boson momentum is contracted with a *massless* one, such as $p^{8\sigma} p_{3\sigma}$, $p^{7\sigma} p_{2\sigma}$, etc. This is analogous to shining light on a billiard ball, rather than hitting it with another ball. By the (5.3) analysis each of the ${p_{3\ldots1}}^{\mu}$

contributes a "zero" because $\sqrt{0}=0$, yet there is still an energy content even in a luminous $p^{\sigma}p_{\sigma}=0$ interaction. To address this, we borrow from Mandelstam, and via (5.1), we *define* a Mandelstam-type variable *s* as follows:

$$\begin{aligned} s &\equiv \left(p^7{}_\mu + p^3{}_\mu\right)\left(p_7{}^u + p_3{}^u\right) = p^7{}_\mu p_7{}^u + p^3{}_\mu p_3{}^u + 2p^7{}_\mu p_3{}^u \\ &= \tfrac{1}{4}v^2 g^2 + 2p^7{}_\mu p_3{}^u = \tfrac{1}{4}v^2 g^2 + 2\tfrac{\sqrt{3}}{2}p^8{}_\mu p_3{}^u \end{aligned}, \tag{5.4}$$

where $p^7{}_\mu p_3{}^u = \frac{\sqrt{3}}{2}p^8{}_\mu p_3{}^u$ in the final line is based on observing that $p^{7\sigma}p_{7\sigma} = \frac{3}{4}p^{8\sigma}p_{8\sigma}$ and again employing the square root Then, we invert this, and use (5.1) to write:

$$\begin{aligned} &p^7{}_\mu p_3{}^u = p^6{}_\mu p_3{}^u = p^5{}_\mu p_3{}^u = p^4{}_\mu p_3{}^u = \tfrac{1}{2}\left(s - \tfrac{1}{4}v^2 g^2\right) \\ &p^8{}_\mu p_3{}^u = \tfrac{1}{\sqrt{3}}\left(s - \tfrac{1}{4}v^2 g^2\right) \end{aligned}, \tag{5.5}$$

where the interaction is the same for all of the equal momenta in the $p^{7\ldots 4}{}_\mu$ sector. Now, we substitute into (4.13) from (5.1) through (5.5) as appropriate, and matrix (4.13) can be reduced to a function, exclusively, of group coupling *g*, Mandelstam-type parameter *s*, mass *M*, and vev *v*.

First, to simplify reduction, because all of the $p^{7\ldots 4}{}_\mu$ have the same effect in the scalar products of (4.13), as do all of the $p^{3\ldots 1}{}_\mu$, we will substitute the highest-indexed momentum in each sector for any lower-indexed momentum, so that the only momenta now appearing are $p^{8\sigma}$, $p^{7\sigma}$, and $p^{3\sigma}$. This yields many factors of $1\pm i$ because, for example, $p^{4\sigma}\pm ip^{5\sigma}$ becomes $p^{7\sigma}\pm ip^{7\sigma}$ which becomes $p^{7\sigma}(1\pm i)$. This intermediate, easy reduction step yields:

$$p^{\sigma}p_{\sigma} - M^2 = \begin{pmatrix} \tfrac{4}{3}p^{8\sigma}p_{8\sigma} + 4p^{7\sigma}p_{7\sigma} - M^2 & \begin{array}{l} \tfrac{2}{\sqrt{3}}(1-i)p^{8\sigma}p_{7\sigma} \\ +(1-i)\left(-\tfrac{1}{\sqrt{3}}p^{7\sigma}p_{8\sigma} + p^{7\sigma}p_{3\sigma}\right) \\ +(1-i)(1+i)p^{7\sigma}p_{3\sigma} \end{array} & \begin{array}{l} \tfrac{2}{\sqrt{3}}(1-i)p^{8\sigma}p_{7\sigma} \\ +(1-i)(1-i)p^{7\sigma}p_{3\sigma} \\ +(1-i)\left(-\tfrac{1}{\sqrt{3}}p^{7\sigma}p_{8\sigma} - p^{7\sigma}p_{3\sigma}\right) \end{array} \\ \begin{array}{l} \tfrac{2}{\sqrt{3}}(1+i)p^{7\sigma}p_{8\sigma} \\ +(1+i)\left(-\tfrac{1}{\sqrt{3}}p^{8\sigma}p_{7\sigma} + p^{3\sigma}p_{7\sigma}\right) \\ +(1-i)(1+i)p^{3\sigma}p_{7\sigma} \end{array} & \begin{array}{l} \tfrac{1}{3}p^{8\sigma}p_{8\sigma} + p^{3\sigma}p_{3\sigma} - 2\tfrac{1}{\sqrt{3}}p^{8\sigma}p_{3\sigma} \\ +2p^{7\sigma}p_{7\sigma} + 2p^{3\sigma}p_{3\sigma} \\ -M^2 \end{array} & \begin{array}{l} (1+i)(1-i)p^{7\sigma}p_{7\sigma} \\ +(1-i)\left(-\tfrac{1}{\sqrt{3}}p^{8\sigma}p_{3\sigma} + p^{3\sigma}p_{3\sigma}\right) \\ +(1-i)\left(-\tfrac{1}{\sqrt{3}}p^{3\sigma}p_{8\sigma} - p^{3\sigma}p_{3\sigma}\right) \end{array} \\ \begin{array}{l} \tfrac{2}{\sqrt{3}}(1+i)p^{7\sigma}p_{8\sigma} \\ +(1+i)(1+i)p^{3\sigma}p_{7\sigma} \\ +(1+i)\left(-\tfrac{1}{\sqrt{3}}p^{8\sigma}p_{7\sigma} - p^{3\sigma}p_{7\sigma}\right) \end{array} & \begin{array}{l} (1+i)(1-i)p^{7\sigma}p_{7\sigma} \\ +(1+i)\left(-\tfrac{1}{\sqrt{3}}p^{3\sigma}p_{8\sigma} + p^{3\sigma}p_{3\sigma}\right) \\ +(1+i)\left(-\tfrac{1}{\sqrt{3}}p^{8\sigma}p_{3\sigma} - p^{3\sigma}p_{3\sigma}\right) \end{array} & \begin{array}{l} \tfrac{1}{3}p^{8\sigma}p_{8\sigma} + p^{3\sigma}p_{3\sigma} + 2\tfrac{1}{\sqrt{3}}p^{8\sigma}p_{3\sigma} \\ +2p^{7\sigma}p_{7\sigma} + 2p^{3\sigma}p_{3\sigma} \\ -M^2 \end{array} \end{pmatrix}. \tag{5.6}$$

Then, we further reduce, and at the same time, we apply (5.1) through (5.5) to obtain:

$$p^{\sigma}p_{\sigma}-M^2=\begin{pmatrix} \frac{13}{9}v^2g^2-M^2 & \begin{array}{l}-\frac{5}{24}v^2g^2+\frac{3}{2}s\\ -i\frac{1}{24}v^2g^2-i\frac{1}{2}s\end{array} & \begin{array}{l}+\frac{7}{24}v^2g^2-\frac{1}{2}s\\ -i\frac{1}{24}v^2g^2-i\frac{1}{2}s\end{array} \\ \begin{array}{l}-\frac{5}{24}v^2g^2+\frac{3}{2}s\\ +i\frac{1}{24}v^2g^2+i\frac{1}{2}s\end{array} & \frac{7}{9}v^2g^2-\frac{2}{3}s-M^2 & \begin{array}{l}\frac{2}{3}v^2g^2-\frac{2}{3}s\\ -i\frac{1}{6}v^2g^2+i\frac{2}{3}s\end{array} \\ \begin{array}{l}\frac{7}{24}v^2g^2-\frac{1}{2}s\\ +i\frac{1}{24}v^2g^2+i\frac{1}{2}s\end{array} & \begin{array}{l}\frac{2}{3}v^2g^2-\frac{2}{3}s\\ +i\frac{1}{6}v^2g^2-i\frac{2}{3}s\end{array} & \frac{4}{9}v^2g^2+\frac{2}{3}s-M^2 \end{pmatrix}. \tag{5.7}$$

Now, making reference to (5.1), we would like to extract the factor $M_{(7\ldots4)}{}^2=\frac{1}{4}v^2g^2$ to the outside of the matrix, so as to in effect, normalize the inside of the matrix about this mass magnitude. Referring to (5.5), we also note that $p^7{}_\mu p_3{}^u=\frac{1}{2}\left(s-\frac{1}{4}v^2g^2\right)$. So we now *define* a dimensionless S, normalized to $s\equiv\frac{1}{4}v^2g^2S$, so that $p^7{}_\mu p_3{}^u=\frac{1}{8}v^2g^2(S-1)$ and:

$$S\equiv1+\frac{p^7{}_\mu p_3{}^u}{\frac{1}{8}v^2g^2}=1+\frac{\frac{\sqrt{3}}{2}p^8{}_\mu p_3{}^u}{\frac{1}{8}v^2g^2}, \tag{5.8}$$

see just after (5.4). Requiring that $p^7{}_\mu p_3{}^u\geq0$, i.e., that this scalar product must never be less than zero lest the interaction energy become negative, we see that $S\geq1$ is a lower, non-negative bound for S. Similarly, using $M_{(7\ldots4)}{}^2=\frac{1}{4}v^2g^2$ as a reference, we define a dimensionless mass ratio $\mu^2\equiv M^2/\frac{1}{4}v^2g^2$. Thus, from (5.1), the *only* three permitted choices for this ratio are $\mu^2=\frac{4}{3}$ (for $M_{(8)}$), $\mu^2=1$ (for $M_{(7\ldots4)}$) and $\mu^2=0$ (for $M_{(3\ldots1)}$), so this is in the nature of a three-valued quantum number (which we suspect may be related to generation replication).

With all of this, the matrix (5.7) now reduces to its simplest, final form:

$$\boxed{p^{\sigma}p_{\sigma}-M^2=\frac{1}{4}v^2g^2\begin{pmatrix} \frac{52}{9}-\mu^2 & -\frac{5}{6}+\frac{3}{2}S-i\left(\frac{1}{6}+\frac{1}{2}S\right) & \frac{7}{6}-\frac{1}{2}S-i\left(\frac{1}{6}+\frac{1}{2}S\right) \\ -\frac{5}{6}+\frac{3}{2}S+i\left(\frac{1}{6}+\frac{1}{2}S\right) & \frac{28}{9}-\frac{8}{3}s-\mu^2 & \frac{8}{3}-\frac{2}{3}S-i\left(\frac{2}{3}-\frac{2}{3}S\right) \\ \frac{7}{6}-\frac{1}{2}S+i\left(\frac{1}{6}+\frac{1}{2}S\right) & \frac{8}{3}-\frac{2}{3}S+i\left(\frac{2}{3}-\frac{2}{3}S\right) & \frac{16}{9}+\frac{2}{3}S-\mu^2 \end{pmatrix}}. \tag{5.9}$$

This is the matrix we now must invert, to obtain $\left(p^{\sigma}p_{\sigma}-M^2\right)^{-1}$ for the propagator.

