Yang-Mill Foundations of Baryons and Confinement Phenomena

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1. Introduction

To be added.

2. A Brief Review of Classical Electromagnetism and Differential Forms

In classical field theory, the Euler Lagrange equations of motion

$$\partial_\mu \frac{\delta \mathfrak{L}}{\delta(\partial_\mu \varphi)} - \frac{\delta \mathfrak{L}}{\delta \varphi} = 0 \tag{2.1}$$

for a given field φ arise by applying a variational principle $\delta S = \delta \int d^4 x \mathfrak{L} = 0$ to the action *S*. In the case of the Maxwell Lagrangian density:

$$\mathfrak{L} = -\tfrac{1}{4} F^{\mu\nu} F_{\mu\nu} + e A_\mu J^\mu , \tag{2.2}$$

where:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \equiv \partial^{[\mu} A^{\nu]} \tag{2.3}$$

is the field strength tensor, A^ν is the vector potential, J^μ is the electric current density and *e* is the electric charge strength. Applying (2.1) to (2.2) using (2.3) yields Maxwell's classical equation for electric charge:

$$e J^\nu = \partial_\mu F^{\mu\nu} . \tag{2.4}$$

If we employ the "dual" field strength tensor $* F^{\mu\nu} \equiv \tfrac{1}{2!} \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$ where $\varepsilon^{\mu\nu\alpha\beta}$ is the totally-antisymmetric contravariant Levi-Civita tensor, then the Faraday Lagrangian density:

$$\mathfrak{L} = -\tfrac{1}{4} F^{\mu\nu} * F_{\mu\nu} \tag{2.5}$$

will, on application of (2.2), yield:

$$\partial_\mu * F^{\mu\nu} = 0 \left(\equiv \mu P^\nu \right), \tag{2.6}$$

which is Faraday's classical equation for a magnetic current density defined above as P^ν with magnetic charge μ. By being equal to zero, such "magnetic monopoles" are thought not to exist, and have never been observed to exist. By Dirac's quantization condition, $e \cdot \mu = 2\pi n \hbar c$.

The language of differential forms, employed in exterior calculus, is especially convenient not only for representing the Maxwell and Faraday equations very concisely without any loss of mathematical rigor, but are also ideally suited for mathematical integration over closed loops, surfaces and volumes. In this language, (2.3), (2.4) and (2.6) are equivalently represented as:

$$F = dA, \tag{2.7}$$

$$e * J = d * F = d * dA, \tag{2.8}$$

$$dF = ddA = 0 \left(\equiv \mu P\right).^{*} \tag{2.9}$$

The latter includes a fundamental identity of exterior calculus, $dd = 0$: "the exterior derivative of an exterior derivative is zero."

Stokes' / Gauss' theorem states that for any closed *d*-dimensional loop / surface / volume:

$$\oint_{d} dH = \oint_{d-1} H \tag{2.10}$$

for any p-form *H*. Thus, the classical Maxwell / Faraday equations (2.8), (2.9) are readily written in integral form, using (2.10), as:

$$\oiiint e * J = \oiint * F = \oiint * dA, \tag{2.11}$$

$$\oiiint \mu P = \oiint F = \oiint dA = \oint A = 0 \ . \tag{2.12}$$

This is the spacetime-covariant formulation of Maxwell's equations in integral form. Now, let us turn to quantum theory.

3. Maxwell's and Faraday's Forms and their Surface Integrals, In Quantum Theory

In the Heisenberg picture of quantum theory, the equation of motion, with respect to an operator observable *O*, is given by:

$$d^{0}O = \frac{i}{\hbar}\left[H, O\right] + \partial^{0}O, \tag{3.1}$$

where *H* is the Hamiltonian and the commutator $\left[H, O\right] = HO - OH$. By Ehrenfest's theorem, designating the expectation value of an operator *O* as $\left\langle O \right\rangle = \left\langle \phi \middle| O \middle| \varphi \right\rangle$, the time derivative of $\left\langle O \right\rangle$ is related to the expectation value $\left\langle \partial^{0}O \right\rangle$ via the similar form:

$$d^{0}\left\langle O \right\rangle = \frac{i}{\hbar}\left\langle \left[H, O\right] \right\rangle + \left\langle \partial^{0}O \right\rangle. \tag{3.2}$$

