Inferring Electrodynamics and Quantum Theory from General Coordinate Invariance

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Abstract:

It is shown how electrodynamics, the quantization of energy, and the Heisenberg formulation of quantum mechanics, may all be inferred from geometrodynamical gravitational theory using the sole symmetry requirement that the laws of nature must remain invariant under general coordinate transformations. “Energy-time” uncertainty is understood based on non-commuting perturbation and time operators similar to the known non-commuting momentum and position operators.

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1. Introduction: A Brief Review of General Coordinate Invariance

During the 20th century, with the advent especially of Noether’s theorem, [1] symmetry principles, and especially local symmetry principles, assumed a central role in theoretical physics. One such symmetry of consequence is the local gauge symmetry of electrodynamics. Another is the global symmetry identified by Minkowski [2] which leaves invariant the linear displacement element \( t^2 - x^2 - y^2 - z^2 \), and which later, when treated on a local basis, provided the geometric foundation for the General Theory of Relativity (GTR) [3]. But perhaps the most fundamental symmetry of all, also articulated with GTR, is the requirement the laws of nature must be formulated to be invariant under general, local transformations of coordinate systems.

We endeavor here, to demonstrate that this principle of general coordinate invariance, as simple yet far reaching as it is already understood to be, leads without anything more, to electrodynamic gauge theory, to Plank’s quantization of energy [4] which also underpins Einstein’s explanation of the photoelectric effect [5], and to the anti-commutating matrix mechanics [6] and uncertainty relationships [7] of Heisenberg, which undergird quantum field theory. Questions regarding the so-called “energy-time” uncertainty are also resolved by better understanding the perturbative role of the electrodynamic potential. (see, e.g., [8], section 5.6)

Therefore, it is helpful to begin by reviewing the basic elements of the principle of general coordinate invariance, and in particular, how coordinate transformations are effectuated in a generally-covariant theory.

Consider the differential coordinate element \( dx^\nu \). Under general coordinate transformations, this is a vector which covariantly transforms as:

\[
\nu dx^\mu \rightarrow \nu' dx'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu.
\]  
(1.1).

It is sometimes helpful to think of this as just a variation of the fundamental calculus statement that total derivatives are related to partial derivatives by \( \frac{dx^\mu}{dx^\nu} = \frac{\partial x^\mu}{\partial x^\nu} \), where one then transforms \( dx^\mu \rightarrow dx'^\mu \) and \( \frac{\partial x^\mu}{\partial x^\nu} \rightarrow \frac{\partial x'^\mu}{\partial x'^\nu} \). A contravariant (upper-indexed) vector \( B^\mu \) follows the identical transformation law, that is:

\[
B^\mu \rightarrow B'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} B^\nu,
\]  
(1.2)

while a covariant (lower-indexed) vector \( B_\mu \) generally transforms as: (see, e.g., [9], section 6.1)
The term “covariant” is often used in two different ways, one in relation to covariance under transformations, and the other in relation to a lower index. Generally, the intended meaning is discernable from context.

Importantly, the $x^\mu$ coordinates themselves, do not form a vector under general coordinate transformations. Rather, they transform generally as: (see, e.g., [10], at 422)

$$x^\mu \rightarrow x'^\mu = x^\mu - \Lambda^\mu (x^\nu),$$

(1.4)

where $\Lambda^\mu (x^\nu)$ is a four-component, quadruplet, local gauge parameter.

In the special case of a linear Poincare transformation which we will examine here in some depth, these coordinates transform as: (see [9], equations (4.10) and (2.16), (2.17))

$$x^\mu \rightarrow x'^\mu = x^\mu - \Lambda^\mu = b^\mu \nu x^\nu + \bar{x}^\mu.$$

(1.5)

Here, $b^\mu \nu$ is a constant matrix which specifies Lorentz boosts and rotational transformations, and $\bar{x}^\mu$ is a constant quadruplet specifying time and space translations. To avoid any possible confusion, because $\bar{x}^\mu$ will appear frequently in much of the development to follow, it is important to explain that we use the arrow above the $x$ in $\bar{x}^\mu$ a) to indicate that we are speaking of translations through time and space, and b) to distinguish these translations $\bar{x}^\mu$ from the coordinates $x^\mu$ themselves. The use of $\rightarrow$ should be interpreted in this way, and should not be taken to signify a “vector,” which is another way in which the $\rightarrow$ symbol is often employed. The coordinates $x^\mu$ from which we are carefully distinguishing the translation $\bar{x}^\mu$, do form a vector under the specialized Poincare transformations, a.k.a., inhomogeneous Lorentz transformations, specified in (1.5). But again, it bears emphasis that the $x^\mu$ do not form a vector under general coordinate transformations. When $\bar{x}^\mu = 0$, (1.5) becomes a homogeneous Lorentz transformation, i.e., a rotation and / or boost.

We now introduce the common notation $\partial_\nu \equiv \partial / \partial x^\nu$ and $\partial'_\nu \equiv \partial / \partial x'^\nu$. It is easy to deduce from (1.4), $x'^\mu = x^\mu - \Lambda^\mu (x^\nu)$, that the transformation matrix $\partial_\nu x'^\mu = \partial x'^\mu / \partial x^\nu$ in the contravariant transformation (1.2) may be written as: (see [10], at 422 just prior to equation (8))

$$\partial_\nu x'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} = \frac{\partial x^\mu}{\partial x^\nu} - \frac{\partial \Lambda^\mu}{\partial x^\nu} = \delta^\mu \nu - \partial_\nu \Lambda^\mu,$$

(1.6)
and that \( \partial'_\mu x^\nu = \partial x^\nu / \partial x'^\mu \) in the covariant transformation (1.3) is:

\[
\partial'_\mu x^\nu = \frac{\partial x^\nu}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} + \frac{\partial \Lambda}{\partial x'^\mu} = \delta^\nu_\mu + \partial'_\mu \Lambda^\nu, \tag{1.7}
\]

where in both cases we have applied \( \partial_\nu x^\mu = \partial x^\nu / \partial x'^\mu = \delta^\mu_\nu \).

Using (1.6), we see that the quadruplet gauge parameter \( \Lambda^\nu(x^\nu) \) is intimately connected to understanding gravitation as a gauge theory. For example, the contravariant metric tensor \( g^{\mu\nu} = g^{\nu\mu} \), transforms as:

\[
g^{\mu\nu} \rightarrow g'^{\mu\nu} = \frac{\partial x'^\sigma}{\partial x^\sigma} g^{\sigma\tau} = \left( \delta^\nu_\tau - \partial_\tau \Lambda^\nu \right) g^{\sigma\tau} = g^{\nu\mu} - \partial_\sigma \Lambda^{\nu\sigma} + \partial_\sigma \Lambda^{\nu\tau} g^{\sigma\tau}, \tag{1.8}
\]

which for small \( \partial^\nu \Lambda^\mu \), becomes \( g^{\mu\nu} \rightarrow g'^{\nu\mu} = g^{\mu\nu} - \partial^{[\nu} \Lambda^{\mu]}(x^\nu) \). Here, \( \{A, B\} = AB + BA \) is the anticommutator, and we have symmetrised the \( \partial_\sigma \Lambda^{\nu\sigma} g^{\sigma\tau} \) term in the second line to accord with the transposition symmetry \( g^{\mu\nu} = g^{\nu\mu} \). This is analogous to the local transformation \( A^\mu \rightarrow A'^\mu + \partial'^\mu \alpha(x^\mu) \) of the electromagnetic gauge field \( A^\mu \), where \( \alpha(x^\mu) \) is a local singlet phase, and explains why gravitation and electromagnetism are both regarded as gauge theories. ([10], at 422) To date, however, the gauge nature of gravitation and the gauge nature of electrodynamics goes no further than this analogy. It would be desirable to find a direct physical connection between these two forms of gauge symmetry, if one exists.

2. The Underlying Connection Between General Coordinate and Gauge Transformations

Let us now consider the behavior of the scalar product \( B_\mu x^\mu \) between a generally-covariant, covariant (lower-indexed) vector \( B_\mu \), and the coordinates \( x^\mu \) which as noted above comprise a Poincare but not a general four-vector. Formally, we take \( B_\mu \) to be, without limitation, any covariant (lower-indexed) four-vector transforming generally-covariantly as \( B_\mu \rightarrow B'_\mu = \left( \partial x^\nu / \partial x'^\mu \right) B_\nu \) under the covariant (lower-indexed) transformation (1.3).

Additionally, we consider that \( B_\mu \) may or may not be a function \( B_\mu(x^\nu) \) of the spacetime
coordinates $x^\mu$, and so for the moment will consider that $B_\mu$ is a function of $x^\mu$, i.e., that $\partial_\alpha B_\mu \neq 0$. One may then always specialize to $\partial_\alpha B_\mu = 0$. Later, we shall substitute specific, physically-meaningful four vectors for $B_\mu$, but for now, we are seeking to examine the formal, mathematical transformation properties of $B_\mu x^\mu$, for any generally-covariant four-vector $B_\mu$ subsisting in a curved four-dimensional Riemannian spacetime geometry.

Combining (1.3), (1.5) and (1.7), we now subject $B_\mu x^\mu$ to the general coordinate transformation:

$$B_\mu x^\mu \rightarrow B'_\mu x'^\mu = \frac{\partial x^\nu}{\partial x'^\mu} B_\nu \left(x^\mu - \Lambda^\mu\right) = B_\nu \left(\delta^\nu_\mu + \partial_\nu \Lambda^\mu\right) \left(x^\mu - \Lambda^\mu\right) = B_\mu x^\mu - B_\mu \Lambda^\mu + B_\mu \partial_\nu \Lambda^\mu \left(x^\nu - \Lambda^\nu\right) = B_\mu x^\mu - \alpha_B(x^\sigma),$$

(2.1)

where, in the last line of the above, we have defined the locally-varying, real, singlet parameter:

$$\alpha_B(x^\sigma) \equiv B_\mu \Lambda^\mu(x^\sigma) - B_\mu \partial_\nu \Lambda^\mu \left(x^\nu - \Lambda^\nu (x^\sigma)\right).$$

(2.2)

We attach the subscript “$B$” to $\alpha_B(x^\sigma)$, to make clear that this parameter is a function not only of the gauge quadruplet $\Lambda^\mu(x^\nu)$, but also of the vector $B_\mu$. Observing that the scalar product $s = B_\mu A^\mu = B'_\mu A'^\mu = s'$ of any two generally covariant four vectors $A^\mu$ and $B_\mu$ is an invariant scalar $s = s'$ under general coordinate transformations, what makes (2.1) distinctive is that $B_\mu x^\mu \neq B'_\mu x'^\mu$, but that these differ from one another by this singlet local parameter $\alpha_B(x^\sigma)$, and that $\alpha_B(x^\sigma)$ arises because the coordinates $x^\nu$ do not form a vector under general coordinate transformations.

Unitary factors $U_B = e^{-iB_\mu x^\mu} = \cos B_\mu x^\mu - i \sin B_\mu x^\mu$, similarly with a subscript “$B$,” appear frequently in a variety of physical situations. As one example, the factor $U_p = e^{-ip_\mu x^\mu}$ appears as part of many wavefunction formulations, e.g., $\psi = u(p^\sigma) e^{-ip_\mu x^\mu} = u(p^\sigma) U_p$ (e.g. [11], equation (5.18), also (6.63) and (6.90) for boson wavefunctions), and is also the “kernel” used in many Fourier transformations. Therefore, there is some motivation to examine what transpires
when the scalar product $B_\mu x^\mu$ is employed in the exponent of $U_B = e^{-iB_\mu x^\mu}$, and to examine the behavior of $U_B$ under general coordinate transformations. Using (2.1), we see that:

$$U_B = e^{-iB_\mu x^\mu} \rightarrow U'_B = e^{-iB'_\mu x'^\mu} = e^{-i[B_\mu x^\mu - \alpha(x^\sigma)]} = e^{i\alpha_\mu(x^\sigma)} e^{-iB_\mu x^\mu} = e^{i\alpha_\mu(x^\sigma)} U_B,$$

(2.3)

with $\alpha_\mu(x^\sigma)$ as specified in (2.2). That is, we see that the parameter $\alpha_\mu(x^\sigma)$ is actually a local phase (gauge), and that the general coordinate transformation $U_B \rightarrow U'_B = e^{i\alpha_\mu(x^\sigma)} U_B$ in the above looks similar to a local gauge transformation of $U_B$, with $\alpha_\mu(x^\sigma)$ determined according to (2.2) in relation to $B_\mu$ and to the particular general coordinate transformation being applied, and with $e^{i\alpha_\mu(x^\sigma)}$ operating to transform $U_B$.

This is of interest because in electrodynamic gauge theory, one simply postulates the local gauge parameter $\alpha(x^\mu)$ and the gauge operator $e^{i\alpha(x^\sigma)}$ and then develops electrodynamics by requiring that, e.g., the Lagrangian density must remain invariant under local gauge transformations. Here, in contrast, $e^{i\alpha_\mu(x^\sigma)}$ is naturally forced upon us by, and directly corresponds to, a general coordinate transformation. For any given general coordinate transformation characterized by $\Lambda^\mu (x^\nu)$ in $x^\mu \rightarrow x'^\mu = x^\mu - \Lambda^\mu (x^\nu)$ of (1.4), in reference to any vector $B_\mu$, there is generated a corresponding local gauge transformation with gauge parameter $\alpha_\mu(x^\sigma)$ specified by (2.2). Obviously, it then becomes of interest to understand whether $\alpha_\mu(x^\sigma)$ here, can somehow be related to the gauge parameter $\alpha(x^\sigma)$ of electrodynamics, and if so, precisely how.