6. Obtaining the SU(3) Momentum Space Inverse

There are two steps required to invert the matrix (5.9). First, we determine the adjugate matrix $\mathbf{A}$. Second, we divide through by determinant $\left|\mathbf{p}^2 - \mathbf{m}^2\right|$. Thus, cross-noting (2.18), $1/\mathbf{M}^2 = \mathrm{Tr}\left(\Gamma\left\{\mathbf{p}^2 - \mathbf{m}^2\right\}^{-1}\Gamma\right)$, via which we wish to find meson masses, we now obtain:

$$\left\{\mathbf{p}^2 - \mathbf{m}^2\right\}^{-1} = \frac{1}{\left|\mathbf{p}^2 - \mathbf{m}^2\right|}\mathbf{A}\,. \tag{6.1}$$

The adjugate calculation is very labor-intensive, but for the reader who attempts this, there is a built in way to check one's calculations for accuracy which it is *very* important to do: Because (5.9) is Hermitian, one may calculate the upper-right off-diagonal terms, and then independently calculate the lower-left off-diagonal terms. If these are not the Hermitian conjugates of one another, then one needs to debug the calculation. For the diagonal elements, of course, this is not an option. But prudence demands repeating these calculations at least twice to make sure one obtains the same results. The embedding of parameters μ and S then ensures, once this calculation is successfully completed, that it does not have to be repeated for different values of these parameters: one simply plugs in the desired parameters and cranks out the results.

Following this very laborious calculation, the *adjugate matrix* turns out to be:

$$\mathbf{A} = \left(\tfrac{1}{4}v^2 g^2\right)^2 \times$$
$$\begin{pmatrix} -\frac{164}{81} + \frac{48}{9}S - \frac{4}{3}S^2 - \frac{1}{3}\mu^2 & \begin{matrix}\frac{127}{27} - 4S - S^2 + \frac{3}{2}S\mu^2 - \frac{5}{6}\mu^2 \\ +i\left[\frac{17}{27} - \frac{4}{3}S + S^2 - \frac{1}{2}S\mu^2 - \frac{1}{6}\mu^2\right]\end{matrix} & \begin{matrix}-\frac{161}{27} + \frac{60}{9}S - S^2 - \frac{1}{2}S\mu^2 + \frac{7}{6}\mu^2 \\ +i\left[\frac{17}{27} - \frac{4}{3}S + S^2 - \frac{1}{2}S\mu^2 - \frac{1}{6}\mu^2\right]\end{matrix} \\ \begin{matrix}\frac{127}{27} - 4S - S^2 + \frac{3}{2}S\mu^2 - \frac{5}{6}\mu^2 \\ -i\left[\frac{17}{27} - \frac{4}{3}S + S^2 - \frac{1}{2}S\mu^2 - \frac{1}{6}\mu^2\right]\end{matrix} & \frac{1439}{162} + \frac{131}{27}S - \frac{1}{2}S^2 - \frac{2}{3}S\mu^2 - \frac{68}{9}\mu^2 + \mu^4 & \begin{matrix}-\frac{449}{27} + \frac{167}{27}S - \frac{1}{2}S^2 - \frac{2}{3}S\mu^2 + \frac{8}{3}\mu^2 \\ +i\left[\frac{113}{27} - \frac{86}{27}S - S^2 + \frac{2}{3}S\mu^2 - \frac{2}{3}\mu^2\right]\end{matrix} \\ \begin{matrix}-\frac{161}{27} + \frac{60}{9}S - S^2 - \frac{1}{2}S\mu^2 + \frac{7}{6}\mu^2 \\ -i\left[\frac{17}{27} - \frac{4}{3}S + S^2 - \frac{1}{2}S\mu^2 - \frac{1}{6}\mu^2\right]\end{matrix} & \begin{matrix}-\frac{449}{27} + \frac{167}{27}S - \frac{1}{2}S^2 - \frac{2}{3}S\mu^2 + \frac{8}{3}\mu^2 \\ -i\left(\frac{113}{27} - \frac{86}{27}S - S^2 + \frac{2}{3}S\mu^2 - \frac{2}{3}\mu^2\right)\end{matrix} & \frac{2795}{162} - \frac{41}{27}S - \frac{5}{2}S^2 + \frac{2}{3}S\mu^2 - \frac{80}{9}\mu^2 + \mu^4 \end{pmatrix} . \tag{6.2}$$

Thereafter, one can make use of the two-term products calculated in the course of finding (6.2), to calculate the *determinant*:

$$\left|\mathbf{p}^2 - \mathbf{m}^2\right| = \left(\tfrac{1}{4}v^2 g^2\right)^3 \times \left(-\tfrac{16610}{729} + \tfrac{1398}{27}S - \tfrac{442}{27}S^2 - 2S^3 - \tfrac{78}{9}S\mu^2 + \tfrac{39}{9}S^2\mu^2 - \tfrac{217}{9}\mu^2 + \tfrac{32}{3}\mu^4 - \mu^6\right). \tag{6.3}$$

To manage mass calculations, it helps to have a total expression for $\left\{\mathbf{p}^2 - \mathbf{m}^2\right\}^{-1}$ in (6.1) *combining the adjugate and the determinant*. The easiest and most general approach is to define:

$$\{\mathbf{p}^2 - \mathbf{m}^2\}^{-1} = \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix}, \tag{6.4}$$

with:

$$A = \frac{1}{\frac{1}{4}v^2 g^2} \frac{-1476 + 3888S - 972S^2 - 243\mu^2}{-16610 + 37746S - 11934S^2 - 1458S^3 - 6318S\mu^2 + 3159S^2\mu^2 - 17577\mu^2 + 7776\mu^4 - 729\mu^6}, \tag{6.5a}$$

$$B = \frac{1}{\frac{1}{4}v^2 g^2} \frac{6858 - 5832S - 1458S^2 + 2187S\mu^2 - 1215\mu^2 + i[918 - 1944S + 1458S^2 - 729S\mu^2 - 243\mu^2]}{-33220 + 75492S - 23868S^2 - 2916S^3 - 12636S\mu^2 + 6318S^2\mu^2 - 35154\mu^2 + 15552\mu^4 - 1458\mu^6}, \tag{6.5b}$$

$$C = \frac{1}{\frac{1}{4}v^2 g^2} \frac{-8694 + 9720S - 1458S^2 - 729S\mu^2 + 1701\mu^2 + i[918 - 1944S + 1458S^2 - 729S\mu^2 - 243\mu^2]}{-33220 + 75492S - 23868S^2 - 2916S^3 - 12636S\mu^2 + 6318S^2\mu^2 - 35154\mu^2 + 15552\mu^4 - 1458\mu^6}, \tag{6.5c}$$

$$D = \frac{1}{\frac{1}{4}v^2 g^2} \frac{6858 - 5832S - 1458S^2 + 2187S\mu^2 - 1215\mu^2 - i[918 - 1944S + 1458S^2 - 729S\mu^2 - 243\mu^2]}{-33220 + 75492S - 23868S^2 - 2916S^3 - 12636S\mu^2 + 6318S^2\mu^2 - 35154\mu^2 + 15552\mu^4 - 1458\mu^6}, \tag{6.5d}$$

$$E = \frac{1}{\frac{1}{4}v^2 g^2} \frac{12951 + 7074S - 729S^2 - 972S\mu^2 - 11016\mu^2 + 1458\mu^4}{-33220 + 75492S - 23868S^2 - 2916S^3 - 12636S\mu^2 + 6318S^2\mu^2 - 35154\mu^2 + 15552\mu^4 - 1458\mu^6}, \tag{6.5e}$$

$$F = \frac{1}{\frac{1}{4}v^2 g^2} \frac{-24246 + 9018S - 729S^2 - 972S\mu^2 + 3888\mu^2 + i[6102 - 4644S - 1458S^2 + 972S\mu^2 - 972\mu^2]}{-33220 + 75492S - 23868S^2 - 2916S^3 - 12636S\mu^2 + 6318S^2\mu^2 - 35154\mu^2 + 15552\mu^4 - 1458\mu^6}, \tag{6.5f}$$

$$G = \frac{1}{\frac{1}{4}v^2 g^2} \frac{-8694 + 9720S - 1458S^2 - 729S\mu^2 + 1701\mu^2 - i[918 - 1944S + 1458S^2 - 729S\mu^2 - 243\mu^2]}{-33220 + 75492S - 23868S^2 - 2916S^3 - 12636S\mu^2 + 6318S^2\mu^2 - 35154\mu^2 + 15552\mu^4 - 1458\mu^6}, \tag{6.5g}$$

$$H = \frac{1}{\frac{1}{4}v^2 g^2} \frac{-24246 + 9018S - 729S^2 - 972S\mu^2 + 3888\mu^2 - i[6102 - 4644S - 1458S^2 + 972S\mu^2 - 972\mu^2]}{-33220 + 75492S - 23868S^2 - 2916S^3 - 12636S\mu^2 + 6318S^2\mu^2 - 35154\mu^2 + 15552\mu^4 - 1458\mu^6}, \tag{6.5h}$$

$$I = \frac{1}{\frac{1}{4}v^2 g^2} \frac{25155 - 2214S - 3645S^2 + 972S\mu^2 - 12960\mu^2 + 1458\mu^4}{-33220 + 75492S - 23868S^2 - 2916S^3 - 12636S\mu^2 + 6318S^2\mu^2 - 35154\mu^2 + 15552\mu^4 - 1458\mu^6}. \tag{6.5i}$$

In the above, we have removed the term-by terms fractions in (6.2) and (6.3) by placing everything over one common denominator ($3^6 = 729$ in A and $2 \cdot 3^6 = 1458$ in the B through I) and then discarding the denominator. Now, we start using the above to find QCD meson masses.

7. Specification of the SU(3) Meson Mass Table

First, we now turn back to (2.18) developed in the context of SU(2) which we shall write here with its internal symmetry indexes as $(1/\mathbf{M}^2)^{ij} = \mathrm{Tr}\left(\Gamma^i \{\mathbf{p}^2 - \mathbf{m}^2\}^{-1} \Gamma^j\right)$. This shows how the vector meson masses are extracted from the matrix (6.4), (6.5). Because (6.5) will vary with the choice of the parameters μ^2 and *S*, and to save the effort of having to repeatedly apply 8x8=56 combinations of indexes to extract out the various $(1/\mathbf{M}^2)^{ij}$ for various choices of μ^2 and *S*, it helps to do this once and for all based on (6.4). Thus, taking (6.4), sandwiching it between the 64 combinations of the i, j indexes in $\mathrm{Tr}\left(\Gamma^i \left\{\mathbf{p}^2 - \mathbf{m}^2\right\}^{-1} \Gamma^j\right)$, we obtain, using $\Gamma^i = \lambda^i$:

$$\left(\frac{1}{\mathbf{M}^2}\right)^{ij} = \mathrm{Tr}\left[\lambda^i \left\{\mathbf{p}^2 - \mathbf{m}^2\right\}^{-1} \lambda^j\right] =$$

$$\begin{array}{cccccccc}
E+I & i(I-E) & H-F & D & -iD & G & -iG & -\frac{1}{\sqrt{3}}(H+F) \\
i(E-I) & E+I & -i(F+H) & iD & D & iG & G & \frac{1}{\sqrt{3}}i(H-F) \\
F-H & i(F+H) & E+I & -G & iG & -H & iH & \frac{1}{\sqrt{3}}(E-I) \\
B & -iB & -C & A+I & i(I-A) & H & -iH & \frac{1}{\sqrt{3}}(2G-C) \\
iB & B & iC & i(A-I) & A+I & iH & H & -\frac{1}{\sqrt{3}}i(2G+C) \\
C & -iC & -F & F & -iF & A+E & i(E-A) & \frac{1}{\sqrt{3}}(2D-B) \\
iC & C & iF & iF & F & i(A-E) & A+E & -\frac{1}{\sqrt{3}}i(2D+B) \\
-\frac{1}{\sqrt{3}}(H+F) & \frac{1}{\sqrt{3}}i(H-F) & \frac{1}{\sqrt{3}}(I-E) & \frac{1}{\sqrt{3}}(2C-G) & \frac{1}{\sqrt{3}}i(2C+G) & \frac{1}{\sqrt{3}}(2B-D) & \frac{1}{\sqrt{3}}i(2B+D) & \frac{4}{3}A+\frac{1}{3}E+\frac{1}{3}I
\end{array} \quad .(7.1)$$