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[*] In the foregoing, $A \equiv A^{\mu} dx_{\mu}$, $F \equiv \frac{1}{2!} F^{\mu\nu} dx_{\mu} \wedge dx_{\nu} = F^{\mu\nu} dx_{\mu} dx_{\nu}$, $J \equiv \frac{1}{3!} J^{\mu\nu\alpha} dx_{\mu} \wedge dx_{\nu} \wedge dx_{\alpha} = J^{\mu\nu\alpha} dx_{\mu} dx_{\nu} dx_{\alpha}$, $P \equiv \frac{1}{3!} P^{\mu\nu\alpha} dx_{\mu} \wedge dx_{\nu} \wedge dx_{\alpha} = P^{\mu\nu\alpha} dx_{\mu} dx_{\nu} dx_{\alpha}$, and the differential operator *d* as applied to any p-form *H* is $dH = \frac{1}{p!} \partial_{\nu} H_{\mu_1 \mu_2 \ldots \mu_p} dx^{\nu} dx^{\mu_1} dx^{\mu_2} \ldots dx^{\mu_p}$. Generally it is most convenient to employ forms without wedge products.

Of particular interest here, this means that the Euler-Lagrange equation (2.1) no longer applies in quantum theory, but does remain valid as an expectation value equation, thus:

$$\left\langle \partial_\mu \frac{\delta \mathfrak{L}}{\delta(\partial_\mu \varphi)} - \frac{\delta \mathfrak{L}}{\delta \varphi} \right\rangle = 0 . \tag{3.3}$$

In quantum theory, the differential forms *A*, *F*, *J*, *P*, are no longer scalars, but become operators. The scalars associated with these operators, are now their expectation values $\langle A \rangle = \langle \phi | A | \varphi \rangle$, $\langle F \rangle = \langle \phi | F | \varphi \rangle$, $\langle J \rangle = \langle \phi | J | \varphi \rangle$, and $\langle P \rangle = \langle \phi | P | \varphi \rangle$. Further, for any operator, $d\langle O \rangle = d\langle \phi | O | \varphi \rangle = \langle \phi | dO | \varphi \rangle = \langle dO \rangle$, and, as a general rule $\langle A + B \rangle = \langle A \rangle + \langle B \rangle$. So, Maxwell / Faraday equations (2.8), (2.9) continue to apply in quantum theory, in the "weaker" form:

$$\langle e * J \rangle = \langle d * F \rangle = \langle d * dA \rangle = d\langle * F \rangle = d\langle * dA \rangle , \tag{3.4}$$

$$\langle \mu P \rangle = \langle dF \rangle = \langle ddA \rangle = d\langle F \rangle = d\langle dA \rangle = dd\langle A \rangle = 0 \ . \tag{3.5}$$

It is interesting to note that in quantum theory, the rule $dd = 0$ that "the exterior derivative of an exterior derivative is zero" continues to apply. However, the scalars to which $dd = 0$ is now applied, are not classical scalars, but the expectation values of these classical scalars after they become quantum mechanical operators. Equation (3.5), written as $\langle \mu P_{\text{op}} \rangle = 0$, tells us that the expectation value of the magnetic charge *operator* is zero, but that this operator can assume a range of non-zero values, $\mu P_{\text{op}} \neq 0$. One may be tempted, but would be incorrect, to conclude from this that there might be quantum statistical fluctuations about the expected value by which one could observe a magnetic charge, $\mu P_{\text{obs}} \neq 0$, because one cannot conclude from the value of $P_{\text{op}} \neq 0$, anything definite about P_{obs}.

In quantum theory, Stokes' / Gauss' theorem is not altered. So long as one can satisfy its underlying formal hypotheses, it must apply. Thus, in integral form, keeping in mind that *A*, *F*, *J*, *P*, are all *operators*, the scalar equations (3.4) and (3.5) become:

$$\oiiint \langle e * J \rangle = \oiint \langle * F \rangle = \oiint \langle * dA \rangle , \tag{3.6}$$

$$\oiiint \langle \mu P \rangle = \oiint \langle F \rangle = \oiint \langle dA \rangle = \oint \langle A \rangle = 0 \ . \tag{3.7}$$

When we make use of the integral form, it is possible to make better statements about what is observed, in particular, because the total observable $O_{\text{obs}\,(3)}$ contained within a given volume is specified by $O_{\text{obs (3)}}=\oiiint\langle O_{\text{op}}\rangle d^3x$ over that volume, in relation to the associated operator O_{op}. So for example, we use $\langle e*J_{\text{op}}\rangle=\langle e*J_{\text{op}}{}^{\mu\nu\alpha}dx_\mu dx_\nu dx_\alpha\rangle=\langle e*J_{\text{op}}{}^{\mu\nu\alpha}\rangle dx_\mu dx_\nu dx_\alpha$ to expand in (3.6), where $dx_\mu dx_\nu dx_\alpha$ can be removed from the expectation brackets because this is a volume element specified with respect to a system of spacetime coordinates and so does not have an "expectation." Thus, $\oiiint\langle e*J_{\text{op}}\rangle=\oiiint\langle e*J_{\text{op}}{}^{\mu\nu\alpha}\rangle dx_\mu dx_\nu dx_\alpha$. Transformed into an "at rest" frame, one can show that $Q_{\text{obs (3)}}\equiv\oiiint\langle e*J_{\text{op}}\rangle=\oiiint\langle e\rho_{0\ \text{op}}\rangle d^3x$ is the total charge *observed* within this 3-dimensional volume.*