This is highlighted by considering the special case of a linear Poincare transformation, i.e., a Lorentz boost and / or a rotation, and / or a time or space translation, governed by the respective constant matrices $b^{\mu \nu}$ and $\tilde{x}^\mu$. In this special case, (1.5) applies, so:

$$\Lambda^\mu (x^\nu) = x^\mu - b^{\mu \sigma} x^\sigma - \tilde{x}^\mu.$$

(2.4)

We may then use (2.2) to pose the question: what impact, if any, does such an inhomogeneous Lorentz transformation have on singlet phase $\alpha_\mu(x^\mu)$? The full calculation is illustrative and so is shown below. Substituting $x'^\mu = x^\mu - \Lambda^\mu (x^\nu)$ from (1.4), then (2.4), into (2.2), suppressing
the \((x^\mu)\) to avoid clutter, and keeping in mind that \(\vec{x}^\mu\) represents a constant time and / or space translation, yields what will eventually become a fundamentally-important result:

\[
\alpha_b = B_\mu \Lambda^\mu - B_\nu \partial_\mu \Lambda^\nu (x^\mu - \Lambda^\mu) = B_\mu \Lambda^\mu - B_\nu \frac{\partial \Lambda^\nu}{\partial (x^\mu - \Lambda^\mu)} (x^\mu - \Lambda^\mu) \\
= B_\mu \left( x^\mu - b^\mu \sigma x^\sigma - \vec{x}^\mu \right) - B_\nu \frac{\partial}{\partial (x^\mu - (x^\mu - b^\mu \sigma x^\sigma - \vec{x}^\mu))} \left( x^\mu - (x^\mu - b^\mu \sigma x^\sigma - \vec{x}^\mu) \right) \\
= B_\mu \left( x^\mu - b^\mu \sigma x^\sigma - \vec{x}^\mu \right) - B_\nu \frac{\partial}{\partial \left( \frac{b^\mu \sigma x^\sigma + \vec{x}^\mu}{b^\mu \sigma x^\sigma + \vec{x}^\mu} \right)} \left( b^\mu \sigma x^\sigma + \vec{x}^\mu \right) \\
= B_\mu \left( x^\mu - b^\mu \sigma x^\sigma - \vec{x}^\mu \right) - B_\nu \left( b^\mu \sigma \right)^{-1} \left( b^\mu \sigma x^\sigma + \vec{x}^\mu \right) + B_\nu \left( b^\mu \sigma \right)^{-1} b^\nu \sigma \left( b^\mu \sigma x^\sigma + \vec{x}^\mu \right) \\
= B_\mu x^\mu - B_\sigma \left( b^\mu \sigma \right)^{-1} b^\mu \sigma x^\sigma - B_\sigma \left( b^\mu \sigma \right)^{-1} \vec{x}^\mu \\
= -B_\sigma \left( b^\mu \sigma \right)^{-1} \vec{x}^\mu
\]

In short:

\[
\left[ b^\mu \sigma \alpha_b \right] = -B_\sigma \vec{x}^\mu. \tag{2.6}
\]

In (2.5), we have made use of: \(\partial \vec{x}^\mu = 0\) (\(\vec{x}^\mu\) is a constant translation); \(\partial (b^\mu \sigma x^\sigma) = b^\mu \sigma \partial x^\sigma\) (\(b^\mu \sigma\) is constant); \(\partial x^\nu / \partial x^\sigma = \delta^\nu \sigma\); and \((b^\mu \sigma)^{-1} b^\nu \sigma = \delta^\nu \mu\) (a matrix times its inverse is a unit matrix).

For \(\vec{x}^\mu = 0\), which specifies a homogeneous (translation-free) Lorentz transformation, i.e., a rotation and / or boost without translation, this simplifies to:

\[
\alpha = 0. \tag{2.7}
\]

Here, we have removed the subscript from \(\alpha_b\) in the above, because this result is independent of the vector \(B_\sigma\). For a time or space translation alone (no boost or rotation), \(b^\mu \sigma = \delta^\mu \sigma\), so:

\[
\left[ \delta^\mu \sigma \alpha_b \right] = -B_\sigma \vec{x}^\mu. \tag{2.8}
\]

Equation (2.8), which will be of keen interest throughout, looks “odd” at first impression, because there is no choice of commuting numbers for the components of \(B_\sigma\) and \(\vec{x}^\mu\) which renders all sixteen equations in (2.8) simultaneously valid. As we shall later see after some further development, (2.8) not only contains Planck’s law \(E = nh \nu\) for the quantization of energy, but it also motivates the development of non-commuting position and momentum and
time and perturbation operators, because all sixteen of the equations in (2.8), both on and off the diagonal, cannot be simultaneously valid unless $B_\sigma$ and $\bar{x}^\mu$ are non-commuting operators. We will return to this, beginning in section 5.

While the phase $\alpha_B(x^\mu)$ is not an absolute observable any more than is the velocity of an observer, (2.6) through (2.8) above nonetheless tell us something physically meaningful about this phase: Posit some generally-covariant vector $B_\sigma$ subsisting in spacetime. Start at a given spacetime event E designated with spacetime coordinate point $x_E^\mu$ in some arbitrarily chosen system of coordinates $x^\mu$, in a given reference frame F arbitrarily assigned to be “at rest” and arbitrarily assigned to a particular rotational orientation that we designate as “not rotated.” We use $F(b^\mu_\sigma = \delta^\mu_\sigma)$, or just $F(\delta^\mu_\sigma)$, to indicate that the Lorentz tensor $b^\mu_\sigma = \delta^\mu_\sigma$ in this reference frame, i.e., that this reference frame is “at rest” and “not rotated.” Because the phase $\alpha_B(x^\mu)$ is not an absolute observable, we may arbitrarily assign a phase $\alpha_B$ to coordinate $x_E^\mu$ in reference frame $F(\delta^\mu_\sigma)$, that is, we may arbitrarily assign $\alpha_B(x_E^\mu, F(\delta^\mu_\sigma))$, which we can further trim without showing F to $\alpha_B(x_E^\mu, \delta^\mu_\sigma)$. Let’s assign the phase $\alpha_B(x_E^\mu, \delta^\mu_\sigma) = 0$, just to keep things simple. In words: as defined in reference to vector $B_\sigma$, the phase $\alpha_B$ at the particular coordinate point $x_E^\mu$ in an arbitrary system of coordinates $x^\mu$ in the reference frame F which has the Lorentz tensor $b^\mu_\sigma = \delta^\mu_\sigma$ and so is arbitrarily assigned to be at rest and not rotated, is arbitrarily assigned to be equal to zero. We may think of this as “calibrating the phase.”

Now, transform into a new reference frame F’ which differs from F only by a boost, and/or a rotation and so is related to F by a homogeneous Lorentz transformation. That is, transform from $(x_E^\mu, \delta^\mu_\sigma) \rightarrow (x_E^\mu, b^\mu_\sigma)$. Equation (2.7) informs us that the phase in this new frame of reference will also be zero, that is, $\alpha_B(x_E^\mu, b^\mu_\sigma) = 0$, for any and all Lorentz tensors $b^\mu_\sigma$. Thus, if two different reference frames at the same coordinate event point $x_E^\mu$ differ from one another by no more than a homogeneous Lorentz transformation, i.e., a boost and/or rotation, then those two frames will have the same phase, i.e., will be “in phase” with one another. Of course, if every event coordinate $x_E^\mu$ exists in a spacetime geometry at which the
tangent space is specified by the Minkowski metric tensor \( \eta_{\mu\nu} = (1, -1, -1, -1) \), then even if \( x_E^\mu \) subsists in a region of spacetime curvature, we will still have \( \alpha_B(x_E^\mu, b_\sigma^\mu) = 0 \). In sum: (2.7) teaches, that once a singlet phase number has been assigned to a particular event \( E \) in spacetime, this phase is invariant, i.e., unchanged, under any boost or rotation applied at that event, and with respect to any spacetime curvature to which that event connects the tangent space.

On the other hand, if we translate either in space or in time through \( \tilde{x}^\mu \), even while maintaining \( b_\sigma^\mu = \delta_\sigma^\mu \), we learn from (2.8) that the phase will change, and that this is so even when we consider the subset of transformations governed by Poincare symmetry, in which there is no spacetime curvature. That is, if we transform from \( (x_E^\mu, \delta_\sigma^\mu) \rightarrow (x_E^\mu + \tilde{x}^\mu, \delta_\sigma^\mu) \), even without any rotation or boost or curvature, (2.8) teaches that \( \delta_\sigma^\mu \alpha_B(x_E^\mu + \tilde{x}^\mu, \delta_\sigma^\mu) = -B_\sigma \tilde{x}^\mu \). And, since a rotation and / or boost \( \delta_\sigma^\mu \rightarrow b_\sigma^\mu \) has no effect on this phase, we can also assert more generally that \( b_\sigma^\mu \alpha_B(x_E^\mu + \tilde{x}^\mu, b_\sigma^\mu) = -B_\sigma \tilde{x}^\mu \).

Finally, let us now remove the specialization to inhomogeneous Lorentz transformations, and consider completely general coordinate transformations. Here, we start with (2.2) and substitute \( x^\mu \rightarrow x'^\mu = x^\mu - \Lambda^\mu (x^\nu) \) from (1.4) to obtain:

\[
\begin{align*}
\alpha_B &= B_\mu \Lambda^\mu - B_\nu \partial_\nu \Lambda^\nu (x^\mu - \Lambda^\mu) = B_\mu \Lambda^\mu - B_\nu \frac{\partial \Lambda^\nu}{\partial x'^\mu}(x^\mu - \Lambda^\mu) \\
&= B_\mu (x^\mu - x'^\mu) - B_\nu \frac{\partial (x^\nu - x'^\nu)}{\partial x'^\mu}(x^\mu - (x^\mu - x'^\mu)) \\
&= B_\mu (x^\mu - x'^\mu) - B_\nu \frac{\partial x^\nu}{\partial x'^\mu} x'^\mu + B_\nu \frac{\partial x'^\nu}{\partial x'^\mu} x'^\mu \\
&= B_\mu x^\mu - B_\mu x'^\mu 
\end{align*}
\]

In the above, we have used \( B_\mu' = \left( \frac{\partial x^\nu}{\partial x'^\mu} \right) B_\nu \) from (1.3), as well as \( \frac{\partial x^\nu}{\partial x'^\mu} = \delta_\nu^\mu \). This just returns us to, and restates (2.1).

To summarize: In general, when we take the scalar product \( B_\mu x^\mu \) of a vector \( B_\mu \) with the spacetime coordinates \( x^\mu \), and then subject this to the general coordinate transformation \( B_\mu x^\mu \rightarrow B'_\mu x'^\mu \), there is a naturally-occurring local singlet phase \( \alpha_B(x^\mu) \) which characterizes
this transformation, and it is specified by the difference \( \alpha_B = B_\mu x^\mu - B'_\mu x'^\mu \). This phase at any given spacetime event \( E \) referred to the coordinate \( x_E^\mu \) in a system of coordinates \( x^\mu \), is unaffected by boosts and rotations at that event, and by any spacetime curvature at that event. However, for inhomogeneous Lorentz transformations, this singlet phase \( \alpha_B \) is specified in (2.6) by \( b_\sigma^\mu \alpha_B = -B_\sigma \bar{x}^\mu \) where \( \bar{x}^\mu \) measures the translation in time and/or space of said inhomogeneous Lorentz transformation.

As such, we now have a completely geometrodynamic interpretation of this phase/gauge parameter \( \alpha_B \). Now, we seek to connect this with the local gauge parameter \( \alpha(x^\mu) \) used to generate electrodynamic gauge theory.

3. Marrying Geometrodynamics to Electrodynamics, at the Gauge Level

Thus far we have taken \( B_\mu \) to be any general four vector subsisting in spacetime, and have not sought to associate \( B_\mu \) with any particular physical four vector. Further, we have shown how a local gauge parameter \( \alpha_B(x^\mu) \) associated with \( B_\mu \) naturally emerges when we consider the behavior of \( B_\mu x^\mu \) under general coordinate transformations \( B_\mu x^\mu \rightarrow B'_\mu x'^\mu \). But, we do not yet know how this might relate to the gauge parameter, which we designate as \( \alpha_G \), which appears in electrodynamic theory. Now, it is time to bridge both of these connections.