It is important to keep in mind that (7.1) is not really a "matrix," but rather is simply a "map" or 8x8 "table" of the square inverses of the various masses which appear in the invariant amplitude, see (2.16). Thus, we omit the () which are ordinarily used to enclose a matrix subject to the rule for performing mathematical operations with matrices, to show that this is merely a table and *not* a matrix. As such, one does not need to "matrix invert" this any longer: that work is now done. One may simply invert and then take the square root (or take the square root and invert) each entry in the above, entry-by-entry, to obtain the SU(3) meson "mass table":

$$\boxed{\begin{array}{cccccccc}
\multicolumn{8}{l}{\mathbf{M}^{ij} =} \\
(E+I)^{-.5} & i^{.5}(E-I)^{-.5} & i(F-H)^{-.5} & D^{-.5} & i^{.5}D^{-.5} & G^{-.5} & i^{.5}G^{-.5} & \pm i\sqrt[4]{3}(H+F)^{-.5} \\
i^{-.5}(E-I)^{-.5} & (E+I)^{-.5} & i^{.5}(F+H)^{-.5} & i^{-.5}D^{-.5} & D^{-.5} & i^{-.5}G^{-.5} & G^{-.5} & i^{-.5}\sqrt[4]{3}(H-F)^{-.5} \\
(F-H)^{-.5} & i^{-.5}(F+H)^{-.5} & (E+I)^{-.5} & iG^{-.5} & i^{-.5}G^{-.5} & \pm iH^{-.5} & i^{-.5}H^{-.5} & \sqrt[4]{3}(E-I)^{-.5} \\
B^{-.5} & i^{.5}B^{-.5} & iC^{-.5} & (A+I)^{-.5} & i^{.5}(A-I)^{-.5} & H^{-.5} & i^{.5}H^{-.5} & \sqrt[4]{3}(2G-C)^{-.5} \\
i^{-.5}B^{-.5} & B^{-.5} & i^{-.5}C^{-.5} & i^{-.5}(A-I)^{-.5} & (A+I)^{-.5} & i^{-.5}H^{-.5} & H^{-.5} & i^{.5}\sqrt[4]{3}(2G+C)^{-.5} \\
C^{-.5} & i^{.5}C^{-.5} & \pm iF^{-.5} & F^{-.5} & i^{.5}F^{-.5} & (A+E)^{-.5} & i^{.5}(A-E)^{-.5} & \sqrt[4]{3}(2D-B)^{-.5} \\
i^{-.5}C^{-.5} & C^{-.5} & i^{-.5}F^{-.5} & i^{-.5}F^{.5} & F^{-.5} & i^{-.5}(A-E)^{-.5} & (A+E)^{-.5} & i^{.5}\sqrt[4]{3}(2D+B)^{-.5} \\
\pm i\sqrt[4]{3}(H+F)^{-.5} & i^{-.5}\sqrt[4]{3}(H-F)^{-.5} & i\sqrt[4]{3}(E-I)^{-.5} & \sqrt[4]{3}(2C-G)^{-.5} & i^{-.5}\sqrt[4]{3}(2C+G)^{-.5} & \sqrt[4]{3}(2B-D)^{-.5} & i^{-.5}\sqrt[4]{3}(2B+D)^{-.5} & \left(\frac{4}{3}A+\frac{1}{3}E+\frac{1}{3}I\right)^{-.5}
\end{array}} \quad .(7.2)$$

Of course, $-i = 1/i = i^{-1}$, by definition, so $i = 1/-i = (-i)^{-1}$. Recall also, that $i^{.5} = \pm\frac{1}{\sqrt{2}}(1+i)$. From these, we derive and apply the useful square root expressions $i^{.5} = (-i)^{-.5} = \pm\frac{1}{\sqrt{2}}(1+i)$ and thus $i^{-.5} = (1/i)^{.5} = (-i)^{.5} = i \cdot i^{.5} = \pm i \cdot \frac{1}{\sqrt{2}}(1+i) = \pm\frac{1}{\sqrt{2}}(1-i)$. Further, we find it helpful to obtain and use $i \cdot i^{-.5} = \pm\frac{1}{\sqrt{2}} i(1-i) = \pm\frac{1}{\sqrt{2}}(1+i) = i^{.5}$.

Now, to obtain vector meson masses, all we need to do is make choose μ^2 and *S*, plug those choices into (6.4), (6.5), and then plug those, in turn, into (7.2). Because of the various imaginary and complex as well as real factors in the above, and this "square root of *i*" mathematics which permeates the expressions in (7.2), and given that an "imaginary mass" is understood to be a "real half-life," we expect that (7.2) will have something to say not only about the vector meson masses, but also about their *lifetimes*, and that the pure mathematics of $\sqrt{i}$ is central to understanding particle lifetimes.

8. Calculation of the SU(3) Inverse for μ=0 and S=1

From (6.5) we see that there is an overall factor of $\frac{1}{4}v^2 g^2$ in the denominator which sets the mass scale, as is to be expected. There are also two parameters in (6.5), namely, μ^2 which is restricted to take on one of the three values $\mu^2 = 0, 1, \frac{4}{3}$, see just before (5.9), and *S*, which we know from just after (5.8) is restricted to $S \geq 1$. Because μ^2 is so restricted, it is, in effect, a three-valued quantum number. Further, because each value of μ^2 causes a *wholesale scaling* of the magnitudes of $\{\mathbf{p}^2 - \mathbf{m}^2\}^{-1}$ via the determinant $|\mathbf{p}^2 - \mathbf{m}^2|$, see (6.1), (6.3), which in turn will cause a wholesale scaling of the vector meson masses in (7.1), we will keep an eye on the possibility that $\mu^2 = 0, 1, \frac{4}{3}$, in particular, may be a *generation quantum number*, since there also happen to be precisely three generation which are distinguished *solely* by their masses which also scale on a wholesale basis from one generation to the next.

While we do not yet know a great deal about *S*, we do know that both μ^2 and *S* serve to a) shift the magnitude of *each component of the adjugate (5.2) relative to the other components* and b) *alter the magnitude of the determinant (5.3) to cause an overall scaling* of $\{\mathbf{p}^2 - \mathbf{m}^2\}^{-1}$. Because experimentally-observed mesons do not have a continuous spectrum of masses, but are restricted to discrete mass values, we expect that *S* is either a quantum number restricted to discrete values just like μ, or is a parameter which has a single, unique value. In either case, *S* is to be set in accordance with matching up to experimental observation, because were *S* to be continuous rather than discrete, we would end up with a continuous spectrum of vector meson mass values which is not what is experimentally observed.

At the outset, let us use (6.5) to determine the mass predictions for $\mu^2 = 0$, while setting $S \geq 1$ to is minimum value $S = 1$. In setting $S = 1$, we are likely making an unphysical, but very simplifying choice, because via (5.8), $S = 1$ means $p^7{}_\mu p_3{}^u = \frac{\sqrt{3}}{2} p^8{}_\mu p_3{}^u = 0$, and so we are effectively "turning off" the effects of the scalar product between the $G^{3..1\mu}$ and the $G^{8..4\mu}$. Nonetheless this choice does yields certain reductions which enable us to gain our bearings and to see how this all works in specific detail. Another choice of *S* which may be of interest for a "hand calculation" (as opposed to a computer calculation which can sample many *S* possibilities) is $S=2$, which sets $p^7{}_\mu p_3{}^u = \frac{1}{8}v^2g^2$, and $p^{8\sigma}p_{3\sigma} = \frac{1}{4\sqrt{3}}v^2g^2$, contrast $p^{7\sigma}p_{7\sigma} = \frac{1}{4}v^2g^2$ from (5.1) and $p^{8\sigma}p_{7\sigma} = \frac{1}{2\sqrt{3}}v^2g^2$ from (5.3). Thus, $S=2$ effectively halves the v^2g^2 coefficient when $p^{7\sigma}$ or $p^{8\sigma}$ is "mixed-contracted" with $p_3{}^u$, and may make physical sense.

So, calculating for the very simplest case, $\mu^2=0$, $S=1$, from (6.5) we obtain:

$$A = \frac{1}{\frac{1}{4}v^2g^2}\frac{-1476+3888-972}{(-16610+37746-11934-1458)} = \frac{1}{\frac{1}{4}v^2g^2}\frac{45}{242} = \frac{1}{\frac{1}{4}v^2g^2}\frac{7\cdot 5}{2\cdot 11^2}, \tag{8.1a}$$

$$B = \frac{1}{\frac{1}{4}v^2g^2}\frac{6858-5832-1458+i[918-1944+1458]}{-33220+75492-23868-2916} = -\frac{1}{\frac{1}{4}v^2g^2}\frac{27}{968}(1-i) = \pm\sqrt{2}i^{-.5}\frac{1}{\frac{1}{4}v^2g^2}\frac{3^3}{2^3 11^2}, \tag{8.1b}$$

$$C = \frac{1}{\frac{1}{4}v^2g^2}\frac{-8694+9720-1458+i[918-1944+1458]}{-33220+75492-23868-2916} = -\frac{1}{\frac{1}{4}v^2g^2}\frac{27}{968}(1-i) = \pm\sqrt{2}i^{-.5}\frac{1}{\frac{1}{4}v^2g^2}\frac{3^3}{2^3 11^2}, \tag{8.1c}$$

$$D = \frac{1}{\frac{1}{4}v^2g^2}\frac{6858-5832-1458-i[918-1944+1458]}{-33220+75492-23868-2916} = -\frac{1}{\frac{1}{4}v^2g^2}\frac{27}{968}(1+i) = \pm\sqrt{2}i^{.5}\frac{1}{\frac{1}{4}v^2g^2}\frac{3^3}{2^3 11^2}, \tag{8.1d}$$

$$E = \frac{1}{\frac{1}{4}v^2g^2}\frac{12951+7074-729}{-33220+75492-23868-2916} = \frac{1}{\frac{1}{4}v^2g^2}\frac{603}{484} = \frac{1}{\frac{1}{4}v^2g^2}\frac{603}{484} = \frac{1}{\frac{1}{4}v^2g^2}\frac{3^2\cdot 67}{2^2\cdot 11^2}, \tag{8.1e}$$

$$F = \frac{1}{\frac{1}{4}v^2g^2}\frac{-24246+9018-729+i[6102-4644-1458]}{-33220+75492-23868-2916} = -\frac{1}{\frac{1}{4}v^2g^2}\frac{15957}{15488} = -\frac{1}{\frac{1}{4}v^2g^2}\frac{3^4\cdot 197}{2^7\cdot 11^2}, \tag{8.1f}$$