By similar rationale, the total observable $O_{\text{obs (2)}}$ contained over a given surface area is specified by $O_{\text{obs (2)}}=\oiint\langle O_{\text{op}}\rangle d^2x$ over that area, in relation to O_{op}. Thus, again in a rest frame, one can show that $\mathbf{E}_{\text{obs (2)}}=\oiint\langle *F_{\text{op}}\rangle=\oiint\langle\mathbf{E}_{\text{op}}\rangle\cdot\mathbf{dA}$ is the total electric field *observed* to flow across the two-dimensional volume which contains the charge $Q_{\text{obs (3)}}$.** Thus, transformed to a rest frame, (3.6), $\oiiint\langle e*J\rangle=\oiint\langle *F\rangle$, recovers the classical result $Q=\oiint\mathbf{E}\cdot\mathbf{dA}$.

Similarly, at rest in the Faraday equation (3.7), $M_{\text{obs (3)}}\equiv\oiiint\langle\mu P_{\text{op}}\rangle$ is the total magnetic charge *observed* within a 3-dimensional volume, $B_{\text{obs (2)}}=\oiint\langle F_{\text{op}}\rangle$ is the total magnetic field *observed* to flow across the two-dimensional volume which contains the magnetic charge, and *these both, in turn, remain equal to zero*. Thus, despite initial contrary appearances, we recover the classical result that magnetic charges do not exist in nature, $M=\oiint\mathbf{B}\cdot\mathbf{dA}=0$. One then arrives at the balance of classical electromagnetism, by the Lorentz transformation of the foregoing.

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* In explicit form, the current density operator $J_{\text{op}}{}^{\mu}=\frac{1}{3!}\varepsilon^{\mu\nu\alpha\beta}*J_{\text{op}\ \nu\alpha\beta}$. Represent the components of this as $J_{\text{op}}{}^{\mu}=(\rho_{\text{op}},\mathbf{J}_{\text{op}})$. Transform to the rest frame, so that $J_{\text{op}}{}^{\mu}=(\rho_{0\ \text{op}},0)$, where $\rho_{0\ \text{op}}$ is the *proper* charge density operator. Then, $*J_{\text{op}\ 123}=\rho_{0\ \text{op}}$, and $*J_{\text{op}\ 230}=*J_{\text{op}\ 301}=*J_{\text{op}\ 012}=0$, in all permutations, since $*J_{\text{op}}{}^{\mu\nu\alpha}$ is totally antisymmetric. Index permutations are already accounted for, via $J=\frac{1}{3!}J^{\mu\nu\alpha}dx_\mu\wedge dx_\nu\wedge dx_\alpha=J^{\mu\nu\alpha}dx_\mu dx_\nu dx_\alpha$.

** Here, keep in mind that $*F^{\mu\nu}\equiv\frac{1}{2!}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$ and $F\equiv\frac{1}{2!}F^{\mu\nu}dx_\mu\wedge dx_\nu=F^{\mu\nu}dx_\mu dx_\nu$. Because duality effectuates $\mathbf{E}\to\mathbf{B}$ and $\mathbf{B}\to-\mathbf{E}$ transformations, the components of interest are now $*F^{12}=E_z$, $*F^{23}=E_x$, and $*F^{31}=E_y$.

4. Yang-Mills Gauge Theory, Cast in the Manner of the Maxwell and Faraday Equations

In Yang-Mills gauge theory, the vector potential is denoted by G^μ rather than by A^μ, and its associated differential form is $G \equiv G^\mu dx_\mu$. However, G^μ is in general a non-commuting object, $[G^\mu, G^\nu] \neq 0$, because it, in turn, is specified by $G^\mu \equiv T^i G_i^{\ \mu}$, where T^i are the traceless, Hermitian, $N \times N$ group generators for the associated gauge group SU(N). The group structure is specified by $f^{ijk} T_i = -i[T^j, T^k]$ and the Latin internal symmetry index $i = 1,2,3 \ldots N^2 - 1$ is raised and lowered with the unit matrix δ_{ij}. The differential from associated with the non-commuting $[G^\mu, G^\nu] \neq 0$ is represented by:

$$G^2 \equiv G \wedge G = \tfrac{1}{2!}[G^\mu, G^\nu] dx_\mu \wedge dx_\nu = [G^\mu, G^\nu] dx_\mu dx_\nu, \tag{4.1}$$

and to ensure the gauge invariance of any such theory, it is necessary for the Yang-Mills field strength two-form to be given by:

$$F = dG + igG^2, ^* \tag{4.2}$$

where *g* is the charge strength of the Yang-Mills group. Contrasting with $F = dA$ from (1.2), we see that the only real difference between Yang-Mill theory and U(1) electromagnetic theory, whether classical or quantum, is the existence of this extra term G^2 in the field strength. *Aside from the existence of this extra, non-linear G^2 term in (4.2), the development of Yang-Mills theory can proceed in an identical fashion to that of electromagnetic theory.*

First, in the same way that one can think about electromagnetism in the classical sense of section 2 and then in the quantum sense of section 3 with expectation values, one can make a similar sharp distinction between "classical Yang-Mills theory" and "quantum Yang-Mills theory." This is not usually done, likely because of the historical accident that classical electromagnetism was known before quantum theory, and the reversed historical accident that quantum theory had undergone substantial development before Yang-Mills theory became known. Nonetheless, it is a useful heuristic tool, if one is thinking for instance about strong interactions, to think not only about "quantum chromo dynamics" (QCD), but also about "classical chromo dynamics," which is what strong interaction theory might have first been, had

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[*] By way of review, Equation (4.2) can thus be expanded using the foregoing group relationships to the $N \times N$ equation $F^{\mu\nu} = \partial^\mu G^\nu - \partial^\nu G^\mu + ig[G^\mu, G^\nu]$, and, with all components explicit, to the commonly-written, most familiar $F^{i\,\mu\nu} = \partial^\mu G^{i\,\nu} - \partial^\nu G^{i\,\mu} - gf^{ijk} G_j^{\ \mu} G_k^{\ \nu}$. In most instances, it is easiest to retain the compacted form (4.2).

Yang-Mills theory also undergone the same historical accident as electromagnetic theory, of coming before quantum theory. In the discussion here, we will frequently use the prefix "chromo" to broadly refer to Yang-Mills theory in general, rather than in the narrower sense of only the particular color group SU(3).

Now, in Yang-Mills theory, classical or quantum, the Lagrangian density which corresponds to (2.2), is simply:

$$\mathfrak{L} = -\tfrac{1}{4} F_i^{\ \mu\nu} F^i_{\ \mu\nu} + g G^i_{\ \mu} J_i^{\ \mu} , \tag{4.3}$$

or, with the normalization $\delta^i_{\ j} = 2\mathrm{Tr}\left(T^i T_j\right)$ and the internal symmetry indexes suppressed:

$$\mathfrak{L} = -\tfrac{1}{2}\mathrm{Tr}\left(F^{\mu\nu} F_{\mu\nu}\right) + 2g\mathrm{Tr}\left(G_\mu J^\mu\right). \tag{4.4}$$

Thus, thinking in terms of *classical* Yang-Mill theory, and compressing some of the detailed stapes taken in section 2, we may use (4.2) to write the "classical" Yang-Mills equation of motion:

$$g * J = d * F = d * dG + id\left(g * G^2\right), \tag{4.5}$$

$$\mu P = dF = ddG + id\left(gG^2\right) = id\left(gG^2\right) \neq 0 \ , \tag{4.6}$$

where, in the latter equation, we have continued to apply the exterior calculus relationship:

$$ddG = 0 \ . \tag{4.7}$$

To go over to quantum Yang-Mills theory, *G*, *F*, *J*, *P* once again become operators, and merely take the expectation values of these operators as in section (3), to arrive at:

$$\left\langle g * J\right\rangle = \left\langle d * F\right\rangle = \left\langle d * dG\right\rangle + i\left\langle d\left(g * G^2\right)\right\rangle = d\left\langle * F\right\rangle = d\left\langle * dG\right\rangle + id\left\langle g * G^2\right\rangle , \tag{4.8}$$

$$\left\langle \mu P\right\rangle = \left\langle dF\right\rangle = i\left\langle d\left(gG^2\right)\right\rangle = d\left\langle F\right\rangle = id\left\langle gG^2\right\rangle \neq 0 \ , \tag{4.9}$$

$$\left\langle ddG\right\rangle = d\left\langle dG\right\rangle = dd\left\langle G\right\rangle = 0 \ . \tag{4.10}$$

These should be contrasted to (3.4) and (3.5).