First, we associate \( B_\mu \) with a physical energy-momentum four-vector \( p_\mu \) with the contravariant components \( p^\mu = (E, p_1, p_2, p_3) \). This is a known four-vector transforming generally covariantly as \( p_\mu \rightarrow p'_\mu = (\partial x'/\partial x^\mu)p_\nu \), see (1.3). From (2.1), this means that under a general coordinate transformation:

\[
p'_\mu x'^\mu = p_\mu x^\mu - \alpha_p(x^\mu),
\]

with a local gauge parameter \( \alpha_p(x^\sigma) \) specified according to (2.2), (2.9) as:

\[
\alpha_p(x^\sigma) \equiv p_\mu \Lambda^\mu(x^\sigma) - p_\mu \partial'_\nu \Lambda^\nu(x^\sigma)(x^\nu - \Lambda^\nu(x^\sigma)) = p_\mu x^\mu - p'_\mu x'^\mu.
\]
Given this genesis as a function of $p_\mu$, we shall refer to $\alpha_p(x^\sigma)$ as the “momentum gauge parameter.” The unitary factor introduced in the previous section now becomes $U_p = e^{-ip_\mu x^\mu}$, which we also designate with a momentum subscript.

Second, we introduce a Dirac wavefunction $\psi$, which we define in the usual manner in terms of a four-complex-component Dirac spinor $u(p^\sigma)$ as: (e.g. [11], equation (5.18))

$$\psi = u(p^\sigma) e^{-ip_\mu x^\mu} = u(p^\sigma) U_p.$$  

(3.3)

As shown, this includes the momentum-based unitary factor $U_p$. As usual, the spinor $u(p^\sigma)$ is a function only of momentum and not of spacetime, i.e., $\partial_\mu u(p^\sigma) = 0$, and it subsists in a four-dimensional complex Hilbert space.

Now, let us subject this Dirac wavefunction $\psi$ to two distinct types of transformation, which we need to carefully distinguish:

First, we subject $\psi$ to a general coordinate transformation of the nature that we have been discussing in sections 1 and 2, such that, via (3.1):

$$\psi \rightarrow \psi' = u'(p'^\sigma) e^{-ip'_\mu x'^\mu} = e^{ia_\mu} u'(p'^\sigma) e^{-ip'_\mu x'^\mu}.$$  

(3.4)

Second, we subject $\psi$ to a local gauge transformation of the sort which is used in electrodynamic gauge theory. We designate the electrodynamic gauge parameter as $a_\alpha$ to distinguish it from the momentum gauge parameter $\alpha_p$ in (3.4), and we use “double primes” for this electrodynamic gauge transformation to distinguish it from the general coordinate transformation denoted with a “single prime” in (3.4) above. Thus:

$$\psi \rightarrow \psi'' = e^{ia_\alpha} \psi = e^{ia_\alpha} u(p^\sigma) e^{-ip_\mu x^\mu} \equiv u''(p^\sigma) e^{-ip_\mu x^\mu}.$$  

(3.5)

In the final term, this includes a definition for the gauge-transformed Dirac spinor:

$$u''(p^\sigma) \equiv e^{ia_\alpha} u(p^\sigma).$$  

(3.6)

Contrasting, we see that in (3.4) the transformation originates in the factor $e^{-ip_\mu x^\mu}$, while in (3.5), (3.6) it is merely imposed to operate on $u(p^\sigma)$ and is independent of the factor $e^{-ip_\mu x^\mu}$.

We now introduce two postulates which will marry together these two similar, but as yet disconnected transformations:
First, we postulate the momentum gauge parameter $\alpha_p$ and the electrodynamic gauge parameter $\alpha_G$ to be one and the same, and so to represent this postulate, define:

$$\alpha_G \equiv \alpha_p.$$  \hspace{1cm} (3.7)

Second, we postulate a general coordinate transformation acting on the wavefunction $\psi$ to be synonymous with an electrodynamic gauge transformation acting on this same wavefunction $\psi$. We represent this postulate by defining these two transformations (3.4), (3.5) to be synonymous:

$$\psi' \equiv \psi''.$$  \hspace{1cm} (3.8)

In effect, we have given an geometrodynamic basis to the electrodynamic gauge parameter $\alpha_G$, because $\alpha_G \equiv \alpha_p$ may be described in relation to general coordinate transformations via (3.2). Further, we have given a geometrodynamic basis to electrodynamic gauge theory in general, because now a gauge transformation acting on a Dirac wavefunction is synonymous with a general coordinate transformation acting on that same Dirac wavefunction.

Using these two definitions, we go back to combine (3.4) through (3.6) into one expression:

$$e^{i \alpha_p} \psi = \psi' = \psi'' = e^{i \alpha_p} \psi(p^\sigma)e^{-i p_{\alpha} x^\alpha} = e^{i \alpha_p} \psi(p^\sigma')e^{-i p_{\alpha} x^\alpha'} = u^*(p^\sigma)e^{-i p_{\alpha} x^\alpha},$$  \hspace{1cm} (3.9)

from which we obtain a relationship among the three Dirac spinors:

$$u(p^\sigma) = u'(p'^\sigma) = e^{-i \alpha_p} u''(p^\sigma).$$  \hspace{1cm} (3.10)

It also helps, from (3.9), to write out the adjoint (conjugate) relationships ($\bar{\psi} = \psi^\dagger \gamma^0$):

$$e^{-i \alpha_p} \bar{\psi} = \bar{\psi}' = \bar{\psi}'' = e^{-i \alpha_p} \bar{\psi}(p^\sigma)e^{i p_{\alpha} x^\alpha} = e^{-i \alpha_p} \bar{\psi}'(p'^\sigma)e^{i p_{\alpha} x^\alpha'} = u^*(p^\sigma)e^{i p_{\alpha} x^\alpha},$$  \hspace{1cm} (3.11)

As we shall now see, an immediate consequence of this, is Planck’s law for the quantization of energy.

4. Quantization of Charge and Energy

In the discussion in this section, we make use of Dirac’s condition for the quantization of electric and magnetic charge [12] to demonstrate that the foregoing development leads directly to the quantization of energy. In so doing, we shall closely follow the derivation of “Dirac’s

Based on the connections established by the postulates (3.7) and (3.8), let us now follow the usual steps employed in electrodynamic theory to develop the gauge field $A_\mu$. Using (3.4) and (3.10), we first subject the Dirac wavefunction $\psi$ to the general coordinate, now a.k.a. local gauge transformation:

$$\psi = e^{-ip_\sigma} u(p^\sigma) \rightarrow \psi' = e^{i\alpha_p} e^{-ip_\sigma} u'(p^\sigma) = e^{i\alpha_p} e^{-ip_\sigma} u(p^\sigma). \quad (4.1)$$

We then take the spacetime gradient $\partial_\mu \psi'^* \psi'$ of this transformed wavefunction:

$$\partial_\mu \psi' = -i(p_\mu - \partial_\mu \alpha_p) \psi' . \quad (4.2)$$

and then move the phase term over to the left hand side, thus:

$$(\partial_\mu - i\partial_\mu \alpha_p) \psi' = -ip_\mu \psi' . \quad (4.3)$$

Then, to compensate for this phase and maintain an invariant Lagrangian density, we introduce the covariant derivative $\partial_\mu \rightarrow D_\mu = \partial_\mu - ieA_{p\mu}$ containing both a gauge field $A_{p\mu}$ and the (negative) electric charge $e$. We include the $p$ subscript in $A_{p\mu}$, to serve as a reminder that this is a gauge field which is being used to compensate for the momentum gauge parameter $\alpha_p$.

Then, (4.3) becomes:

$$(D_\mu - i\partial_\mu \alpha_p) \psi' = (\partial_\mu - ieA_{p\mu} - i\partial_\mu \alpha_p) \psi' = (\partial_\mu - ieA'_{p\mu}) \psi' . \quad (4.4)$$

where the gauge field transforms as:

$$eA_{p\mu} \rightarrow eA'_{p\mu} = eA_{p\mu} + \partial_\mu \alpha_p . \quad (4.5)$$

Now, let us recast this in the language of differential forms, and meet up with Wu and Yang’s derivation of Dirac’s quantization condition, referenced above. The gauge field one-form is $A_p = A_{p\mu} dx^\mu$, and so written as a differential form equation, (4.5) becomes: (see [10], equation (IV.4.8))

$$A'_{p} = A_p + \frac{1}{ie} e^{-i\alpha_p} de^{i\alpha_p} . \quad (4.6)$$

It is also of interest to take $\partial_\mu \psi'$, with $\partial_\mu = \partial / \partial x^\mu$ containing the transformed coordinates $x'^\mu$, though at the moment we wish to explore this conventional path by which gauge theory is developed.
Based on the connections developed in the last section, \( A_p \) is now a physical gauge field, and it may be related to the field strength two-form \( F_p = \frac{1}{2} F_{p \mu \nu} dx^\mu \wedge dx^\nu \) by \( F_p = dA_p \) (we carry the momentum subscript through to \( F \) as well). According to Gauss’ law, if we posit a Dirac magnetic charge \( g \), then the field flux across any closed surface containing this charge is
\[
\oint \oint F_p = \oint \oint dA_p = g.
\]
Of course, because \( dF_p = ddA_p = 0 \) for an Abelian gauge theory, \(^{*}\) there is no such charge, but nonetheless, we plow forward in the usual way recognizing this enigma that has persisted ever Dirac’s first discovered the DQC, then return at the end of this derivation to discuss this and related questions.

We now label points along the surface of integration with polar coordinates \( \theta, \varphi \), so that \( F_p = dA_p = (g / 4\pi) d \cos \theta d \varphi \), where \( g \) is Dirac’s magnetic charge. Thus, \( A_p = (g / 4\pi) \cos \theta d \varphi \) because \( dd = 0 \). However, because \( d\varphi \) is not defined on the north and south poles, we define north and south coordinate patches \( A_S \equiv A'_p \) and \( A_N \equiv A_p \), and relate them to one another via the gauge transformation in (4.6), thus:
\[
A_S - A_N = 2 \frac{g}{4\pi} d\varphi = \frac{1}{ie} e^{-i\alpha_p} d e^{i\alpha_p}.
\] (4.7)

This means that
\[
e^{i\alpha_p} = e^{i 2\pi g}.
\] (4.8)

However, because the exact same point is described by \( \varphi = 0 \) and \( \varphi = 2\pi \):
\[
e^{i\alpha_p} = e^{i 2\pi g} = e^{i 2\pi g} e^{i \frac{2\pi}{4\pi} (0)} = 1.
\] (4.9)

This simultaneously yields two relationships:
\[
eg g = \pm 2\pi m,
\] (4.10)
\[
\alpha_p = \pm 2\pi m.
\] (4.11)

The former is Dirac’s quantization condition for electric charge, but what can we say about the latter \( \alpha_p = \pm 2\pi m \), which is really just a restatement of the definition (4.7), given the development in sections 2 and 3?

\(^{*}\) Though not so for non-Abelian theories where \( F = dG + ig G^2 \) with \( G \) being the non-Abelian gauge field.
To keep things as simple as possible, let us return to equation (2.8), which applies to inhomogeneous Lorentz transformations in the $F(\delta^{\mu}_{\sigma})$ frame, using $p_\sigma$ for $B_\sigma$, to write:

$$p_\sigma \tilde{x}^\mu = -\delta^{\mu}_{\sigma} \alpha_\rho .$$  \hfill (4.12)

Now, let us combine (4.11) and (4.12), transform to a rest frame with no rotation $b^{\mu}_{\sigma} \rightarrow \delta^{\mu}_{\sigma}$, and because there is a $\pm$ in (4.11), flip the sign as a choice of sign convention. We also note that the left side of (4.12) is in dimensions of angular momentum, in a system of units where $\hbar = 1$, so we also restore $\hbar$. With all of this, (4.11) and (4.12) combine to yield:

$$p_\sigma \tilde{x}^\mu = \pm \delta^{\mu}_{\sigma} 2\pi \hbar = \pm \delta^{\mu}_{\sigma} \hbar .$$  \hfill (4.13)

This contains sixteen (16) equations, four (4) along the diagonal, and twelve (12) off the diagonal, as well as an enigma that emerges when one tries to reconcile the on-diagonal to the off-diagonal equations. For the moment, we focus on the four diagonal equations. In the next section, we will examine the enigma, which for its resolution, requires that the components of $\tilde{x}^\mu$ and $p_\sigma$ be non-commuting operators.