$$G = \frac{1}{\frac{1}{4}v^2g^2}\frac{-8694+9720-1458-i[918-1944+1458]}{-33220+75492-23868-2916} = -\frac{1}{\frac{1}{4}v^2g^2}\frac{27}{968}(1+i) = \pm\sqrt{2}i^{.5}\frac{1}{\frac{1}{4}v^2g^2}\frac{3^3}{2^3 11^2}, \tag{8.1g}$$

$$H = \frac{1}{\frac{1}{4}v^2g^2}\frac{-24246+9018-729-i[6102-4644-1458]}{-33220+75492-23868-2916} = -\frac{1}{\frac{1}{4}v^2g^2}\frac{15957}{15488} = -\frac{1}{\frac{1}{4}v^2g^2}\frac{3^4\cdot 197}{2^7\cdot 11^2}, \tag{8.1h}$$

$$I = \frac{1}{\frac{1}{4}v^2g^2}\frac{25155-2214-3645}{-33220+75492-23868-2916} = \frac{1}{\frac{1}{4}v^2g^2}\frac{19296}{15488} = \frac{1}{\frac{1}{4}v^2g^2}\frac{603}{484} = \frac{1}{\frac{1}{4}v^2g^2}\frac{3^2\cdot 67}{2^2\cdot 11^2}, \tag{8.1i}$$

where we have reduced using $1+i=\pm\sqrt{2}i^{.5}$ and $1-i=\pm\sqrt{2}i^{-.5}$, and also decomposed each ratio into its prime number factors. It is noteworthy that the complex terms *B*, *C*, *D*, *G* all reduce down to the same fraction 27/968, times the simple $1\pm i$, while in the other complex terms *F* and *H*, the imaginary portion cancels identically. We also note that the *E* and *I* terms on the diagonal turn out to be equal. All of these result from the simplifying choice of $\mu^2 = 0$, $S = 1$, and in general, these simplifications will not occur. Finally, we also note the appearance of the prime factor "11" in the denominators throughout, which commonly appears in the renormalization equations for Yang-Mills groups, see. e.g. [3], equation (15.54). Using (6.4), we now summarize the above by:

$$\left\{\mathbf{p}^2 - \mathbf{m}^2\right\}^{-1}(\mu = 0, S = 1) = \frac{1}{\frac{1}{4}v^2 g^2}\begin{pmatrix} \frac{45}{242} & \pm\sqrt{2}i^{-.5}\frac{27}{968} & \pm\sqrt{2}i^{-.5}\frac{27}{968} \\ \pm\sqrt{2}i^{.5}\frac{27}{968} & \frac{603}{484} & \frac{15957}{15488} \\ \pm\sqrt{2}i^{.5}\frac{27}{968} & \frac{15957}{15488} & \frac{603}{484} \end{pmatrix}. \tag{8.2}$$

To follow up the earlier discussion toward the end of section 2, it is worth noting that because $\mu^2 = 0$, this originates from the $g^{\mu\nu}\partial^\sigma\partial_\sigma - \partial^\nu\partial^\mu$ term in the Lagrangian density (4.7), which in Abelian theory has no inverse. Normally, one needs Faddeev-Popov or some analogous gauge fixing process obtain a finite result. Here, however, because we use SU(N>2), we obtain a finite results in due course without any special measures, simply by applying ordinary matrix methods.

Now, we simply insert values from the above into (7.2), to see what masses and lifetimes result. However, there is some additional groundwork required, which can be seen, for example, when we try to obtain $\mathbf{M}^{84} = \sqrt[4]{3}(2C - G)^{-.5}$ and $\mathbf{M}^{85} = i^{-.5}\sqrt[4]{3}(2C + G)^{-.5}$ in (7.2). Specifically, using (8.2) in (7.2), we find, for example, that:

$$\frac{\mathbf{M}^{84}}{\frac{1}{2}vg} = \frac{\sqrt[4]{3}(2C - G)^{-.5}}{\frac{1}{2}vg} = \sqrt[4]{3}\left(2\left(\pm\sqrt{2}i^{-.5}\frac{27}{968}\right) \mp \sqrt{2}i^{.5}\frac{27}{968}\right)^{-.5} = \sqrt[4]{6}\left(\frac{968}{27}\right)^{.5}\left(\pm 2i^{-.5} \mp i^{.5}\right)^{-.5}, \tag{8.3a}$$

$$\frac{\mathbf{M}^{85}}{\frac{1}{2}vg} = \frac{i^{-.5}\sqrt[4]{3}(2C + G)^{-.5}}{\frac{1}{2}vg} = i^{-.5}\sqrt[4]{3}\left(2\left(\pm\sqrt{2}i^{-.5}\frac{27}{968}\right) \pm \sqrt{2}i^{.5}\frac{27}{968}\right)^{-.5} = \sqrt[4]{6}\left(\frac{968}{27}\right)^{.5}\left(\pm 2i^{.5} \pm i^{-.5}\right)^{-.5}, \tag{8.3b}$$

Noting the extra factor $i^{-.5}$ in $\mathbf{M}^{85}$ and elsewhere in (7.2), we have used both $i^{.5} = \pm\frac{1}{\sqrt{2}}(1 + i)$ and $i^{-.5} = \pm\frac{1}{\sqrt{2}}(1 - i)$, see after (7.2), to obtain and use $i^{1.5} = i \cdot i^{.5} = \pm\frac{1}{\sqrt{2}}i(1 + i) = \pm\frac{1}{\sqrt{2}}(i - 1) = i^{-.5}$. This implies that these "square root of *i* terms" conjugate at each order and so return to their original form *every second order*, i.e., $i^{(n+.5)+2} = i^{(n+.5)}$, where $n = -\infty \ldots -2, -1, 0, 1, 1 \ldots \infty$, versus the four-order cycle that is typical of the usual complex math, i.e., $i^n = i^{n+4}$. However, terms such as

$\left(\pm 2i^{-.5} \mp i^{.5}\right)^{-.5}$ and $\left(\pm 2i^{.5} \pm i^{-.5}\right)^{-.5}$ are somewhat unusual, and if these appear in the simplest case where $\mu^2 = 0$ and $S = 1$, then ever more unusual versions of these terms will appear in other, more complicated cases. Further, these terms will be at the heart of what we anticipate will be the calculation of particle lifetimes. Thus, we need to digress briefly into some mathematics of imaginary numbers, to determine how to evaluate terms such as $\left(\pm 2i^{-.5} \mp i^{.5}\right)^{-.5}$ and $\left(\pm 2i^{.5} \pm i^{-.5}\right)^{-.5}$.

9. The Imaginary Mathematics of Particle Lifetimes

In order to deal with expressions such a (8.3), it is best to find the general form expression for $\left(Ai^{.5} + Bi^{-.5}\right)^{-.5}$, which we shall *define* as $P + iQ \equiv \left(Ai^{.5} + Bi^{-.5}\right)^{-.5}$. The goal is to find *P* and *Q* in terms of *A* and *B*, and so deduce the "answer" $P + iQ$. To start out, we again use $i^{.5} = \pm\frac{1}{\sqrt{2}}(1+i)$ and $i^{-.5} = \pm\frac{1}{\sqrt{2}}(1-i)$ to write:

$$\left(Ai^{.5} + Bi^{-.5}\right)^{-.5} = \left(\pm\tfrac{1}{\sqrt{2}}A(1+i) \pm \tfrac{1}{\sqrt{2}}B(1-i)\right)^{-.5} = \tfrac{1}{\sqrt{2}}\left(\pm A \pm B + i(\pm A \mp B)\right)^{-.5} \equiv \tfrac{1}{\sqrt{2}}\left(C + Di\right)^{-.5}, \quad (9.1)$$

and we now *define* $C \equiv \pm A \pm B$ and $D \equiv \pm A \mp B$, using the inverted $\mp$, which is important to track the $1-i$ coefficient of *B* versus that of $1+i$ for *A*. Now, we need to obtain $(C + Di)^{-.5}$.

First, let us invert $C + Di$, to obtain $(C + Di)^{-1} \equiv (M + iN)$. Then, finally, we will take the square root. For the inversion, we thus need to calculate:

$$(C + Di)(M + iN) = CM - ND + i(DM + CN) = 1. \quad (9.2)$$

This yields the simultaneous equations:

$$\begin{cases} CM - ND = 1 \\ DM + CN = 0 \end{cases}, \quad (9.3)$$

which have the well-known solution:

$$\begin{cases} M = \dfrac{C}{C^2 + D^2} \\ N = -\dfrac{D}{C^2 + D^2} \end{cases}, \quad (9.4)$$

Now that we have $(C + Di)^{-1} \equiv (M + iN)$, the next step is to find $(C + Di)^{-.5} \equiv (M + iN)^{.5}$. This now specifies the desired "answer" expression $P + iQ$ such that:

$$M + iN = (P + iQ)^2 = P^2 - Q^2 + 2iPQ \ . \tag{9.5}$$

This has the simultaneous equations:

$$\begin{cases} M = P^2 - Q^2 \\ N = 2PQ \end{cases} , \tag{9.6}$$

and via the intermediate quadratic $Q^4 + MQ^2 - \frac{1}{4}N^2 = 0$, this is solved by:

$$\begin{cases} P = \pm\frac{\sqrt{2}}{2}\left(\frac{N^2}{-M \pm \sqrt{M^2 + N^2}}\right)^{.5} \\ Q = \pm\left(\frac{-M \pm \sqrt{M^2 + N^2}}{2}\right)^{.5} \end{cases} . \tag{9.7}$$

Now, we go backwards, since $P + iQ$ is our answer. Serially using (9.7), (9.4) and (9.1), we make appropriate substitutions and reduce to find that:

$$\begin{aligned} \left(Ai^{.5} + Bi^{-.5}\right)^{-.5} &\equiv P + iQ \\ &= \pm\frac{1}{\sqrt{2}}\left(\frac{(\pm A \mp B)}{(\pm A \pm B)^2 + (\pm A \mp B)^2}\right)\left(\frac{(\pm A \pm B)^2 + (\pm A \mp B)^2}{-(\pm A \pm B) \pm \sqrt{(\pm A \pm B)^2 + (\pm A \mp B)^2}}\right)^{.5} \\ &\quad \pm i\frac{1}{\sqrt{2}}\left(\frac{-(\pm A \pm B) \pm \sqrt{(\pm A \pm B)^2 + (\pm A \mp B)^2}}{(\pm A \pm B)^2 + (\pm A \mp B)^2}\right)^{.5} \end{aligned} . \tag{9.8}$$