Then, we apply Stokes' / Gauss' theorem, as before, to establish the integral forms of these three equations, while simultaneously moving the expectation brackets outside the integrals as discussed previously, to arrive at:

$$\left\langle \oint\!\!\!\iiint g * J\right\rangle = \left\langle \oint\!\!\!\iint * F\right\rangle = \left\langle \oint\!\!\!\iint * dG\right\rangle + i\left\langle \oint\!\!\!\iint g * G^2\right\rangle , \tag{4.11}$$

$$\left\langle \oint\!\!\!\iiint \mu P\right\rangle = \left\langle \oint\!\!\!\iint F\right\rangle = i\left\langle \oint\!\!\!\iint gG^2\right\rangle \neq 0 \ , \tag{4.12}$$

$$\left\langle \oiiint ddG \right\rangle = \left\langle \oiint dG \right\rangle = \left\langle \oint G \right\rangle = 0 \quad . \tag{4.13}$$

These three equations should be carefully compared with their two quantum electromagnetic counterparts (3.8), (3.9), for these are the Yang-Mill analogs of the "expectationized" Maxwell-Faraday equations in integral form.

With these equations, our preliminary development is complete. From this point forward, we shall devote our attentions to trying to *interpret* what equations (4.8) through (4.13) tell us about the character of any system of currents and fields and potentials which might be governed by these equations, and to engaging in further *calculation* based on one or more of these equations, where warranted, to help us better understand and interpret these equations.

5. Chromo-Electric Charge and Current Confinement

Let first examine equation (4.8), in contrast with its electromagnetic counterpart (3.4). Each of these equations contains an "electric" source current density J. In electromagnetic theory, the expectation value of the potentials for which the current expectation vanishes, $\langle J \rangle = 0$, is specified by the second order linear differential equation $d\langle * dA \rangle = 0$. In tensor form, the equivalent equation is $\partial_\sigma \partial^\sigma \langle A^\nu \rangle = 0$, and this is a comparatively simple differential equation.

In Yang-Mills theory, this is different. Here, the expectation values of the potentials for which the chromo-electric current expectation value vanishes, $\langle J \rangle = 0$, is arrived at from (4.8), and is specified by:

$$d\langle * dG \rangle + id\langle g * G^2 \rangle = d\langle * dG \rangle + i\langle (dg) * G^2 \rangle + i\langle g\left(d * G^2\right)\rangle = 0 \, . \tag{5.1}$$

Here, given the asymptotic freedom and infrared slavery known to subsist in QCD, we work from the supposition that $dg \neq 0$. If g and G are *uncorrelated*, i.e., if they have separate probability distribution functions, the one can use $\langle g * G^2 \rangle = \langle g \rangle \langle * G^2 \rangle$ to separate the nonlinear term above, but one ought not assume this to be the case. That is, one must consider the possibility that the interaction charge and the potentials will correlate in some fashion, unless and until it is shown otherwise.

Equation (5.1) is a highly nontrivial differential equation with all of zero, first and second order terms in the potential G, and to boot, it also contains terms which are linear and of first order in the charge strength g, and to further boot, one has to consider that g and G may be probabilistically correlated.

Yet, we know from electrodynamics that potentials are often formulated as a function of spatial distance (r), and that in QCD, one frequently attempts to understand confinement in confinement in terms of an effective potential which, similarly, once would also like to cast as a function of both space (r) and time (t). Equation (5.1) implicitly establishes a non-trivial differential equation for the N^2-1 potentials $\langle G_i^{\ \nu} \rangle$ of $\langle G^\nu \rangle = T^i \langle G_i^{\ \nu} \rangle$, and, what is most important, the potentials which are specified by (5.1), *are the potentials for which the current expectation vanishes*. In terms of space and time, once (5.1) is solved for $\langle G_i^{\ \nu} \rangle$ and the spatial distribution $\langle G_i^{\ \nu}(t, \mathbf{x}) \rangle$ is either know or postulated, (5.1) then establishes those events in spacetime at which the there is no expected chromo-electric *charge* density (J^0) and at which there is no expected chromo-electric *current* density (J^k). Where solutions to (5.1) may turn out to describe a closed surface, such a $\langle J \rangle = 0$ "expected surface," *by definition* will exhibit the properties of a surface within which the chromo-electric charges and currents are expected to be *confined*, because, by definition, there can neither be any charge density at, nor any charge density flowing across, this surface.

Thus, one may regard (5.1) as a differential equation which can be used to identify what are, *by definition*, effective expected confinement potentials and surfaces.

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