For the energy momentum vector we employ the components $p^\sigma \equiv (E, p^1, p^2, p^3)$, making no a priori assumption about the nature of each of $E, p^1, p^2, p^3$, i.e., about whether these are ordinary numbers or non-commuting operators. For the translation quadruplet, again making no a priori commutativity assumptions, we employ the components $\tilde{x}^\mu \equiv (\tilde{t}, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3)$. Using these components, the 00 component of (4.13) introduces a time displacement $\tilde{t}$, and is written in component form as:

$$E \tilde{t} = \pm \hbar ,$$  \hfill (4.14)

while the space components of (4.13) read:

$$p_1 \tilde{x}^1 = \pm \hbar ; \quad p_2 \tilde{x}^2 = \pm \hbar ; \quad p_3 \tilde{x}^3 = \pm \hbar ,$$  \hfill (4.15)

Now, continuing to ignore the off-diagonal elements of (4.13), we assume for just the moment that each component of $p^\sigma \equiv (E, p^1, p^2, p^3)$ and $\tilde{x}^\mu \equiv (\tilde{t}, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3)$ is an ordinary commuting number. We are reminded that $\tilde{t}, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3$ are time and space translations arising from an inhomogeneous Lorentz transformation, and so choose to physically measure the time translation with reference to a frequency $\nu \equiv 1/\tilde{t}$ of an oscillating signal and to measure space
translations with reference to wavelengths $\lambda' = \tilde{x}'$ of an oscillating signal. Measured on this basis, (4.14) and (4.15) become:

\[ E = \pm nh \nu. \quad (4.16) \]
\[ p = \pm nh / \lambda. \quad (4.17) \]

This tells us that the energy is an integral multiple of the frequency times Planck’s constant, and that the energy may be positive or negative. As to the negative energy, we employ the customary Feynman and Stückelberg interpretation, define negative energy particles as positive energy antiparticles, and thereby work with only positive energies. This enables us to discard the minus sign in the above thereby maintaining only zero or positive energy states, so to write:

\[ E = nh \nu. \quad (4.18) \]
\[ p = nh / \lambda. \quad (4.19) \]

Following this, we also rewrite (4.13) to accord with permitting only positive energies, as:

\[ p_\sigma \tilde{x}'^\mu = \delta^\mu_\sigma 2m \hbar = \delta^\mu_\sigma nh. \quad (4.20) \]

Now, despite these recognizable results (4.18) and (4.19), it is fair to ask about the degree to which the DQC can really be applied here, and especially, the degree to which we can take (4.18) and (4.19) as a statement that physical energy is quantized. Gauss’ law for magnetism \( \iiint F_p = \iiint dA_p = g \) upon which this derivation is based, which also contains Faraday’s law, is really a simultaneous statement about two things, by virtue of Stokes’ theorem. First, it is a statement about the magnetic charge enclosed within a closed surface, which we know is zero for electromagnetism which is an Abelian interaction, and which is why the DQC remains something of an enigma to this day. But secondly, and inseparably linked by Stokes, in the \( \iiint dA_p \) term, Gauss’ law is also a statement about the gauge field potential \( A_p \) which flows across the surface. So when we define the coordinate patches \( A_x \equiv A'_p \) and \( A_y \equiv A_p \), which in effect constrains the gauge parameter to the quantized values \( \alpha_p = \pm 2m \), we are referring the surface over which this gauge field flows to a definite system of coordinates so that we can find out about the physical gauge field flows across this surface. After all, what is Gauss’ law other than a statement about charge enclosed within a surface, fields flowing across that surface, and how these are both related? What we find in (4.18), (4.19), is that the flux of energy of the gauge potential \( A_p \) across this Gaussian surface covered by the two coordinate patches, is quantized in
integer multiples of $h\nu$, and as a result, we come to associate this gauge field $A_p$, quantum mechanically, with photons.

It is also fair to observe that $\alpha_p = \pm 2\pi n$ in (4.11) is really just synonymous with the definition (4.7), not anything which is independently derived. But normally, in gauge theory, we simply postulate the gauge parameter $\alpha$ (no $p$ subscript here), and that parameter is disconnected from any geometrodynamic interpretation. So, the DQC is derived by Wu and Yang by defining coordinate patches which in effect quantize the gauge parameter to $\alpha = \pm 2\pi n$, and that is the end of the story. Once a gauge parameter generates the gauge field, no one cares any longer about the gauge parameter. Here, what makes the key difference is equation (4.12), which now connects the magnitude of the momentum gauge parameter $\alpha_p$, which has its origins in general coordinate invariance, to the product $p_\sigma \tilde{x}^\mu$, via $p_\sigma \tilde{x}^\mu = -\delta^\mu_\sigma \alpha_p$. Therefore, as soon as the momentum gauge parameter becomes quantized, this quantization will also affect the energy momentum, which is precisely what we see in (4.18) and (4.19).

To place this in context: We start with an energy-momentum vector $p_\mu$ subsisting in spacetime. Via $p_\mu x^\mu \rightarrow p_\mu' x'^\mu$, we generate an energy-momentum phase $a_p$ for which $\delta^\mu_\sigma \alpha_p = -p_\sigma \tilde{x}^\mu$. From the phase, we generate an energy-momentum gauge field $A_{p\mu}$. We can symbolize this by $p_\mu \Rightarrow \alpha_p \Rightarrow A_{p\mu}$. If by some means $\alpha_p$ should become quantized, $\alpha_p \propto n$ – as has occurred here from the DQC – then the $p_\mu$ of the $A_{p\mu}$, that is, the energy-momentum of the gauge field, $p_\mu (A_{p\mu})$ will also become quantized. We use $\alpha_p \propto n \Rightarrow p_\mu (A_{p\mu}) \propto n$ to symbolize this. Because $\delta^\mu_\sigma \alpha_B = -B_\sigma \tilde{x}^\mu$ is perfectly general for any $B_\sigma$, see (2.8), this means that we may deduce the general theorem that for any like progression $B_\mu \Rightarrow \alpha_B \Rightarrow A_{B\mu}$, if by some means $\alpha_B \propto n$, then $\alpha_B \propto n \Rightarrow B_\mu (A_{B\mu}) \propto n$.

Finally, this still does not address the enigma of the Dirac quantization condition vis-a-vis the existence of magnetic charges, but rather deepens and spreads the enigma. The DQC applied to charges tells us that if magnetic charges were to exist, then both electric and magnetic charges would have to be quantized. We have never found any magnetic charges, yet we know that electric charge is nonetheless quantized. Above, we now also learn that if magnetic charges
were to exist, then the gauge field $A_\mu$ would have to cross the Gaussian surface in integer multiples of $h\nu$. Again, we have never found any magnetic charges, yet we know that energy is nonetheless quantized in this fashion, and have long recognized that the gauge field $A_\mu$, which comprises electromagnetic light energy, does in fact flow with energies quantized by $E = nh\nu$.

Thus, in (4.18) and (4.19), the quantization of energy-momentum is now seen to be directly connected to the quantization of electric charge as first understood by Dirac, as a consequence of the principle of general coordinate invariance. We now have not only the connection between gravitational geometrodynamics and electrodynamics made in the postulates (3.7), (3.8), but a further connection to Planck’s 1901 quantization formula [4] used to explain the blackbody spectrum, which was later used by Einstein in 1905 [5] to explain the photoelectric effect based upon quantization of light energy. These quantization formulae precipitated the development of, and stand at the root of, quantum physics. And, fundamentally, both the gravitational / electrodynamic connection of section 3, and energy quantization developed in this section, arise from the purely geometric circumstance that in Riemannian geometry, spacetime coordinates $x^\mu$ do not transform as a four vector.

However, this energy quantization arises by considering only the diagonal components of (4.13), while ignoring the off-diagonal components. A difficulty still looms when we try to reconcile the off-diagonal components of (4.13) with the on-diagonal components discussed just above. Resolving this, requires us to now regard $p_\sigma$ and $x^\mu$ as non-commuting numbers, and leads us to directly to the canonical commutation relations of quantum theory based on Heisenberg’s matrix mechanics, [6] and to the Heisenberg uncertainty principle [7].

5. Canonical Commutation and Quantum Mechanics as an Outgrowth of General Coordinate Invariance

Let us return to (4.20), and for the moment, focus only on the space component equations:

$$p_j \bar{x}^i = \delta^i_j 2\pi \hbar. \quad (5.1)$$

The three on-diagonal equations are:

$$p_1 \bar{x}^1 = 2\pi \hbar; \quad p_2 \bar{x}^2 = 2\pi \hbar; \quad p_3 \bar{x}^3 = 2\pi \hbar. \quad (5.2)$$

The six off-diagonal equations are:
\[ p_1 \hat{x}^2 = 0; \quad p_2 \hat{x}^3 = 0; \quad p_3 \hat{x}^1 = 0; \]
\[ p_2 \hat{x}^1 = 0; \quad p_3 \hat{x}^2 = 0; \quad p_4 \hat{x}^3 = 0. \tag{5.3} \]

Now, let us postulate for disproof by contradiction that the components of \( p_j \) and \( \hat{x}^i \) are ordinary commuting numbers. And, let us consider the non-trivial situation where the quantum number \( n \neq 0 \). Then, by (5.2), each of \( p_1, p_2, p_3 \) and \( \hat{x}^1, \hat{x}^2, \hat{x}^3 \) must be non-zero. Simultaneously, by (5.3), at least some of the \( p_j \) and \( \hat{x}^i \) must be zero. This is an absolute contradiction, and it means that we have falsely postulated that the components of \( p_j \) and \( \hat{x}^i \) are ordinary commuting numbers. To reconcile (5.2) and (5.3), the components of \( p_j \) and \( \hat{x}^i \) must be non-commuting numbers. This is what was “odd” about equation (2.8), for which (5.1) is a downstream consequence.

To explore this further, let us turn to the canonical commutation relationship of Heisenberg matrix theory, which we write as:
\[ i\hbar \delta_{ij} = [\hat{x}^i, \hat{p}_j], \tag{5.4} \]
where \( \hat{x}^i \) and \( \hat{p}_j \) designate the canonical position and momentum operators, respectively. In general, we shall use the “hat” notation such as \( \hat{x}^i \) and \( \hat{p}_j \) to designate these Heisenberg operators. Further, let us assume that \( p_j \propto \hat{p}_j \) and \( \hat{x}^i \propto \hat{x}^i \) by some linear constants of proportionality (possible including \( n \)) to be determined, which will eventually be determined in section 10. Then, we simply combine (5.4) with (5.1) via the common term \( \hbar \delta_{ij} \), to write:
\[ p_j \hat{x}^i = i2\mathfrak{m}_n [\hat{p}_j, \hat{x}^i] = -i2\mathfrak{m}_n [\hat{x}^i, \hat{p}_j]. \tag{5.5} \]

If we now commute \( p_j \) and \( \hat{x}^i \), which via \( p_j \propto \hat{p}_j \) and \( \hat{x}^i \propto \hat{x}^i \) means that we are simultaneously commuting \( \hat{p}_j \) and \( \hat{x}^i \), we obtain:
\[ \hat{x}^i p_j = i2\mathfrak{m}_n [\hat{x}^i, \hat{p}_j] = -i2\mathfrak{m}_n [\hat{p}_j, \hat{x}^i]. \tag{5.6} \]
Then, subtracting (5.5) from (5.6) and combining in (5.4), we obtain:
\[ [\hat{x}^i, p_j] = i4\mathfrak{m}_n [\hat{x}^i, \hat{p}_j] = -4\mathfrak{m}_n \hbar \delta_{ij}. \tag{5.7} \]
That is, (5.7) is a solution to (5.1), simultaneously satisfying the on-diagonal equations (5.2) and the off-diagonal equations (5.3). In other words, to resolve that which is “odd” about equations (2.8) and (5.1), we are actually forced into the canonical commutation relationship (5.7).

After referral to Cartesian coordinates, it is well known [14] [15] that the Heisenberg momentum / position uncertainty principles:

\[ \Delta \hat{x} \Delta \hat{p}_x \geq \frac{\hbar}{2}, \]  \hspace{1cm} (5.8)
\[ \Delta \hat{y} \Delta \hat{p}_y \geq \frac{\hbar}{2}, \]  \hspace{1cm} (5.9)
\[ \Delta \hat{z} \Delta \hat{p}_z \geq \frac{\hbar}{2}, \]  \hspace{1cm} (5.10)

are an immediate consequence of (5.7). Here, \( \Delta \hat{x} \) is the standard deviation of the observable operator \( \hat{x} \), as is for \( \Delta \hat{p}_x \) for \( \hat{p}_x \), and similarly for and the other operators in the above. Thus, we find that the position / momentum uncertainty is actually one of the features which emerges from requiring general coordinate invariance. Via the connection \( \hat{x}^i \propto \hat{r}^i \) in (5.7), we should however be a little bit more careful about the verbal formation of (5.8) through (5.10), because \( \hat{x}^i \propto \hat{r}^i \) is a space “translation” which originated from the inhomogeneous Lorentz transformation of (1.5), and so (5.8) through (5.10) technically specify a “momentum / spatial translation uncertainty,” as opposed to a momentum / position uncertainty.

To put this all in context: We began in (1.4) by simply observing that the spacetime coordinates \( x^\mu \) are not a vector under general coordinate transformations. At the same time, we insist that any theory which describes nature must remain invariant with respect to general coordinate transformations. Yet, it is impossible to maintain general coordinate invariance if one assumes that the components of spacetime translations \( \hat{x}^\mu \) and momentum vectors \( \hat{p}_\mu \) are simple numbers. Rather – at least for space translations \( \hat{x}^i \) and the three space components \( \hat{p}_j \) of the momentum vector, as evidenced by (5.7) – \( \hat{x}^\mu \) and \( \hat{p}_\mu \) must be regarded as non-commuting operators, operating on an infinite-dimensional Hilbert space, in the manner of Heisenberg matrix mechanics.

Put succinctly: try as one might, and simply because \( x^\mu \) is not a four-vector under general coordinate transformations, one cannot describe nature in a generally coordinate-covariant fashion, without resort to quantum mechanics! \textit{Quantum mechanics is the direct and}
inseparable result of insisting upon a description of nature which is covariant under general coordinate transformations.