Finally, we return to the terms for $\left(\pm 2i^{-.5} \mp i^{.5}\right)^{-.5}$ in $\mathbf{M}^{84}$ in (8.3a), and $\left(\pm 2i^{.5} \pm i^{-.5}\right)^{-.5}$ in $\mathbf{M}^{85}$ in (8.3b), and we keep in mind that these same terms will emerge from other masses in (7.2) as well. We must be careful to recognize with the $\pm,\mp$ signs that the term in (8.3a) really represents *both* $\left(+2i^{-.5} - i^{.5}\right)^{-.5}$ and $\left(-2i^{-.5} + i^{.5}\right)^{-.5}$, and that the term in (8.3b) is both $\left(+2i^{.5} + i^{-.5}\right)^{-.5}$ and $\left(-2i^{.5} - i^{-.5}\right)^{-.5}$. Using (9.8), the term from $\mathbf{M}^{84}$ in (8.3a) evaluates to all of the four expressions:

$$\begin{cases} \left(+2i^{-.5}-i^{.5}\right)^{-.5} = \pm\frac{1}{\sqrt{20}}\left[(+3)\left(\pm\sqrt{10}-1\right)^{-.5} \pm i\left(\pm\sqrt{10}-1\right)^{.5}\right] = \begin{cases} \pm\frac{1}{\sqrt{20}}\left[3\left(-1+\sqrt{10}\right)^{-.5} \pm i\left(-1+\sqrt{10}\right)^{.5}\right] \\ = \pm.4562 \pm .3288i \\ \pm\frac{1}{\sqrt{20}}\left[\mp\left(1+\sqrt{10}\right)^{.5} + i3\left(1+\sqrt{10}\right)^{-.5}\right] \\ = \mp.4562 \pm .3288i \end{cases} \\ \left(-2i^{-.5}+i^{.5}\right)^{-.5} = \pm\frac{1}{\sqrt{20}}\left[(-3)\left(\pm\sqrt{10}+1\right)^{-.5} \pm i\left(\pm\sqrt{10}+1\right)^{.5}\right] = \begin{cases} \pm\frac{1}{\sqrt{20}}\left[-3\left(1+\sqrt{10}\right)^{-.5} \pm i\left(1+\sqrt{10}\right)^{.5}\right] \\ = \mp.3288 \pm .4562i \\ \pm\frac{1}{\sqrt{20}}\left[\mp\left(1-\sqrt{10}\right)^{.5} - i3\left(1-\sqrt{10}\right)^{-.5}\right] \\ = \mp.3288 \mp .4562i \end{cases} \end{cases} ,(9.9)$$

while that from $\mathbf{M}^{85}$ evaluates to:

$$\begin{cases} \left(+2i^{.5}+i^{-.5}\right)^{-.5} = \pm\frac{1}{\sqrt{20}}\left[(+1)\left(\pm\sqrt{10}-3\right)^{-.5} \pm i\left(\pm\sqrt{10}-3\right)^{.5}\right] = \begin{cases} \pm\frac{1}{\sqrt{20}}\left[\left(-3+\sqrt{10}\right)^{-.5} \pm i\left(-3+\sqrt{10}\right)^{.5}\right] \\ = \pm.5551 \pm .0901i \\ \pm\frac{1}{\sqrt{20}}\left[\mp\left(3+\sqrt{10}\right)^{.5} + i\left(3+\sqrt{10}\right)^{-.5}\right] \\ = \mp.5551 \pm .0901i \end{cases} \\ \left(-2i^{.5}-i^{-.5}\right)^{-.5} = \pm\frac{1}{\sqrt{20}}\left[(-1)\left(\pm\sqrt{10}+3\right)^{-.5} \pm i\left(\pm\sqrt{10}+3\right)^{.5}\right] = \begin{cases} \pm\frac{1}{\sqrt{20}}\left[\left(3+\sqrt{10}\right)^{-.5} \pm i\left(3+\sqrt{10}\right)^{.5}\right] \\ = \pm.0901 \pm .5551i \\ \pm\frac{1}{\sqrt{20}}\left[\mp\left(-3+\sqrt{10}\right)^{.5} - i\left(-3+\sqrt{10}\right)^{-.5}\right] \\ = \mp.0901 \mp .5551i \end{cases} \end{cases} ,(9.10)$$

Keep in mind that the terms such as $\left(\pm\sqrt{10}-1\right)^{-.5}$, etc., to the extent that they are related to particle life, provide additional variation in permissible lifetimes. For, if one uses the choice of $\left(-\sqrt{10}-1\right)^{-.5} = -i\left(\sqrt{10}+1\right)^{-.5}$, then the term which is real will become imaginary and vice versa. And, there is further freedom in the $\pm i(\ldots)$ factors in the above, indicating that there will be several permissible "lifetime / mass" combinations for particles which contain these sorts of unusual mathematical factors $\left(Ai^{.5}+Bi^{-.5}\right)^{-.5}$.

10. Detailed Calculation for the SU(3) Mass Table for μ=0 and S=1

Finally, after all of this preparatory work, we are ready to calculate the "mass / lifetime table" in (7.2). Specifically, we now take all of (8.1) for, $\mu^2 = 0$, $S = 1$, and substitute them into

(7.2). While this yields an 8x8 table, for space saving on the page, it will be more convenient to show the 7x7 table corresponding to the indexes 1,2,3,4,5,6,7 related to off-diagonal generators, and to separately show the terms related to the 8 generators which contain the $\left(Ai^{.5}+Bi^{-.5}\right)^{-.5}$ terms discussed in the prior section and so are more complicated. The 7x7 table for the 1-7 indexes, when extracted from (7.2) using (8.2), originates as:

$\frac{\mathbf{M}^{ij}}{\frac{1}{2}vg}$	1	2	3	4	5	6	7
1	$\left(\frac{1206}{484}\right)^{-.5}$	0	0	$\left(\pm\sqrt{2}i^{.5}\frac{27}{968}\right)^{-.5}$	$i^{.5}\left(\pm\sqrt{2}i^{.5}\frac{27}{968}\right)^{-.5}$	$\left(\pm\sqrt{2}i^{.5}\frac{27}{968}\right)^{-.5}$	$i^{.5}\left(\pm\sqrt{2}i^{.5}\frac{27}{968}\right)^{-.5}$
2	0	$\left(\frac{1206}{484}\right)^{-.5}$	$i^{.5}\left(\frac{31914}{15488}\right)^{-.5}$	$i^{-.5}\left(\pm\sqrt{2}i^{.5}\frac{27}{968}\right)^{-.5}$	$\left(\pm\sqrt{2}i^{.5}\frac{27}{968}\right)^{-.5}$	$i^{-.5}\left(\pm\sqrt{2}i^{.5}\frac{27}{968}\right)^{-.5}$	$\left(\pm\sqrt{2}i^{.5}\frac{27}{968}\right)^{-.5}$
3	0	$i^{-.5}\left(\frac{31914}{15488}\right)^{-.5}$	$\left(\frac{1206}{484}\right)^{-.5}$	$i\left(\pm\sqrt{2}i^{.5}\frac{27}{968}\right)^{-.5}$	$i^{-.5}\left(\pm\sqrt{2}i^{.5}\frac{27}{968}\right)^{-.5}$	$i\left(\frac{15957}{15488}\right)^{-.5}$	$i^{-.5}\left(\frac{15957}{15488}\right)^{-.5}$
4	$\left(\pm\sqrt{2}i^{-.5}\frac{27}{968}\right)^{-.5}$	$i^{.5}\left(\pm\sqrt{2}i^{-.5}\frac{27}{968}\right)^{-.5}$	$i\left(\pm\sqrt{2}i^{-.5}\frac{27}{968}\right)^{-.5}$	$\left(\frac{693}{484}\right)^{-.5}$	$i^{.5}\left(-\frac{513}{484}\right)^{-.5}$	$\left(\frac{15957}{15488}\right)^{-.5}$	$i^{.5}\left(\frac{15957}{15488}\right)^{-.5}$
5	$i^{-.5}\left(\pm\sqrt{2}i^{-.5}\frac{27}{968}\right)^{-.5}$	$\left(\pm\sqrt{2}i^{-.5}\frac{27}{968}\right)^{-.5}$	$i^{-.5}\left(\pm\sqrt{2}i^{-.5}\frac{27}{968}\right)^{-.5}$	$i^{-.5}\left(-\frac{513}{484}\right)^{-.5}$	$\left(\frac{693}{484}\right)^{-.5}$	$i^{-.5}\left(\frac{15957}{15488}\right)^{-.5}$	$\left(\frac{15957}{15488}\right)^{-.5}$
6	$\left(\pm\sqrt{2}i^{-.5}\frac{27}{968}\right)^{-.5}$	$i^{.5}\left(\pm\sqrt{2}i^{-.5}\frac{27}{968}\right)^{-.5}$	$i\left(\frac{15957}{15488}\right)^{-.5}$	$\left(\frac{15957}{15488}\right)^{-.5}$	$i^{.5}\left(\frac{15957}{15488}\right)^{-.5}$	$\left(\frac{693}{484}\right)^{-.5}$	$i^{.5}\left(-\frac{513}{484}\right)^{-.5}$
7	$i^{-.5}\left(\pm\sqrt{2}i^{-.5}\frac{27}{968}\right)^{-.5}$	$\left(\pm\sqrt{2}i^{-.5}\frac{27}{968}\right)^{-.5}$	$i^{-.5}\left(\frac{15957}{15488}\right)^{-.5}$	$i^{-.5}\left(\frac{15957}{15488}\right)^{-.5}$	$\left(\frac{15957}{15488}\right)^{-.5}$	$i^{-.5}\left(-\frac{513}{484}\right)^{-.5}$	$\left(\frac{693}{484}\right)^{-.5}$

.(10.1)

Next, we wish to invert each term from $(\ldots)^{-.5}$ into $(\ldots)^{.5}$, which can be done using the $\sqrt{i}$ relationships outlined following (7.2) and (8.3) and especially $i^{(n+.5)+2}=i^{(n+.5)}$. Even after these reductions, however, there is yet another bit of complex mathematics which we need to consider, because after reduction, not only does the above contain $\sqrt{i}$, it also yields factors of $\sqrt[4]{i}=i^{.25}$ and $\sqrt[-4]{i}=i^{-.25}$, see, e.g., the 4-3 and 3-4 terms in which this is manifest even before these reductions. We must now seek explicit expressions for these fourth root terms.

Following a similar procedure to that used in section 9, we define $\sqrt[4]{i}=i^{.25}\equiv A+Bi$. Then, we obtain A and B such that $(A+Bi)^2=i^{.5}=\pm\frac{1}{\sqrt{2}}(1+i)$ The simultaneous equations to be solved are $A^2-B^2=\frac{1}{\sqrt{2}}$ and $2AB=\frac{1}{\sqrt{2}}$, and the solution obtained for both $+\frac{1}{\sqrt{2}}(1+i)$ and $-\frac{1}{\sqrt{2}}(1+i)$ is:

$$\begin{cases} i^{.25} = \sqrt{+\tfrac{1}{\sqrt{2}}(1+i)} = \pm\dfrac{\left(\sqrt{2}\pm 2\right)^{.5}}{2} \pm i\dfrac{\sqrt{2}\left(\sqrt{2}\pm 2\right)^{-.5}}{2} = \begin{cases} \pm\tfrac{1}{2}\left(2+\sqrt{2}\right)^{.5} \pm \tfrac{1}{2}i\sqrt{2}\left(2+\sqrt{2}\right)^{-.5} \\ = \pm.9239 \pm .3827i \\ \mp\tfrac{1}{2}\sqrt{2}\left(2-\sqrt{2}\right)^{-.5} \pm \tfrac{1}{2}i\left(2-\sqrt{2}\right)^{.5} \\ = \mp.9239 \pm .3827i \end{cases} \\ i^{.25} = \sqrt{-\tfrac{1}{\sqrt{2}}(1+i)} = \mp\dfrac{\sqrt{2}\left(\sqrt{2}\pm 2\right)^{-.5}}{2} \pm i\dfrac{\left(\sqrt{2}\pm 2\right)^{.5}}{2} = \begin{cases} \pm\tfrac{1}{2}\sqrt{2}\left(\sqrt{2}+2\right)^{-.5} \mp \tfrac{1}{2}i\left(\sqrt{2}+2\right)^{.5} \\ = \pm.3827 \mp .9239i \\ \mp\tfrac{1}{2}\left(2-\sqrt{2}\right)^{.5} \pm \tfrac{1}{2}i\sqrt{2}\left(2-\sqrt{2}\right)^{-.5} \\ = \pm.3827 \pm .9239i \end{cases} \end{cases}, \tag{10.2}$$

where the upper terms within the sets on the right use "+" and the lower terms use "–" in the $\sqrt{2}\pm 2$ terms.

For $i^{-.25}$, similarly to what we did above, we now set $(A+Bi)^2 = i^{-.5} = \pm\frac{1}{\sqrt{2}}(1-i)$, thus:

$$\begin{cases} i^{-.25} = \sqrt{+\tfrac{1}{\sqrt{2}}(1-i)} = \pm\dfrac{\left(\sqrt{2}\pm 2\right)^{.5}}{2} \mp i\dfrac{\sqrt{2}\left(\sqrt{2}\pm 2\right)^{-.5}}{2} = \begin{cases} \pm\tfrac{1}{2}\left(2+\sqrt{2}\right)^{.5} \mp \tfrac{1}{2}i\sqrt{2}\left(2+\sqrt{2}\right)^{-.5} \\ = \pm.9239 \mp .3827i \\ \pm\tfrac{1}{2}\sqrt{2}\left(2-\sqrt{2}\right)^{-.5} \pm \tfrac{1}{2}i\left(2-\sqrt{2}\right)^{.5} \\ = \pm.9239 \pm .3827i \end{cases} \\ i^{-.25} = \sqrt{-\tfrac{1}{\sqrt{2}}(1-i)} = \pm\dfrac{\sqrt{2}\left(\sqrt{2}\pm 2\right)^{-.5}}{2} \pm i\dfrac{\left(\sqrt{2}\pm 2\right)^{.5}}{2} = \begin{cases} \pm\tfrac{1}{2}\sqrt{2}\left(2+\sqrt{2}\right)^{-.5} \mp \tfrac{1}{2}i\left(2+\sqrt{2}\right)^{.5} \\ = \pm.3827 \mp .9239i \\ \mp\tfrac{1}{2}\left(2-\sqrt{2}\right)^{.5} \pm \tfrac{1}{2} - \sqrt{2}\left(2-\sqrt{2}\right)^{-.5} \\ = \pm.3827 \pm .9239i \end{cases} \end{cases}, \tag{10.3}$$

Comparing the four values (10.2) and (10.3) which are identical, we see that $i^{.25} = i^{-.25}$ is its own self-inverse, though these are not equal to 1 or -1. One might say that in this sense, $i^{.25}$ is the "1" of imaginary mathematics.

In addition, these fourth root terms always appear multiplying a term with $\pm^{.5}$, e.g., $\mathbf{M}^{14} / \frac{1}{2}vg = i^{-.25}\left(\pm\frac{1}{\sqrt{2}}\frac{968}{27}\right)^{.5}$ and $\mathbf{M}^{15} / \frac{1}{2}vg = i^{.25}\left(\pm\frac{1}{\sqrt{2}}\frac{968}{27}\right)^{.5}$. The root mathematical terms are $i^{-.25}(\pm 1)^{.5}$ and $i^{.25}(\pm 1)^{.5}$. For the "+", we have $i^{-.25}(+1)^{.5} = i^{-.25}$ and $i^{.25}(1)^{.5} = i^{.25}$, which is simple. But, for the "–" selection we have $i^{-.25}(-1)^{.5} = i^{-.25}i = i^{.75}$ and $i^{.25}(-1)^{.5} = i^{.25}i = i^{1.25}$. If, however, we multiply each of (10.2) and (10.3) by *i*, we obtain the exact same set of four values, and this will continue indefinitely. That is, $i^{.5n+.25} = i^{.5(n+1)+.25}$ for $n = \infty \ldots -2,-1,0,1,2 \ldots \infty$. This half-power cycling allows us to replace $\left(\pm i^{-.5}\right)^{.5}$ with $i^{.25}$, wherever it appears.

So, with the foregoing in mind, following term-by-term inversion from $(\ldots)^{-.5}$ into $(\ldots)^{.5}$, and expressing every term as a real fraction (and possible real root) times one of the foregoing complex factors, the mass / lifetime table (10.1) for the 1-7 indexes becomes:

$\frac{\mathbf{M}^{ij}}{\frac{1}{2}vg}$	1	2	3	4	5	6	7
1	$\left(\frac{242}{603}\right)^{.5}$	0	0	$i^{.25}\left(\frac{968}{27\sqrt{2}}\right)^{.5}$	$i^{.25}\left(\frac{968}{27\sqrt{2}}\right)^{.5}$	$i^{.25}\left(\frac{968}{27\sqrt{2}}\right)^{.5}$	$i^{.25}\left(\frac{968}{27\sqrt{2}}\right)^{.5}$
2	0	$\left(\frac{242}{603}\right)^{.5}$	$i^{.5}\left(\frac{7744}{15957}\right)^{.5}$	$i^{.25}\left(\frac{968}{27\sqrt{2}}\right)^{.5}$	$i^{.25}\left(\frac{968}{27\sqrt{2}}\right)^{.5}$	$i^{.25}\left(\frac{968}{27\sqrt{2}}\right)^{.5}$	$i^{.25}\left(\frac{968}{27\sqrt{2}}\right)^{.5}$
3	0	$i^{-.5}\left(\frac{7744}{15957}\right)^{.5}$	$\left(\frac{242}{603}\right)^{.5}$	$i^{.25}\left(\frac{968}{27\sqrt{2}}\right)^{.5}$	$i^{.25}\left(\frac{968}{27\sqrt{2}}\right)^{.5}$	$\pm i\left(\frac{15488}{15957}\right)^{.5}$	$i^{-.5}\left(\frac{15488}{15957}\right)^{.5}$
4	$i^{.25}\left(\frac{968}{27\sqrt{2}}\right)^{.5}$	$i^{.25}\left(\frac{968}{27\sqrt{2}}\right)^{.5}$	$i^{.25}\left(\frac{968}{27\sqrt{2}}\right)^{.5}$	$\left(\frac{484}{693}\right)^{.5}$	$i^{-.5}\left(\frac{484}{513}\right)^{.5}$	$\left(\frac{15488}{15957}\right)^{.5}$	$i^{.5}\left(\frac{15488}{15957}\right)^{.5}$
5	$i^{.25}\left(\frac{968}{27\sqrt{2}}\right)^{.5}$	$i^{.25}\left(\frac{968}{27\sqrt{2}}\right)^{.5}$	$i^{.25}\left(\frac{968}{27\sqrt{2}}\right)^{.5}$	$i^{.5}\left(\frac{484}{513}\right)^{.5}$	$\left(\frac{484}{693}\right)^{.5}$	$i^{-.5}\left(\frac{15488}{15957}\right)^{.5}$	$\left(\frac{15488}{15957}\right)^{.5}$
6	$i^{.25}\left(\frac{968}{27\sqrt{2}}\right)^{.5}$	$i^{.25}\left(\frac{968}{27\sqrt{2}}\right)^{.5}$	$\pm i\left(\frac{15488}{15957}\right)^{.5}$	$\left(\frac{15488}{15957}\right)^{.5}$	$i^{.5}\left(\frac{15488}{15957}\right)^{.5}$	$\left(\frac{484}{693}\right)^{.5}$	$i^{-.5}\left(\frac{484}{513}\right)^{.5}$
7	$i^{.25}\left(\frac{968}{27\sqrt{2}}\right)^{.5}$	$i^{.25}\left(\frac{968}{27\sqrt{2}}\right)^{.5}$	$i^{-.5}\left(\frac{15488}{15957}\right)^{.5}$	$i^{-.5}\left(\frac{15488}{15957}\right)^{.5}$	$\left(\frac{15488}{15957}\right)^{.5}$	$i^{.5}\left(\frac{484}{513}\right)^{.5}$	$\left(\frac{484}{693}\right)^{.5}$

,(10.8A)

while the $\mathbf{M}^{i8}, \mathbf{M}^{8j}$ terms are:

$$\frac{\mathbf{M}^{8,4\ldots7}}{\frac{1}{2}vg} = \left(\left(\pm 2i^{-.5} \mp i^{.5}\right)^{-.5} \quad \left(\pm 2i^{.5} \pm i^{-.5}\right)^{-.5} \quad \left(\pm 2i^{-.5} \mp i^{.5}\right)^{-.5} \quad \left(\pm 2i^{.5} \pm i^{-.5}\right)^{-.5}\right)\sqrt[4]{6}\left(\frac{986}{27}\right)^{.5}$$

$$\frac{\mathbf{M}^{4\ldots7,8}}{\frac{1}{2}vg} = \left(\left(\pm 2i^{.5} \mp i^{-.5}\right)^{-.5} \quad \left(\mp 2i^{-.5} \mp i^{.5}\right)^{-.5} \quad \left(\pm 2i^{.5} \mp i^{-.5}\right)^{-.5} \quad \left(\mp 2i^{-.5} \mp i^{.5}\right)^{-.5}\right)\sqrt[4]{6}\left(\frac{986}{27}\right)^{.5}$$

$$\frac{\mathbf{M}^{1\ldots3,8}}{\frac{1}{2}vg} = \frac{\mathbf{M}^{8,1\ldots3}}{\frac{1}{2}vg} = \left(\pm i\sqrt[4]{3}\left(\frac{15488}{31914}\right)^{.5} \quad 0 \quad 0\right)$$

$$\frac{\mathbf{M}^{88}}{\frac{1}{2}vg} = \left(\frac{242}{261}\right)^{.5}$$

,(10.8B)

i.e., these would in the eighth row and column of (10.8A) if these was enough space on the page. Clearly, with their 2:1 ratios, the $\left(\pm 2i^{-.5} \mp i^{.5}\right)^{-.5}$ and $\left(\pm 2i^{.5} \pm i^{-.5}\right)^{-.5}$ factors, which we developed at length in section 9 (see (9.9) and (9.10)) are descended from the λ^8 generator of SU(2).

We see that most of the foregoing mass factors in (10.8) are complex factors, i.e., that these masses have both real and imaginary portions and so have lifetimes which can be deduced together with mass values. Further, for many of the $\mathbf{M}^{ij}$, there are several mass values and lifetimes which can be deduced from each of the 64 terms, because of the various $\pm$ factors together with the various $\sqrt{i}$ and $\sqrt[4]{i}$ expressions we have developed above. One may think of this complex factor mathematics, as being the underlying mathematics of particle lifetimes.

All of the foregoing is determined up to the overall $\frac{1}{2}vg$ factor for which g is the strong interaction coupling presuming we wish to apply this to QCD, and the vev v which, based on the real numeric ratios in the above which are within an order of magnitude of unity, is likely to be different than the v=246.220 GeV of electroweak interactions. We leave this *vev* for now as an experimental parameter to be determined, and focus on characterizing the ratios and their complex coefficients in (10.8).