6. The Mass Commutation Paradox

Now let us return to equation (4.20), which we reproduce again here:

\[ p_\sigma \dot{x}^\mu = \delta^\mu_\sigma 2\pi \hbar . \]  \hfill (6.1)

Because (5.7) satisfies (5.1) and (5.1) contains the space components of the covariant equation (6.1) above, it is immediately clear that (6.1) which also contains time / energy components will be satisfied if \( p_\sigma, \dot{x}^\mu \) satisfy the four-covariant commutativity relationships:

\[ \left[ \dot{x}^\mu, p_\nu \right] = -4\pi \hbar \delta^\mu_\nu \equiv i4\pi \hbar \left[ \dot{x}^\mu, \hat{p}_\nu \right] . \]  \hfill (6.2)

That is, (6.1) forces us toward a commutation relationship which maintains covariant form over the entire four-dimensional geometry, and serves to define an energy operator \( \hat{\rho}_0 \), and a time operator (not parameter!) \( \hat{x}^0 \). However, this runs us into the following known paradox (which explains why time is still often treated as a parameter):

Write part of equation (6.2) in the two commuted forms:

\[ \left[ \dot{x}^\mu, \hat{p}_\nu \right] = i\hbar \delta^\mu_\nu , \]  \hfill (6.3)

\[ \left[ \hat{p}_\nu, \dot{x}^\mu \right] = -i\hbar \delta^\mu_\nu . \]

Then, multiply the upper equation from the right and the lower equation from the left by \( \hat{p}^\nu \):

\[ \left[ \dot{x}^\mu, \hat{p}_\nu \right] \hat{p}^\nu = i\hbar \delta^\mu_\nu \hat{p}^\nu = i\hbar \hat{p}^\mu \hat{p}^\nu - \hat{p}_\nu \dot{x}^\mu \hat{p}^\nu = \dot{x}^\mu m^2 - \hat{p}_\nu \dot{x}^\mu \hat{p}^\nu \]

\[ \hat{p}^\nu \left[ \hat{p}_\nu, \dot{x}^\mu \right] = -i\hbar \delta^\mu_\nu \hat{p}^\nu = -i\hbar \hat{p}^\mu \hat{p}_\nu \dot{x}^\mu - \hat{p}^\nu \dot{x}^\mu \hat{p}_\nu = m^2 \dot{x}^\mu - \hat{p}^\nu \dot{x}^\mu \hat{p}_\nu , \]  \hfill (6.4)

where the scalar product of the two vectors \( m^2 = \hat{p}_\nu \hat{p}^\nu \) is a scalar rest mass (multiplied by what is now an infinite-dimensional unit matrix). From the above, we calculate both the anticommutator:

\[ \frac{i}{2} \left\{ \dot{x}^\mu, m^2 \right\} = \hat{p}_\nu \dot{x}^\mu \hat{p}^\nu , \]  \hfill (6.5)

and the commutator:

\[ \frac{i}{2} \left[ \dot{x}^\mu, m^2 \right] = i\hbar \hat{p}^\mu \neq 0 . \]  \hfill (6.6)

Equation (6.6), in particular, highlights the known paradox: if, as in (6.2), one tries to extend the Heisenberg commutation relations of the three space dimensions, into all four
dimensions including time, one encounters a problem with the commutativity of the rest mass. Yet, (6.2) is a solution to (6.1), and (6.1) has brought us energy quantization and canonical commutation in the three space dimensions not to mention electrodynamic gauge theory solely on the basis of considerations of general covariance. So, we ought not abandon (6.1) very lightly.

At the same time, it would be problematic were we to suddenly find that even rest mass is a non-commuting operator, \[ [\hat{x}^\mu, m^2] \neq 0 \]. This would mean that the rest mass of, say, an electron, is not a reliable scalar number, but an eigenvalue of an eigenstate of an infinite-dimensional mass matrix operator, taken from among an infinite number of eigenvalue observables along a continuous spectrum of possible observable masses, complete with an expected value for the observed mass, and with variability in the observed mass from one experiment to the next. This flatly contradicts what is observed, namely, that amidst all the uncertainties and probabilities of quantum theory, the one thing we can count on is that every time we measure the electron’s mass, that mass will be 0.511 MeV and that this does not have a probability or a variance for one observation to the next.

Furthermore, the mass relation \[ m^2 = p_\sigma p^\sigma = g_{\mu\nu} p^\mu p^\nu \] bears a very-direct, known correlation to the expression \[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu \] for the invariant differential interval over a spacetime metric. To abandon the rest mass as an object which commutes with translation \( \hat{x}^\mu \) would be akin to abandoning the invariant metric interval \( ds \) itself, and would forfeit the very notion that nature can eventually be described as varying manifestations of geometry in a coordinate invariant system of measurement. One can introduce uncertainties and statistical standard deviations everywhere else, but to abandon the metric interval \( ds \) and the rest mass \( m \) as objects which commute with displacement and with everything else, would be to abandon all possible definite and invariant references against which to take measurements of natural phenomena, and would contradict what is well established from experimental observations, that particle rest masses are invariant and predictable from one observation to the next.

To resolve this paradox without foregoing either (6.1), or the commutativity of the scalar rest mass and metric differential interval and our ability to eventually refer to scalar invariants to take sensible measurements, we need another way forward. Let us now see if there is a resolution to this paradox, which simultaneously solves (6.1), and preserves mass commutativity.
In the next section, we develop some tools to approach this paradox, and in the section following, we show how this is resolved. Two sections hence, we employ this resolution to finally bring a proper understanding to the so-called “energy-time uncertainty.”

7. The Hamiltonian Four-Vector

To start the present discussion, we introduce the Dirac $\gamma^\mu$ matrices, and use these to form $j^\mu = \bar{\psi} \gamma^\mu \psi$ which we know is the probability density / flux four vector. Because $j^\mu$ is a known four-vector, under general coordinate transformations this will transform as:

$$
j^\mu = \bar{\psi} \gamma^\mu \psi \rightarrow j'^\mu = \bar{\psi'} \gamma'^\mu \psi' = \frac{\partial x'^\mu}{\partial x^\nu} j^\nu = \frac{\partial x'^\mu}{\partial x^\nu} \bar{\psi'} \gamma'^\nu \psi'. \quad (7.1)
$$

Now, we simply use the relationships $\psi = e^{-iA} \psi'$ in (3.9) and the adjoint in (3.11) to write:

$$
\bar{\psi'} \gamma'^\nu \psi' = \frac{\partial x'^\mu}{\partial x^\nu} \bar{\psi'} \gamma'^\nu \psi' = \frac{\partial x'^\mu}{\partial x^\nu} e^{iA} \gamma^\nu e^{-iA} \psi' = \frac{\partial x'^\mu}{\partial x^\nu} \gamma^\nu \psi'. \quad (7.2)
$$

From the first and last terms above, we then extract:

$$
\gamma'^\nu = \frac{\partial x'^\mu}{\partial x^\nu} \gamma^\nu. \quad (7.3)
$$

This is the transformation requirement for a generally-covariant four-vector, so we have established the result which we shall shortly employ that the Dirac $\gamma^\nu$, in this formal sense, is a generally-covariant four vector.

Now, let us turn to Dirac’s equation for a free (non-interacting) electron:

$$
m \psi = \gamma^\mu p_\mu \psi = \left( \gamma^0 p_0 + \gamma^k p_k \right) \psi, \quad (7.4)
$$

which we rewrite in the customary manner:

$$
p_0 \psi = \gamma^0 m \psi - \gamma^0 \gamma^k p_k \psi \equiv H_0 \psi, \quad (7.5)
$$

in the process, defining the Dirac Hamiltonian operator:

$$
H_0 \equiv \gamma^0 m - \gamma^0 \gamma^k p_k. \quad (7.6)
$$

We include a “0” subscript in $H_0$, as a reminder that this is associated with the time component of the four-vector $p_\mu$, for which the components $p^\mu = \left( E^0, p^1, p^2, p^3 \right)$. More formally, because $H_0 \psi = p_0 \psi$ in (7.5), the energy component $p_0 = E_0$ represents the eigenvalues of the
Hamiltonian $H_0$. Similarly, because $H_0 = H_0^\dagger$, the adjoint relationship $\overline{\psi} H_0 = \overline{\psi} p_0$, as well. Thus, in general, for anything $\bullet$ with is commuted with the Hamiltonian:

$$\overline{\psi}[\bullet, H_0] \psi = \overline{\psi}[\bullet H_0 - H_0 \bullet] \psi = \overline{\psi}[\bullet E_0 - E_0 \bullet] \psi = \overline{\psi}[\bullet, E_0] \psi.$$ \quad (7.7)

Thus, so long as we sandwich $H_0$ and $E_0$ between wavefunctions, we can trade $H_0$ and $E_0$ pretty much with impunity, and when we wish to discard the sandwich, we can extract that portion of the equation which contains $H_0$, and thereby work with an operator equation.

Let us now take $H_0^\dagger$, together with $p^1, p^2, p^3$, and form them into the four-component quadruplet $H^\mu = \left(H_0^\dagger, p^1, p^2, p^3\right)$. Query: is $H^\mu$ a four-vector? That is, does $H^\mu$ satisfy the general coordinate transformation law (1.2) in the form $H^\mu \rightarrow H'^\nu = \left(\partial x'^\mu / \partial x^\nu\right) H^\nu$?

There is more than one way to prove $H^\mu$ is a vector, but using (7.3) for the Dirac $\gamma^\nu$, we can prove this quite simply. If we contract $H_\mu$ with $\gamma^\nu$, we find that:

$$\not{H} = \gamma^\mu H_\mu = \gamma^0 H_0 + \gamma^k H_k = \gamma^0 (\gamma^0 m - \gamma^b \gamma^b p^b) + \gamma^k p_k = m - \gamma^k p_k + \gamma^k p_k = m.$$ \quad (7.8)

That is, $\not{H} = m$. We know that the rest mass $m$ is a scalar invariant under general coordinate transformations. We also know from (7.3) that $\gamma^\mu$ transforms as a four-vector under general coordinate transformations. Because $m$ is known to be a scalar and $\gamma^\mu$ is known to be a vector and $\gamma^\mu H_\mu = m$, we deduce that $H_\mu$ must be a vector as well. We may and shall now refer to $H^\mu$ as the “Hamiltonian four-vector.” Importantly, $\not{H} = m$ is the mass at the center of the mass commutation paradox we are seeking to resolve.

8. Resolving the Mass Commutation Paradox Using the Four-Vector Potential

Using this Hamiltonian four-vector, $H_\mu$, we are now better equipped to resolve the paradox highlighted in section 6. Recall, it is our goal to find a solution to (6.1) while simultaneously maintaining a scalar rest mass which commutes with position, $\left[\hat{x}^\mu, m\right] = 0$. This is in contrast to the problematic result (6.6) which resulted from the attempted, but failed solution (6.2).

Using the Hamiltonian four-vector $H^\mu$, let us return to the paradox presented in section 6, and let us return to the four-dimensional equation (6.1). Because the space components $H^k$
are identical to $p^k$ and the time component $p^0$ specifies the energy eigenvalues of $H^0$, we may substitute $H^\mu$ for $p^\mu$, and so write (4.13) as:

$$H_\sigma \hat{x}^\mu = 2\hbar \delta^\mu_\sigma .$$  \hfill (8.1)

This is now the equation we wish to solve.

Similarly to section 5 (see after (5.4)), let us suppose that there exist operators $\hat{H}_\sigma \propto H_\sigma$ and $\hat{x}^\mu \propto \hat{x}^\mu$, with the proportionalities consisting of strictly constant factors to be determined. Also, keeping in mind that $H = \gamma^\mu H_\mu = m$, see (7.8), we also define some scalar invariant $\hat{m} \equiv \gamma^\sigma \hat{H}_\sigma \propto m$, also proportional by a constant factor. Here – to be clear and not create confusion – we are placing the “hat” on $m$, although it is not an operator, in order to a) maintain the correspondence $\hat{m} \equiv \gamma^\sigma \hat{H}_\sigma$ which $\hat{m}$ has with $\hat{H}_\sigma$, and b) to distinguish $\hat{m}$ from $m$, to which it is related by an unknown constant factor.

It is clear that we could solve (8.1) if we were to extend the canonical commutation relationship to include the time component, i.e., if we were to employ:

$$[\hat{x}^\mu, \hat{H}_\nu] = i\hbar \delta^\mu_\nu .$$ \hfill (8.2)

Specifically, as we did in section 5, we could use $\delta^\mu_\nu$ to connect (8.1) and (8.2) to obtain:

$$H_\sigma \hat{x}^\mu = -i2\hbar \delta^\mu_\nu [\hat{x}^\mu, \hat{H}_\nu] .$$ \hfill (8.3)

contrast (5.5). Then, by the same route we obtained (5.7), we could obtain:

$$[\hat{x}^\mu, H_\nu] = i4\hbar \delta^\mu_\nu [\hat{x}^\mu, \hat{H}_\nu] = -4\hbar \delta^\mu_\nu .$$ \hfill (8.4)

On its own terms, with nothing else considered, (8.4) perfectly solves (8.2). Now, we turn back to the paradox of section 6, which becomes especially transparent in equations (8.2) and (8.4).