11. Preliminary Comparison with Observed Phenomenological Data

Studying (10.8), we see that that there are a total of nine (9) distinct real number ratios some of which further contain a real square or fourth root coefficient. We now wish to simply see how the numeric ratios and lifetimes appear when these are all numerically evaluated.

There are a total of four different ratios which appear as a real number without any complex or imaginary coefficient, three of which appear in the diagonal of $\mathbf{M}^{ij}$. These are:

$$\left(\frac{242}{603}\right)^{.5}=.6335\,,\ \left(\frac{484}{693}\right)^{.5}=.8357\,,\ \left(\frac{242}{261}\right)^{.5}=.9629\,,\text{ and }\left(\frac{15488}{15957}\right)^{.5}=.9852\,. \tag{11.1}$$

The last of these appears in (10.8) not only as a real number, but also multiplied by i alone (and so is purely imaginary), and also multiplied by $i^{\pm.5}$ which is a complex number.

Including $(15488/15957)^{.5}$ noted above, there are two strictly imaginary numbers, with no real component. These are:

$$\pm i\left(\frac{15488}{15957}\right)^{.5}=\pm.9852i\ \text{ and }\ \pm i\sqrt[4]{3}\left(\frac{15488}{31914}\right)^{.5}=\pm.9168i\,. \tag{11.2}$$

These pure imaginary terms would represent a massless meson with finite lifetime.

In addition to $(15488/15957)^{.5}$ mentioned above which has several guises, there are two more ratios multiplied by $i^{\pm.5}$. With $i^{.5}=\pm\frac{1}{\sqrt{2}}(1+i)$ and $i^{-.5}=\pm\frac{1}{\sqrt{2}}(1-i)$, these are all, in all permitted sign combinations:

$$i^{\pm.5}\left(\frac{15488}{15957}\right)^{.5}=\left(\frac{15488}{15957}\right)^{.5}\times\begin{cases}\pm\frac{1}{\sqrt{2}}(1+i)\\\pm\frac{1}{\sqrt{2}}(1-i)\end{cases}=\begin{cases}.6966+.6966i\\-.6966-.6966i\\.6966-.6966i\\-.6966+.6966i\end{cases}, \tag{11.3a}$$

$$i^{\pm.5}\left(\frac{7744}{15957}\right)^{.5}=\left(\frac{7744}{15957}\right)^{.5}\times\begin{cases}\pm\frac{1}{\sqrt{2}}(1+i)\\\pm\frac{1}{\sqrt{2}}(1-i)\end{cases}=\begin{cases}.4926+.4926i\\-.4926-.4926i\\.4926-.4926i\\-.4926+.4926i\end{cases}, \tag{11.3b}$$

$$i^{\pm.5}\left(\frac{484}{513}\right)^{.5}=\left(\frac{484}{513}\right)^{.5}\times\begin{cases}\pm\frac{1}{\sqrt{2}}(1+i)\\\pm\frac{1}{\sqrt{2}}(1-i)\end{cases}=\begin{cases}.6868+.6868i\\-.6868-.6868i\\.6868-.6868i\\-.6868+.6868i\end{cases}. \tag{11.3c}$$

The factor $i^{.25}$ multiplies only a single number, in the form:

$$i^{.25}\left(\frac{968}{27\sqrt{2}}\right)^{.5}=\left(\frac{968}{27\sqrt{2}}\right)^{.5}\times\begin{cases}\pm\frac{1}{2}\left(2+\sqrt{2}\right)^{.5}\pm\frac{1}{2}i\sqrt{2}\left(2+\sqrt{2}\right)^{-.5}\\\mp\frac{1}{2}\sqrt{2}\left(2-\sqrt{2}\right)^{-.5}\pm\frac{1}{2}i\left(2-\sqrt{2}\right)^{.5}\\\pm\frac{1}{2}\sqrt{2}\left(\sqrt{2}+2\right)^{-.5}\mp\frac{1}{2}i\left(\sqrt{2}+2\right)^{.5}\\\pm\frac{1}{2}\left(2-\sqrt{2}\right)^{.5}\pm\frac{1}{2}i\sqrt{2}\left(2-\sqrt{2}\right)^{-.5}\end{cases}=\begin{cases}\pm.4652\pm1.9269i\\\mp.4652\pm1.9269i\\\pm1.9269\mp.46518i\\\pm1.9269\pm.46518i\end{cases}, \tag{11.4}$$

see (10.2), and appears in numerous positions in (10.8A) (this is the single most prolific term).

Finally, from (9.9) and (9.10):

$$\sqrt[4]{6}\left(\frac{986}{27}\right)^{.5}\times\begin{cases}\left(+2i^{-.5}-i^{.5}\right)^{-.5}=\sqrt[4]{6}\left(\frac{986}{27}\right)^{.5}\times\begin{cases}\pm\frac{1}{\sqrt{20}}\left[3\left(-1+\sqrt{10}\right)^{-.5}\pm i\left(-1+\sqrt{10}\right)^{.5}\right]\\=\pm4.3146\pm3.1098i\\\pm\frac{1}{\sqrt{20}}\left[\mp\left(1+\sqrt{10}\right)^{.5}+i3\left(1+\sqrt{10}\right)^{-.5}\right]\\=\mp4.3146\pm3.1098i\end{cases}\\\left(-2i^{-.5}+i^{.5}\right)^{-.5}=\sqrt[4]{6}\left(\frac{986}{27}\right)^{.5}\times\begin{cases}\pm\frac{1}{\sqrt{20}}\left[-3\left(1+\sqrt{10}\right)^{-.5}\pm i\left(1+\sqrt{10}\right)^{.5}\right]\\=\mp3.1098\pm4.3146i\\\pm\frac{1}{\sqrt{20}}\left[\mp\left(1-\sqrt{10}\right)^{.5}-i3\left(1-\sqrt{10}\right)^{-.5}\right]\\=\mp3.1098\mp4.3146i\end{cases}\end{cases}, \tag{11.5}$$

and:

$$\sqrt[4]{6}\left(\frac{986}{27}\right)^{.5}\times\begin{cases} \left(+2i^{.5}+i^{-.5}\right)^{-.5}=\sqrt[4]{6}\left(\frac{986}{27}\right)^{.5}\times\begin{cases} \pm\frac{1}{\sqrt{20}}\left[\left(-3+\sqrt{10}\right)^{-.5}\pm i\left(-3+\sqrt{10}\right)^{.5}\right] \\ =\pm 5.2501\pm .8522i \\ \pm\frac{1}{\sqrt{20}}\left[\mp\left(3+\sqrt{10}\right)^{.5}+i\left(3+\sqrt{10}\right)^{-.5}\right] \\ =\mp 5.2501\pm .8552i \end{cases} \\ \left(-2i^{.5}-i^{-.5}\right)^{-.5}=\sqrt[4]{6}\left(\frac{986}{27}\right)^{.5}\times\begin{cases} \pm\frac{1}{\sqrt{20}}\left[\left(3+\sqrt{10}\right)^{-.5}\pm i\left(3+\sqrt{10}\right)^{.5}\right] \\ =\pm .8552\pm 5.2501i \\ \pm\frac{1}{\sqrt{20}}\left[\mp\left(-3+\sqrt{10}\right)^{.5}-i\left(-3+\sqrt{10}\right)^{-.5}\right] \\ =\mp .8552\mp 5.2501i \end{cases} \end{cases}, \qquad (11.6)$$

Now, from all of the above, let us list all of the above in ascending order, based on the magnitude of the real component, and reintroduce the positions in which they appear in (10.8). Using $\pm A\{\pm\}Bi$ to represent $\pm A\pm Bi$ or $\pm A\mp Bi$ in all four sign combinations, these dimensionless numbers are multiplied by $\frac{1}{2}vg$ in all cases to arrive at a mass, and are:

$$\begin{array}{ll} \pm .9168i & \left(\mathbf{M}^{81}\text{ and transpose}\right) \\ \pm .9852i & \left(\mathbf{M}^{63}\text{ and transpose}\right) \\ \pm .4652\pm 1.9269i & \left(\mathbf{M}^{4\ldots7,1\ldots2};\mathbf{M}^{4\ldots5,3}\text{ and transpose}\right) \\ \pm .4926\pm .4926i & \left(\mathbf{M}^{32}\text{ and transpose}\right) \\ .6335 & \left(\mathbf{M}^{11},\mathbf{M}^{22},\mathbf{M}^{33}\right) \\ \pm .6868\pm .6868i & \left(\mathbf{M}^{54}\text{ and transpose}\right) \\ .6966+.6966i & \left(\mathbf{M}^{7,3\ldots4},\mathbf{M}^{65}\text{ and transpose}\right) \\ .8357 & \left(\mathbf{M}^{44},\mathbf{M}^{55},\mathbf{M}^{66},\mathbf{M}^{77}\right) \\ \pm .8552\pm 5.2501i & \left(\mathbf{M}^{85},\mathbf{M}^{87}\text{ and transpose}\right) \\ .9629 & \left(\mathbf{M}^{88}\right) \\ .9852 & \left(\mathbf{M}^{75}\text{ and transpose}\right) \\ \pm 1.9269\pm .46518i & \left(\mathbf{M}^{4\ldots7,1\ldots2};\mathbf{M}^{4\ldots5,3}\text{ and transpose}\right) \\ \pm 3.1098\pm 4.3146i & \left(\mathbf{M}^{84},\mathbf{M}^{86}\text{ and transpose}\right) \\ \pm 4.3146\pm 3.1098i & \left(\mathbf{M}^{84},\mathbf{M}^{86}\text{ and transpose}\right) \\ \pm 5.2501\pm .8522i & \left(\mathbf{M}^{85},\mathbf{M}^{87}\text{ and transpose}\right) \end{array}. \qquad (11.7)$$

These are to be compared with the experimentally observed meson masses, and the goal, of course, is to obtain an exact fit with experiment. We recall that these were generated by setting the parameters $\mu=0$ and $S=1$ back in (8.1), and that while μ is one of three physical values permitted for $\mu^2 = 0, 1, \frac{4}{3}$, the choice of $S = 1$ was to provide the simplest set of numbers though is likely an "unphysical" choice because it entirely turns off any energy contributions from the massless gauge bosons $G^{1\ldots3\mu}$. Nonetheless, analogues of the above obtained with other S choices including perhaps $S = 2$ (see discussion preceding (8.1)), should be compared with experimental meson tables such as those at [1], with the goal of obtaining a precise match with observed data. For reference, if one chooses $\mu = 0$ and $S = 2$, then via (6.4), (6.5):

$$\{\mathbf{p}^2 - \mathbf{m}^2\}^{-1}(\mu = 0, S = 2) = \frac{1}{\frac{1}{4}v^2 g^2}\begin{pmatrix} -\frac{1206}{259} & \frac{5319 - 1431i}{518} & \frac{-2457 - 1431i}{518} \\ \frac{5319 + 1431i}{518} & -\frac{24183}{1036} & \frac{4563 + 4509i}{518} \\ \frac{-2457 + 1431i}{518} & \frac{4563 - 4509i}{518} & -\frac{6147}{1036} \end{pmatrix}, \qquad (11.8)$$

contrast (8.2). We will not take (11.8) any further here, but will save that for a separate effort.

If one examines (11.7) above in relation to the experimental data for the light, unflavored (S=C=B=T=0) meson at [1], while the ratios for $\mu = 0$ and $S = 1$ do not match the data with precision, there are a number of features in the theoretical data of (11.7) in relation to the experimental data which are striking, and which suggest that the foregoing is on the correct course in relation to observable particle mass phenomenology.

First, keep in mind that although we started out with the Lagrangian density (2.1) to examine *vector* (spin 1) particles, we ended up making the *approximation* just before (2.13), that $p_\nu \cong (M,0,0,0)$ so that we could approximate $\mathrm{diag}\left(-g_{\nu\lambda} + \frac{p_\lambda p_\nu}{M^2}\right) \cong (0,-1,-1,-1)$, and we have been using this ever since. This enabled us to factor out the spin sum and focus on taking $\mathrm{Tr}\left[\Gamma\{\mathbf{p}^2 - \mathbf{m}^2\}^{-1}\Gamma\right]$, see (1.17), which is still, however, for vector particles, an approximate result. Yet, for *scalar* particles, the usual propagator, modulo *i*, is, in fact, just $1/\left(p^\sigma p_\sigma - m^2\right)$. That means, in the current context, that (6.4), (6.5) for $\{\mathbf{p}^2 - \mathbf{m}^2\}^{-1}$ is the *exact* propagator for *scalar* mesons, but only approximate for *vector* mesons due to the $p_\nu \cong (M,0,0,0)$ approximation for the latter. *Thus, it is the scalar mesons at [1] which should be the focus of our present attention.* Additionally, since we have set $\mu = 0$, which is the least energetic choice among $\mu^2 = 0, 1, \frac{4}{3}$, and corresponding in fact merely to $\{\mathbf{p}^2\}^{-1}$ with $\mathbf{m} = 0$, and given our wish to explore whether

$\mu^2 = 0, 1, \frac{4}{3}$ is somehow connected with generation replication, let us compare (11.7) with the light, unflavored (S=C=B=T=0) *scalar* (J=0) mesons in the listing at [1].

From that listing, setting aside lifetimes for the moment, we find the following masses in MeV, also in ascending order:

$$\begin{aligned}
&\pi^0 = 134.9766 \pm 0.0006 \text{ MeV} \\
&\pi^\pm = 139.5701 \pm 0.00006 \text{ MeV} \\
&\eta = 547.853 \pm 0.024 \text{ MeV} \\
&f_0(600) = 400 - 1200 \text{ MeV} \\
&\eta'(958) = 957.66 \pm 0.24 \text{ MeV} \\
&f_0(980) = 980 \pm 10 \text{ MeV} \\
&a_0(980) = 984.7 \text{ MeV} \\
&\eta(1295) = 1294 \pm 4 \text{ MeV} \\
&\pi(1300) = 1300 \pm 100 \text{ MeV} \\
&f_0(1370) = 1200 - 1500 \text{ MeV} \\
&\eta(1405) = 1409.8 \pm 2.5 \text{ MeV} \\
&a_0' = 1474 \pm 19 \text{ MeV} \\
&\eta(1475) = 1476 \pm 4 \text{ MeV} \\
&f_0(1505) = 1505 \pm 6 \text{ MeV} \\
&f_0(1710) = 1724 \pm 7 \text{ MeV} \\
&\pi = 1816 \pm 14 \text{ MeV}
\end{aligned} \quad . \qquad (11.9)$$

Contrasting with (11.7), several features immediately emerge with clarity. First, with $\mu = 0$ and $S = 1$, we obtain a total of 13 non-zero, real mass numbers. The above shows a total of 16 observed experimental masses numbers, but if we look at (11.8) for $\mu = 0$ and $S = 2$, which would then have to be employed in (7.2), it is apparent that this less simple choice of parameters will, in fact, generate a few more masses than we have already. The point here, is that the number of distinct masses which emerge from theoretically-based (11.7), appears to be just about *the right number of masses needed to fit the experimental data*.

Second, we note that the spread from the lowest to highest mass in (11.7) is 11.286 to 1. The spread in the experimental data is 13.159 to 1. So it is clear that not only do we generate the correct number of distinct mass values, we also generate *the right overall theoretical spread of data points which matches closely to the experimental data, with a distribution of predicted data that does bear striking similarities to the overall character of the QCD meson mass spectrum.*

Third, we note from (11.3) through (11.6) that the imaginary mathematics naturally generates a twofold ($i^{.25}$ factor) and fourfold ($\left(\pm 2i^{-.5} \mp i^{.5}\right)^{-.5}$ and $\left(\pm 2i^{.5} \pm i^{-.5}\right)^{-.5}$ factor) "splitting" of the mass for similar underlying states. Thus, some of the observed states should cluster into subsets of two or four masses. In the experimental data, we also see a fourfold set of η mesons, a fivefold set of f_0 mesons, and a twofold set of a_0. This is off by one mass for the f_0, but could perhaps be resolved by a suitable re-characterization of one of these f_0. Fundamentally, however, the imaginary math of (11.4) through (11.6) in particular, seems to suggest *the right structure for the overall mass splitting of common underlying particle states*.

Fourth, it seems very clear that the vev *v* is *not* be that of Fermi, i.e., 246.220 GeV. Rather, a contrast of (11.7) and (11.9) and knowledge of the strong coupling strength suggests that for SU(3) vev required to match the experimental data will turn out to be on the order of 1 GeV, which motivates us to entertain the prospect that perhaps the so-called "Λ_{QCD}" may actually be the vacuum expectation for SU(3).

12. Conclusion

Fundamentally, all of the results here flow from a single observation, coupled with an extension of the spontaneous symmetry breaking which is successfully utilized in SU(2), to larger Yang-Mills groups. The single observation, is that the term $p^\sigma p_\sigma - m^2$ must be treated in any given Yang-Mills SU(N) theory as an NxN matrix, with $p^\sigma = T_i p^{i\sigma}$, and therefore inverted according to established principles for matrix inversion, as $\left(p^\sigma p_\sigma - m^2\right)^{-1}$, rather than simply forced into a denominator as the reciprocal $1/\left(p^\sigma p_\sigma - m^2\right)$. In particular, in section 2, we saw that in SU(2) and by extension SU(2)xU(1), this sort of simple reciprocal inversion is permitted and is implicitly utilized in established electroweak theory, but only due to the special properties of SU(2). *This does not, however, extend to higher order Yang Mills groups.* Thus, all else is simply a detailed calculation and utilization of the SU(3) inverse $\left(p^\sigma p_\sigma - m^2\right)^{-1}$. Of definite interest, in the course of carrying out the $p^\sigma p_\sigma - m^2$ inversion in this manner, all of the problems normally associated with propagator formation including the need to work around infinite poles, simply evaporate. Even a matrix $\left(p^\sigma p_\sigma\right)^{-1}$, which is what is represented by the

$\mu = 0$ example that we worked through in detail here for SU(3), and which yielded all of the finite mass numbers in (11.7), inverts without difficulty and without the need for any special measures.

The extension of spontaneous symmetry breaking which we have employed and would propose to employ for larger groups as well, which was detailed generally in section 3 and applied specifically to SU(3) in section 4, gives mass to the gauge bosons which are "new" to any SU(N) when one moves up from SU(N-1), while leaving massless, all the gauge bosons associated with the SU(N-1) subgroup. Thus, the SU(N-1) vacuum remains unbroken, and so cam later be broken with a different vev. This provides for *any* SU(N), just enough degrees of freedom to properly give mass and a longitudinal polarization state to its "new" gauge bosons, while leaving over one degree of freedom for the Higgs field. However, once we are using SU(3) or higher, the masses of the gauge bosons are *not* synonymous with the meson masses which arise from the inverse term $\left(p^\sigma p_\sigma - m^2\right)^{-1}$. This is why the gauge bosons in SU(3) appear to be confined, i.e., not directly observed, while only massive mesons are observed. In this process, we fill the "mass gap" by giving rise to over a dozen meson masses >0 just for the parameter choice $\mu = 0$, see (11.7), with a second and third set of >0 masses arising from the other permitted quantum numbers $\mu = 1, \frac{4}{3}$, which one would wish to examine to see if a foundation for generation replication might be obtained. While more exploration is needed, and in particular more complicated but possibly physically on-target parameterizations such as $S = 2$ should also be calculated out, optimally by computer so large sets of parameters can be tried for optimum fit with experimental data, the general approach laid out herein does seem to point in a fruitful direction.

The question of confinement is perhaps best understood in the context of Quantum Field Theory, and for simplicity, with what Zee refers to in Appendix A of [4] as the "Central Identity of Quantum Field Theory":

$$\int_{-\infty}^{+\infty} D\phi\, e^{-\frac{1}{2}\phi\cdot K\cdot\phi + J\cdot\phi - V(\phi)} = \mathcal{C}\, e^{-V\left(\frac{\delta}{\delta J}\right)} e^{\frac{1}{2}J\cdot K^{-1}\cdot J}. \qquad (12.1)$$

Above, the exponent on the left-hand-side of this Gaussian-based identity represents a Lagrangian / action such as that in (2.1), which includes $p^\sigma p_\sigma - m^2$ in the K term, and in which interactions of higher than second order in the field are subsumed into $V(\phi)$. The exponent

$J \cdot K^{-1} \cdot J$ on the right hand side, represents the invariant amplitude for which an example is shown in (2.16), and K^{-1} represents $\left(p^{\sigma} p_{\sigma} - m^2\right)^{-1}$ which has been central to the development here, which is multiplied by a spin sum suitable to whatever particles are under consideration, and which especially includes non-zero *observed* masses $\mathbf{M}^{ij}$. In this context, the question of confinement is summarized very simply: *We cannot directly observe anything on the left hand side of the Central Identity(12.1). We do directly observe the masses occurs on the right hand side of the Central Identity.* Only in the special case of SU(2) or SU(2)xU(1), does it look *as if* we are observing the mass in $p^{\sigma} p_{\sigma} - m^2$ on the left hand side, because these turn out *only in this special case* to be identical with the $\mathbf{M}^{ij}$ garnered from right hand side, see (2.16). But it is the right hand side which is the mainspring of our direct observation of particle mass. In general, for any SU(N) with N>2, masses on the left hand side of (12.1) are "confined" from being directly observed, and what is on the right hand side is what we can and do directly observe.

Beyond all of the foregoing, the validation or falsification of this approach rests in matching the meson masses which are predicted, with those which are observed. Whether this approach is or is not eventually validated, it certainly puts up numerous prospects for numeric mass prediction which can be matched to phenomenological data, witness the simplest-case example of (11.7).

In this regard, it bears emphasis that the masses emergent from (11.7) and more generally from (7.2) via (6.4) and (6.5) are, in the context of SU(3), *completely correspondent* with what in electroweak theory become the $W^{\pm}$ mass in (2.16), via the right hand side of (12.1). Hopefulness that the observed QCD mesons can be theoretically characterized by a suitable choice of the parameter *S* in combination with the theoretically-imposed, three-valued quantum number $\mu = 0, 1, \frac{4}{3}$, rests on this carefully-constructed correspondence to the demonstrably-successfully theory of electroweak interactions.

References

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