Now, let’s just multiply (8.2) from the left by $\gamma^\nu$ and use $\hat{m} = \gamma^\sigma \hat{H}_\sigma$ to obtain the stark contradiction:

$$0 = [\hat{x}^\mu, \hat{m}] = \gamma^\nu [\hat{x}^\mu, \hat{H}_\nu] = i\hbar \delta^\mu_\nu \gamma^\nu = i\hbar \gamma^\mu .$$ \hfill (8.5)

This, in a nutshell, tells us why equation (8.2), though an appealing and seemingly-obvious extension of the three dimensional canonical commutation relationships, simply does not work.

---

* To be precise about it, we put (6.1) into a wavefunction sandwich, see, e.g., (7.7), replace $p_\sigma$ with $H_\sigma$ and then remove the sandwich to obtain the operator equation (8.1).
Since $\hat{m} \propto m$, (8.2) would require that $[\hat{x}^\mu, \hat{m}] = i\hbar \gamma^\nu$, when in fact we must have $[\hat{x}^\mu, \hat{m}] = 0$.

So, what do we do?

First, let’s try the simple expedient of adding some unknown vector term $Y^\mu \gamma^\nu$ to right hand side of (8.5), and then solving for the unknown tensor $Y^\mu \gamma^\nu$. Thus:

$$0 = [\hat{x}^\mu, \hat{m}] = \gamma^\nu [\hat{x}^\mu, \hat{H}_\nu] = i\hbar \delta^\mu_\nu \gamma^\nu + Y^\mu \gamma^\nu = i\hbar \gamma^\mu + Y^\mu \gamma^\nu. \tag{8.6}$$

The solution which makes (8.6) work, is $Y^\mu \gamma^\nu = -i\hbar \delta^\mu_\nu$. But that would mean $[\hat{x}^\mu, \hat{H}_\nu] = 0$, which is a non-starter.

Let’s now take different approach, and this will work. Let us add some unknown vector $\hat{Y}_\nu$ to inside the commutator, thus trying out the solution:

$$[\hat{x}^\mu, \hat{H}_\nu + Y_\nu] = [\hat{x}^\mu, \hat{H}_\nu] + [\hat{x}^\mu, Y_\nu] = i\hbar \delta^\mu_\nu, \tag{8.7}$$

contrast (8.2). Now, multiplying by $\gamma^\nu$ and forcing the issue by setting $[\hat{x}^\mu, \hat{m}] = 0$ yields:

$$0 + \gamma^\nu [\hat{x}^\mu, \hat{Y}_\nu] = \gamma^\nu [\hat{x}^\mu, \hat{H}_\nu] + \gamma^\nu [\hat{x}^\mu, Y_\nu] = i\hbar \delta^\mu_\nu \gamma^\nu = i\hbar \gamma^\mu. \tag{8.8}$$

From this, stripping off the $\gamma^\nu$ from $\gamma^\nu [\hat{x}^\mu, \hat{Y}_\nu] = i\hbar \delta^\mu_\nu \gamma^\nu$, the solution is seen to be:

$$[\hat{x}^\mu, \hat{Y}_\nu] = i\hbar \delta^\mu_\nu. \tag{8.9}$$

That is, we can solve (8.1), and we can simultaneously maintain $[\hat{x}^\mu, \hat{m}] = 0$, if we add another vector $\hat{Y}_\nu$ to $\hat{H}_\nu$ inside the commutator. But what should be this new vector $\hat{Y}_\nu$?

This new vector $\hat{Y}_\nu$ would of course have to have some physical meaning and it would have to be a vector which can sensibly be added to $\hat{H}_\nu$. But $\hat{H}_\nu \psi = \hat{p}_\nu \psi$, because the space components $H_k \equiv p_k$ and the eigenvalues of $H_0$ are $p_0$, i.e., $H_0 \psi = p_0 \psi$. So, what is a natural vector to add to the momentum vector $p^\mu$? None other than the gauge potential $eA^\mu$! If one starts with free electrons and then wishes to know how they interact, one makes the rule-of-thumb substitution $p^\mu \rightarrow p^\mu + eA^\mu$. This is one of the most basic substitutions of QED gauge theory, and it emanates from the covariant derivative $\partial_\mu \rightarrow D_\mu = \partial_\mu - ieA_\mu$, see after (4.3), that is used to maintain local gauge symmetry in electrodynamics. So, the answer to our paradox is merely this: replace $p^\mu \rightarrow p^\mu + eA^\mu$ in the usual way! Stop working with only free electrons, and start working with interacting electrons.
Thus, the solution to (8.1), which simultaneously maintains $[\hat{x}^\mu, m] = 0$, is:

$$[\hat{x}^\mu, H_\nu + eA_\nu] = i4\pi [\hat{x}^\mu, \hat{H}_\nu + e\hat{A}_\nu] = -4\pi \hbar \delta^{\mu \nu}.$$  \hspace{1cm} (8.10)

where we have also defined $\hat{A}_\nu \propto A_\nu$. However, the use of $H_\nu$ in the above, really turns out to be “scaffolding” which allows us to get to (8.5) to display the section 6 paradox in the starkest way possible, so that would could figure out how to resolve the paradox. Keeping in mind that $H_0$ has eigenvalues $p_0$, if we take a step back, we realize that the equation for which we should be really obtaining a solution is:

$$(p_\sigma + eA_\sigma)\hat{x}^\mu = 2\pi \hbar \delta^{\mu \sigma}.$$  \hspace{1cm} (8.11)

In other words, just go all the way back to (6.1) before all the trouble with mass commutation began, and start dealing with interacting rather than free electrons by taking $p_\sigma \rightarrow p_\sigma + eA_\sigma$.

Modeled on (8.10), the solution to (8.11) with $\hat{p}_\nu \propto p_\nu$, and introducing $\hat{A}_\nu$ such that $A_\nu \propto \hat{A}_\nu$, is:

$$[\hat{x}^\mu, p_\nu + eA_\nu] = i4\pi [\hat{x}^\mu, \hat{p}_\nu + e\hat{A}_\nu] = -4\pi \hbar \delta^{\mu \nu}.$$  \hspace{1cm} (8.12)

This runs over indexes in all four spacetime dimensions and is fully compatible with $[\hat{x}^\mu, m] = 0$, thereby resolving the paradox in section 6.

There are many reasons why one must introduce a vector potential in electrodynamic gauge theory. Now, we have one more: to maintain a commuting rest mass $[\hat{x}^\mu, m] = 0$, and at the same time have a four-dimensional commutation relationship (8.12). Now, let’s us study (8.12), which for working purposes, is best written as:

$$[\hat{x}^\mu, \hat{p}_\nu] + e[\hat{x}^\mu, \hat{A}_\nu] = i\hbar \delta^{\mu \nu}.$$  \hspace{1cm} (8.13)

9. Energy-Time Uncertainty as Perturbation-Time Uncertainty

Let us first examine the space components of (8.13):

$$[\hat{x}^i, \hat{p}_j] + e[\hat{x}^i, \hat{A}_j] = i\hbar \delta^{ij}.$$  \hspace{1cm} (9.1)

This, of course, contains the usual commutation relationship $i\hbar \delta^{ij} = [\hat{x}^i, \hat{p}_j]$, see (5.4), from which we immediate deduce that:

$$[\hat{x}^i, \hat{A}_j] = 0.$$  \hspace{1cm} (9.2)
The three space dimension components $A_j$ of the vector potential operator are seen to commute with their corresponding spatial displacement operators $\hat{x}^j$. There is no “uncertainty” relationship between position and the three-vector potential components $A^k = (A^1, A^2, A^3)$. These can be measured simultaneously with $\hat{x}^j$. Where things become especially interesting, is when one considers the time components of (8.13).

The commutation relationship $i\hbar \delta^i_j = [\hat{x}^i, \hat{\rho}_j]$ over the three space dimensions of course leads in a well-known way to the uncertainty principle (5.8) through (5.10). [14] [15] It is known that there is also an energy / time uncertainty relationship given by $\Delta t \Delta E \geq \hbar / 2$, but the interpretation of this relationship has always been somewhat fraught with difficulty. Particularly, $\Delta t$ in the energy / time relationship pertains not to the amount of time over which one attempts to experimentally resolve an observation, but rather to the amount of time during which a physical perturbation does not occur. But how do we measure perturbations? Why, with the vector potential $A^\mu$, of course. The very same vector potential $A^\mu$ that is now so fortuitously in (9.1)!

Let us now examine the 00 component of (8.13). With $p^\mu = (E^0, p^1, p^2, p^3)$, this reads:

$$\left[\hat{x}^0, \hat{E}_0\right] + e\hat{\xi}^0, \hat{A}_0 = i\hbar.$$ (9.3)

The perturbative interpretation of energy / time uncertainty suggests that this is most properly separated into:

$$\left[\hat{x}^0, \hat{E}_0\right] = 0,$$ (9.4)

$$e\hat{\xi}^0, \hat{A}_0 = i\hbar.$$ (9.5)

In particular, in Dirac theory, perturbations are given by $V^0 \equiv -\gamma^0 \gamma^\mu A^\mu$, see, e.g. [11], equation (6.3), where we attach the 0 superscript to keep track of the vector component with which $V$ is associated, and the negative sign is based on the negative definition of the charge of the electron.

Defining $\hat{V}^0 \equiv V^0$, the commutation of $\hat{V}^0 = -\gamma^0 \gamma^\mu A^\mu = -\gamma^0 \gamma^0 A_0 - \gamma^0 \gamma^j A_j$ with $\hat{x}^\mu$, is:

$$\left[\hat{x}^\mu, \hat{V}^0\right] = -e\hat{x}^\mu, \hat{A}_0 - e\hat{\xi}^\mu, \gamma^0 \gamma^j \hat{A}_j = -e\hat{x}^\mu, \hat{A}_0.$$ (9.6)

We can set the space component equation $\left[\hat{x}^i, \gamma^0 \gamma^j \hat{A}_j\right] = \gamma^0 \gamma^j \left[\hat{x}^i, \hat{A}_j\right] = 0$ in the above, because we know from (9.2) that $\left[\hat{x}^i, \hat{A}_j\right] = 0$, leaving only $\left[\hat{x}^0, \hat{A}_j\right]$. Further, for we may set $\left[\hat{x}^0, \hat{A}_j\right] = 0$,
because no matter what might be the nature of $\hat{x}^0$, operators along different spacetime dimensions generally do commute.

Equation (9.6) contains four equations. For the three space equations, again because operators along different spacetime dimensions generally do commute, we may also set $[\hat{x}^k, \hat{V}] = -e[\hat{x}^k, \hat{A}_0] = 0$. Thus, the only possibly non-zero component of (9.6), is the time component:

$$-[\hat{x}^0, \hat{V}] = e[\hat{x}^0, \hat{A}_0].$$

(9.7)

Combining this with (9.5), and using the four-vector potential components $A^\mu = (\phi, A^1, A^2, A^3)$, we can now make the connection:

$$-\hat{x}^0, \hat{V} = e[\hat{x}^0, \hat{A}_0] = e[\hat{x}^0, \hat{\phi}] = ih.$$

(9.8)

But (8.13) tells us that the component equation:

$$[\hat{x}^0, \hat{p}_0] + e[\hat{x}^0, \hat{A}_0] = ih.$$

(9.9)

This means, combining (9.8) and (9.9), that:

$$[\hat{x}^0, \hat{p}_0] = [\hat{t}, \hat{E}] = 0.$$

(9.10)

This is why the so-called energy-time uncertainty can cause conceptual fits. One the one hand, $[\hat{t}, \hat{E}] = 0$. On the other hand, it is supposed that $\Delta t \Delta E \geq \hbar / 2$. Yet, Robertson [14] [15] tells us that these two relationships are incompatible.

Taken together, (9.8) and (9.10) give us a greater insight into the energy time uncertainty, and resolve these deep dilemmas. First, from (9.8), time displacement is not a parameter, it is an operator $\hat{x}^0 = \hat{t}$. (This question is another cause for fits.) And, this operator is non-commuting with the perturbation $-[\hat{t}, \hat{V}] = i\hbar$ but fully commuting, $[\hat{t}, \hat{E}] = 0$ with the energy. Therefore, applying Robertson to (9.8), the correct statement of the “energy-time” uncertainty principle is:

$$-\Delta t \Delta V = \Delta t \Delta \hat{\phi} \geq \hbar / 2$$

(9.11)

In words: the shorter the time under consideration, the greater will be standard deviation in the perturbations acting on that system, but there is no such uncertainty as between energy and time. We thus refer to (9.11) as the “perturbation / time uncertainty.”

Now, let’s consolidate all of this. It may be observed in the foregoing that the time components of $p^\mu$ and of $A^\mu$ have in some sense “traded places,” insofar was which operators
are commuting and which operators are non-commuting. We can formalize this “trading,” and
consciously summarize all of foregoing, by defining two “vectors” (we use “quotes” because we
still need to examine general coordinate transformation properties):
\[ -V^\mu \equiv (eA^0, p^1, p^2, p^3) = (-V^0, p^1, p^2, p^3), \]  
\[ E^\mu \equiv (p^0, eA^1, eA^2, eA^3) = (E^0, eA^1, eA^2, eA^3), \]  
where the choice of symbol for the overall “vector” is based on its time component. Based on
these definitions, the four-dimensional commutation relationships reduce in covariant form to:
\[ -[\hat{\xi}^\mu, \hat{V}_\nu] = i\hbar \delta^{\mu}_{\nu}, \]  
\[ [\hat{\xi}^\mu, \hat{E}_\nu] = 0. \]  
As a final check, is should be observed that these are fully consistent with (8.13), that is:
\[ \left[ \hat{\xi}^\mu, \hat{m}_\nu \right] + e\left[ \hat{\xi}^\mu, \hat{A}_\nu \right] = \left[ \hat{\xi}^\mu, \hat{E}_\nu \right] - \left[ \hat{\xi}^\mu, \hat{V}_\nu \right] = i\hbar \delta^{\mu}_{\nu}. \]  

In this light, now we see clearly why the energy / time uncertainty must pertain to
potential energy / perturbations, and can place into perspective the mass commutation paradox
we faced earlier. The time component of the energy-momentum vector, classically, is
\[ Ec = mc^2 + \frac{1}{2}mv^2. \]  
At rest, \( E = mc \). Were we to have \( [\hat{t}, \hat{E}] = i\hbar \) to support an energy / time
uncertainty relationship \( \Delta \hat{t} \Delta \hat{E} \geq \hbar / 2 \), then when transformed to an “at rest” frame, we would
have \( [\hat{t}, \hat{m}] = i\hbar \), which would violate the requirement that \( [\hat{\xi}^\mu, m] = 0 \). Normally, this is one
reason why time is not regarded as an operator. Now, we see that time can be and is an operator,
but one which is fully commuting with energy and rest mass. This is also why \( E^0 \) is shunted
over into the time component of (9.13), and traded with the perturbation \( V^0 \) which now fills the
time component of (9.12). It is OK for the perturbation to be non-commuting with the time
translations. It is not OK for energy and the rest mass to be non-commuting.

Finally, let’s deal with the question of “vectors.” It is clear from the definitions (9.12)
and (9.13) that:
\[ E^\mu - V^\mu = p^\mu + A^\mu. \]  
Because we know \( p^\mu \) and \( A^\mu \) are vectors, we deduce that \( E^\mu - V^\mu \) must also be a vector. But
what of \( E^\mu \) and \( V^\mu \) separately? If \( E^\mu \) and \( V^\mu \) are each to be four vectors under general
coordinate transformations (1.2), then, showing the above-developed “trading places” explicitly in the transformation, we would have:

\[-V'^\mu = -\frac{\partial x'^\mu}{\partial x^\nu} V^\nu = -\frac{\partial x'^\mu}{\partial x^0} V^0 - \frac{\partial x'^\mu}{\partial x^k} V^k = \frac{\partial x'^\mu}{\partial x^0} eA^0 + \frac{\partial x'^\mu}{\partial x^k} p^k,\]

\[E'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} E^\nu = \frac{\partial x'^\mu}{\partial x^0} E^0 + \frac{\partial x'^\mu}{\partial x^k} E^k = \frac{\partial x'^\mu}{\partial x^0} p^0 + \frac{\partial x'^\mu}{\partial x^k} eA^k.\] (9.18)

For the known vectors \(p^\mu\) and \(A^\mu\), the transformation laws are:

\[p'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} p^\nu = \frac{\partial x'^\mu}{\partial x^0} p^0 + \frac{\partial x'^\mu}{\partial x^k} p^k.\] (9.20)

\[A'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} A^\nu = \frac{\partial x'^\mu}{\partial x^0} A^0 + \frac{\partial x'^\mu}{\partial x^k} A^k.\] (9.21)

If we were to transform the known vectors \(p'^\mu = (\partial x'^\mu / \partial x^\nu) p^\nu\) and \(A'^\mu = (\partial x'^\mu / \partial x^\nu) A^\nu\) independently of one another, then examining (9.18) and (9.19) closely, \(E^\mu\) and \(V^\mu\) would not transform as a vector. For example, transforming \(p^\nu \rightarrow p'^\nu\) but not \(A^\nu\), (9.19) would now read as \(E'^\mu = (\partial x'^\mu / \partial x^0) p^0 + eA^k\), which is clearly not the transformation law for a vector.

But, (9.20) and (9.21) describe general coordinate transformations, and if one changes coordinates \(x^\mu \rightarrow x'^\mu\), then all physical vectors \(B^\mu\) being measured with respect to those coordinates also go from \(B^\mu \rightarrow B'^\mu\), and so from the very nature of coordinate transformations, \(p^\nu\) and \(A^\nu\) cannot be transformed independently. If one transforms \(p^\nu \rightarrow p'^\nu\), then one must simultaneously transform \(A^\nu \rightarrow A'^\nu\), otherwise it would be like saying because 1 foot and 1 meter both have the number 1, both lengths are the same length. A system of coordinates, once chosen, must be used consistently with respect to everything it is being used to describe, otherwise the results are nonsense. One cannot just pick and choose, describing some things in one coordinate system and other things in a different coordinate system. The coordinate system is arbitrary, but once chosen, it must be consistently applied to everything it is being used to describe. Therefore, (9.20) and (9.21) will always take place simultaneously, and as a consequence, \(E^\mu\) and \(V^\mu\) will indeed transform as four-vectors under general coordinate transformations, based on (9.18) and (9.19).
Now, let’s examine all of these operators in greater depth, and fix the constants of proportionality that we have been employing since section 6.

10. Explicit Commuting and Non-Commuting Operators

In section 6 and 8, we specified certain operators as being in proportion to one another, for example, \( \tilde{x}^\mu \propto \tilde{x}^\nu, \tilde{p}_\nu \propto p_\nu \). Now we have enough information to be able to establish these operators and these proportions explicitly. We start with the creation operator, defined as usual.

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & \ldots \\
\sqrt{1} & 0 & 0 & 0 & \ldots \\
0 & \sqrt{2} & 0 & 0 & \ldots \\
0 & 0 & \sqrt{3} & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

The annihilation operator \( a = (a^\dagger)^\dagger \). We define one such operator for each dimension of space, and one for the time dimension, thus \( a^\mu \equiv (a^0, a^1, a^2, a^3) = (a, a, a, a)^* \).

The position (really displacement) and momentum operators are generally defined using a harmonic oscillation model as \( \hat{x}^i \equiv \sqrt{\hbar/2m\omega} (\hat{a}^i + \hat{a}^i) \) and \( \hat{p}_j = -i\sqrt{\hbar m\omega/2} (\hat{a}_j - \hat{a}_j) \), where \( \omega = 2\pi\nu \) is the angular frequency of a classical oscillator with frequency \( \nu \) and mass \( m \). (see, e.g., [8], eq. (2.3.2)) The related (reduced) wavelength \( \lambda = \lambda / 2\pi \).

Now, let us turn briefly to review the Planck scale “geometrodynamic vacuum” first introduced by Wheeler. [16], [17] section 43. This vacuum contains fluctuations which on average are expected to be of the Planck mass \( Gm_p^2 = \hbar c \), i.e., \( m_p = \sqrt{\hbar c/G} \), thus Planck energy \( E_p = m_p c^2 = \sqrt{\hbar c^5/G} \), separated on average by the Planck length \( r_p = \hbar / cm_p = \sqrt{G\hbar / c^3} \), coming into and out of existence over time scales which average the Planck time \( t_p = r_p / c = \sqrt{G\hbar / c^5} \). What makes this a vacuum is that when two Planck energy fluctuations are separated by the Planck length, the negative gravitational potential energy

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* On deeper consideration, it may actually be preferred to define \( a^\mu \equiv (a^0, a^1, a^2, a^3) = (a^\gamma, a^\gamma', a^\gamma, a^\gamma') \) where each \( \gamma^\mu \) represents an infinite-dimensional unit matrix of that \( \gamma^\mu \). Keep in mind that for each \( \gamma^\mu \), we have \( \gamma^\mu \gamma^\mu = 1 \), and so this can be thought of with proper accounting for conjugacy to be multiplying each square root in (10.1) by \( \sqrt{1} \). Nonetheless, this is not required for the level of consideration to be given here.
\[ V_p = -Gm_p^2 / r_p = -\sqrt{\hbar c^5 / G} = -E_p. \] Thus, the negative and positive \( E_p \) energies counterbalance, yielding a net zero of energy, thus, a “vacuum.”

Therefore, following Wheeler, let us posit a Planck vacuum full of harmonic oscillators designed (expected) to net out to zero energy. To do this, we take these harmonic oscillators and give them a mass \( m_p = \sqrt{\hbar c / G} \), and an angular oscillation frequency \( \omega = 2\pi \nu = 2\pi / t_p \). Then, we use \( m \omega = 2\pi c^3 / G \) in the displacement and momentum operators.

Because we will need a time operator to carry out the commutation in (9.14), we define the one time and three space operators in this vacuum as:

\[
\hat{x}^\mu = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_\mu^\dagger + \hat{a}_\mu) = \sqrt{\frac{\hbar G}{4\pi c^3}}(\hat{a}_\mu^\dagger + \hat{a}_\mu) = \sqrt{\frac{\hbar}{4\pi}}(\hat{a}_\mu^\dagger + \hat{a}_\mu). \tag{10.2}
\]

In the final term, we use natural units \( \hbar = c = G = 1 \). For momentum operators, we need to keep in mind that the non-commuting operators are contained in the vector (9.12), and so we establish:

\[
-\hat{V}_\mu = -i\sqrt{\frac{m\omega}{2}}(\hat{a}_\mu^\dagger - \hat{a}_\mu) = -i\sqrt{\frac{\pi \hbar c^3}{G}}(\hat{a}_\mu^\dagger - \hat{a}_\mu) = -i\sqrt{\pi}(\hat{a}_\mu^\dagger - \hat{a}_\mu). \tag{10.3}
\]

It is clear that these definitions will satisfy the required non-commuting relationship

\[-[\hat{x}^\mu, \hat{V}_\nu] = i\hbar \delta^\mu_\nu \text{ of (9.14)}.\]

On the other hand, to satisfy \([\hat{x}^\mu, \hat{E}_\nu] = 0 \) from (9.15), we must ensure that the vector \( \hat{E}_\mu \) specified (up to a constant proportionality) in (9.13) does commute with \( \hat{x}^\mu \). The way to do that is to make certain that \( \hat{E}_\mu \) is a properly-dimensioned multiple of the infinite-dimensional unit matrix \( \hat{I}_\mu \). These dimensions are then simply read off the coefficient of (10.3), thus:

\[
\hat{E}_\mu = -i\sqrt{\frac{\pi \hbar c^3}{G}}I_\mu = -i\sqrt{\pi I_\mu}. \tag{10.4}
\]

Now, to finally fix the constants of proportionality, we turn to (8.12), which we write as:

\[
-4\pi \hbar \delta^\mu_\nu
= \left[ \hat{x}^\mu, \hat{p}_\nu \right] + \left[ \hat{x}^\mu, eA_\nu \right] + \left[ \hat{x}^\mu, \hat{V}_\nu \right]
= i4\pi\left[ \hat{x}^\mu, \hat{p}_\nu \right] + i4\pi\left[ \hat{x}^\mu, eA_\nu \right] = i4\pi\left[ \hat{x}^\mu, \hat{E}_\nu \right] - i4\pi\left[ \hat{x}^\mu, \hat{V}_\nu \right]. \tag{10.5}
\]
We must now determine how to assign the constant factor $i4\pi$ inside the commutators, to fix the proportionalities $\bar{x}^{\mu} \propto \hat{x}^{\mu}$, $V_{\nu} \propto \hat{V}_{\nu}$, $p_{\nu} \propto \hat{p}_{\nu}$, $E_{\nu} \propto \hat{E}_{\nu}$, and $A_{\nu} \propto \hat{A}_{\nu}$.

First, let us define a reduced wavelength $\lambda^{\mu} \equiv \bar{x}^{\mu} / 2\pi$, and use this to absorb $2\pi$ from (10.5), thus:

$$-2\hbar \delta^{\mu} \nu$$

$$= \left[ \lambda^{\mu}, p_{\nu} \right] + \left[ \lambda^{\mu}, eA_{\nu} \right] = \left[ \lambda^{\mu}, E_{\nu} \right] + \left[ \lambda^{\mu}, V_{\nu} \right] \quad (10.6)$$

This essentially assigns $2\pi$ out of $i4\pi$ into the spacetime displacement proportion $\bar{x}^{\mu} \propto \hat{x}^{\mu}$.

Next, referring to the Planck energy quantization relationships $E = nh \nu$ and $\mathbf{p} = nh / \lambda$ of (4.18), (4.19), it is clear that the quantum number $n$ should go into $V_{\nu} \propto \hat{V}_{\nu}$, $p_{\nu} \propto \hat{p}_{\nu}$, $E_{\nu} \propto \hat{E}_{\nu}$, and $A_{\nu} \propto \hat{A}_{\nu}$ proportionalities, and not into $\bar{x}^{\mu} \propto \hat{x}^{\mu}$, because it is energy, not spacetime translations, which is quantized.

Now, we turn to the imaginary factor $i$. Given their origin in real coordinates and real energy-momentum, all of $\bar{x}^{\mu}$, $V_{\nu}$, $p_{\nu}$, $E_{\nu}$, and $A_{\nu}$ should be real. We know from (10.2), (10.3) and (10.4) that the translation $\hat{x}^{i}$ operator is real while other “hatted” operators are imaginary. Thus, we will also move the factor $i$ into the proportionalities $V_{\nu} \propto \hat{V}_{\nu}$, $p_{\nu} \propto \hat{p}_{\nu}$, $E_{\nu} \propto \hat{E}_{\nu}$, and $A_{\nu} \propto \hat{A}_{\nu}$, to keep these real. Now, (10.6) boils down to:

$$-2\hbar \delta^{\mu} \nu$$

$$= \left[ \lambda^{\mu}, p_{\nu} \right] + \left[ \lambda^{\mu}, eA_{\nu} \right] = \left[ \lambda^{\mu}, E_{\nu} \right] + \left[ \lambda^{\mu}, V_{\nu} \right] \quad (10.7)$$

All that is left is a final factor of 2. For this, we observe from (10.2) and (10.3) that both $\hat{x}^{\mu}$ and $\hat{V}_{\mu}$ are defined in relation to $\hat{a}^{+ \mu} + \hat{a}^{\mu}$ via a “balanced” factor $\sqrt{\hbar / 2}$. So, we split the factor of $2 = \sqrt{2} \sqrt{2}$ between $\lambda^{i}$ and the other operators, to write (10.7) as:

$$-2\hbar \delta^{\mu} \nu$$

$$= \left[ \lambda^{\mu}, p_{\nu} \right] + \left[ \lambda^{\mu}, eA_{\nu} \right] = \left[ \lambda^{\mu}, E_{\nu} \right] + \left[ \lambda^{\mu}, V_{\nu} \right] \quad (10.8)$$

As a result of $\lambda^{\mu} \equiv \bar{x}^{\mu} / 2\pi$ together with (10.2), (10.3), (10.4), we can pick out from (10.8) the now-defined proportionalities:
\[ \dot{x}^\mu \equiv 2 \pi \hat{x}^\mu = 2 \pi \sqrt{2} \hat{x}^\mu \]
\[ = \sqrt{\frac{2 \pi \hbar c}{G}} (\hat{a}^\dagger \mu + \hat{a}^\mu) = \sqrt{2 \pi} (\hat{\alpha}^\dagger \mu + \hat{\alpha}^\mu). \]  
(10.9)

\[ V_v = i \sqrt{2} n \hat{V}_v = -n \sqrt{\frac{2 \pi \hbar c}{G}} (\hat{a}_v^\dagger - \hat{a}_v) = -n \sqrt{2 \pi} (\hat{\alpha}_v^\dagger - \hat{\alpha}_v). \]
(10.10)

\[ E_v = i \sqrt{2} n \hat{E}_v = n \sqrt{\frac{2 \pi \hbar c}{G}} I_v = n \sqrt{2 \pi} I_v. \]
(10.11)

The operators \( V_v \) and \( E_v \), of course contain the components of \( p_v \) and \( A_v \), via the definitions (9.12), (9.13).

Finally, returning to (9.14) and (9.15), in terms of the original \( \dot{x}^\mu \) which started out as a time and space translation all the way back in the inhomogeneous Lorentz transformation (1.5), and \( V_v \) and \( E_v \) which contain the energy-momentum four-vector \( p_\mu \) first introduced in (3.1) and the gauge field \( A_{\mu} \) first introduced in (4.4), the commutators which satisfy relationship (8.11), \( (p_\sigma + eA_\sigma) \dot{x}^\mu = 2 \hbar \delta^\mu_\sigma \), which originated in (2.6), are given by:

\[ [\dot{x}^\mu, V_v] = 4 \hbar \delta^\mu_\nu \]
(10.12)

\[ [\dot{x}^\mu, E_v] = 0. \]
(10.13)

### 11. Operator Boosts, and Relativistic Operator Behavior

All the operators specified in section 10 were developed to solve equation (8.12), \([\dot{x}^\mu, p_v + eA_v] = -4 \hbar \delta^\mu_\nu \). But the \( \delta^\mu_\nu \) in this equation originated from the Lorentz transformation matrix \( b^\mu_\nu \), see (2.6), and so these operators were developed in a reference frame arbitrarily defined as having “no rotation” and “no boost.” In the notation of section 2, these operators were developed for \( F(b^\mu_\sigma = \delta^\mu_\sigma) \). Because all of this emerged in a generally covariant context, one may now obtain operators in other inertial frames of reference by applying suitable Lorentz transformations and rotations via \( b^\mu_\sigma \).

For example, the Lorentz transformation matrix \( b^\mu_\nu = 0.3 \)
\[ v = 0.3 \]
\[ = \begin{pmatrix} \cosh \varphi & -\sinh \varphi \\ -\sinh \varphi & \cosh \varphi \end{pmatrix} \]
with
\[ \sinh \varphi = v \sqrt{1 - v^2}, \quad \cosh \varphi = 1 / \sqrt{1 - v^2} \]
effectuates boosts \( p^\mu = 0.3 \)
\[ v = 0.3 \]
and \( A^\mu = 0.3 \)
\[ \nu = 0.3 \]
\[ A^\nu = 0.3 \]
along the “3” axis. So, one may combine (9.12), (9.13), (10.1), (10.10)
and (10.11), to obtain the various components of \( p^\nu = (E, p^1, p^2, p^3) \) and 
\( eA^\nu = (\phi, eA^1, eA^2, eA^3) \) among those of \( E^\nu = (E^0, eA^1, eA^2, eA^3) = n\sqrt{2\pi}I^\nu \) and 
\( -V^\nu = (eA^0, p^1, p^2, p^3) = n\sqrt{2\pi}(\hat{a}^\nu - \hat{a}^\nu) \), in explicit matrix form. Thus, for example, the 
boosted time components of each of \( p^\nu \) and \( eA^\nu \) are:

\[
p^0 = E^0 = n\sqrt{2\pi} \begin{pmatrix}
\cosh \phi & -\sqrt{1} \sinh \phi & 0 & 0 & \cdots \\
\sqrt{1} \sinh \phi & \cosh \phi & -\sqrt{2} \sinh \phi & 0 & \cdots \\
0 & \sqrt{2} \sinh \phi & \cosh \phi & -\sqrt{3} \sinh \phi & \cdots \\
0 & 0 & \sqrt{3} \sinh \phi & \cosh \phi & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}, \quad (11.1)
\]

\[
eA^0 = \phi' = -V^0 = n\sqrt{2\pi} \begin{pmatrix}
-\sinh \phi & \sqrt{1} \cosh \phi & 0 & 0 & \cdots \\
-\sqrt{1} \cosh \phi & -\sinh \phi & \sqrt{2} \cosh \phi & 0 & \cdots \\
0 & -\sqrt{2} \cosh \phi & -\sinh \phi & \sqrt{3} \cosh \phi & \cdots \\
0 & 0 & -\sqrt{3} \cosh \phi & -\sinh \phi & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}. \quad (11.2)
\]

As discussed earlier using (9.18) to (9.21), these two transformations (11.1), (11.2) must occur 
simultaneously, i.e., once a reference frame has been boosted, \( F \to F' \), that boost occurs with 
respect to both \( p^\nu \) and \( eA^\nu \) together.

These matrices (11.1), (11.2) further highlight the “trading” of components among \( p^\nu \), 
\( eA^\nu \), \( E^\nu \), and \( V^\nu \), as developed in section 9, specifically equations (9.12), (9.13). At rest, where 
\( \sinh \phi = 0 \), \( \cosh \phi = 1 \), the kinetic energy operator \( p^0 = E^0 \) is purely commuting and the 
potential energy / perturbation operator \( eA^0 = \phi = -V^0 \) is purely non-commuting. Once a boost 
is added, \( p^0 = E^0 \) acquires off-diagonal components and is no longer purely commuting, while 
at the same time, \( eA^0 = \phi' = -V^0 \) starts to acquire on-diagonal components proportional to the 
unit matrix and is no longer purely non-commuting. As \( v_z \to 1 \), \( \sinh \phi \to \infty \), and \( \cosh \phi \to \infty \), 
and so the magnitude of the components of (11.1) and (11.2) grow to be of infinite magnitude.

This is yet another a consequence of the “trading” which occurred in section 9. We see 
from (11.1) that the energy component \( E^0 \) is totally commuting with the time operator, \( \hat{E}, \hat{E} \) = 0 
only in a rest frame where the energy becomes the rest mass. As soon as there is some boost, \( E^0 \)
picks up some components which are non-commuting. Thus, it is rest mass which is commuting with time but once some kinetic energy is added, one starts to see \([\hat{p}, \hat{E}] \neq 0\) because some of the non-commuting momentum components transform into the energy component. This is all the result of relativistic motion, and in this sense, is similar to how magnetic fields arise from electric charges merely as a consequence of relative motion.

12. Conclusion: Toward a General Theory of Quantum Gravitation

Finally, and in conclusion, let consider gravitation, without restriction. In approaching quantum gravitation, it actually helps, as a point of departure, to still consider the special case of (1.5) where there is no rotation and no boost, but merely a translation, so that \(b^{\mu \nu} = \delta^{\mu \nu}\). In this special case, the general coordinate transformation of (1.4) is related to the special case of the inhomogeneous Lorentz transformation (1.5) according to:

\[
\nu \mu \nu \mu \delta \mu = \Lambda = b_x.
\]

That is:

\[
\Lambda^{\mu}(x^\nu) = -\tilde{x}^{\mu} = \text{constant } \equiv \Lambda_0^{\mu}.
\]

The point here is that in the \(b^{\mu \nu} = \delta^{\mu \nu}\) frame of reference, the constant \(\tilde{x}^{\mu}\) operators which we have taken great pains to develop throughout, are synonymous with the (negative of) the gravitational gauge quadruplet \(\Lambda^{\mu}\), for the special case where \(\Lambda^{\mu}(x^\nu) = \Lambda_0^{\mu} = \text{constant}\).

Knowing this, we may rewrite the commutation relationships (10.12) and (10.13) in terms of \(\Lambda_0^{\mu} = -\tilde{x}^{\mu}\), as:

\[
\begin{align*}
[\Lambda_0^{\mu}, V_\nu] &= -4\pi \hbar \delta^{\mu \nu}, \\
[\Lambda_0^{\mu}, E_\nu] &= 0.
\end{align*}
\]

Then we turn to the metric tensor, which by (1.8) transforms for \(\Lambda^{\mu}(x^\nu) = \Lambda_0^{\mu} = \text{constant}\) as:

\[
g^{\mu \nu} \rightarrow g^{\mu \nu} = g^{\mu \nu} - \partial^{[\nu} \Lambda_0^{\mu]} + \frac{i}{2} \partial_\sigma \Lambda_0^{[\mu} \partial^{\sigma} \Lambda_0^{\nu]} = g^{\mu \nu} = \eta^{\mu \nu},
\]

where, from (10.9) via \(\tilde{x}^{\mu} = -\Lambda_0^{\mu}\):

\[
\Lambda_0^{\mu} = -\sqrt{\frac{2\pi G}{c^3}} (\hat{a}^{\dagger \mu} + \hat{a}^{\mu}) = -\sqrt{2\pi} (\hat{a}^{\dagger \mu} + \hat{a}^{\mu}).
\]
Now, gravitation is a local gauge theory where general transformations in the metric tensor are given by (1.8). The special case \( \Lambda^\mu (x^\nu) = \Lambda_0^\mu = \text{constant} \) which is shown in (12.5), is the case in which the metric tensor is constant, \( g^{\mu\nu} \rightarrow g'^{\mu\nu} = g^{\mu\nu} = \text{constant} \). However, since the tangent space to a curved spacetime at each spacetime event is the Minkowski space defined by \( \eta^{\mu\nu} \), and because \( g^{\mu\nu} \rightarrow g'^{\mu\nu} = g^{\mu\nu} \), we know that for \( \Lambda^\mu (x^\nu) = \Lambda_0^\mu = \text{constant} \), we will have \( g^{\mu\nu} = \eta^{\mu\nu} \), everywhere. That is all stated in (12.5).

In effect, therefore, (12.3), (12.4) and (12.6) specify quantum mechanical operators and commutation relationships for a global gauge theory of quantum gravitation, also thought of as a “special theory” of quantum gravitation, because gauge quadruplet \( \Lambda^\mu (x^\nu) = \Lambda_0^\mu = \text{constant} \). So, how do we turn this into a local gauge theory, i.e., a “general theory” of quantum gravitation? That is simple: we just remove the restriction \( \Lambda^\mu (x^\nu) = \Lambda_0^\mu = \text{constant} \). Because (12.3), (12.4) and (12.6) apply in the Minkowski space which is tangent to spacetime at each event, we simply generalize these to allow for a locally-varying \( \Lambda^\mu (x^\nu) \).

Thus, in curved spacetime, (12.3) and (12.4) are generalized to:

\[
\left[ \Lambda^\mu (x^\sigma), V_\nu (x^\sigma) \right] = -4\pi i \hbar \delta^\mu_\nu , 
\]

(12.7)

\[
\left[ \Lambda^\mu (x^\sigma), E_\nu (x^\sigma) \right] = 0 ,
\]

(12.8)

while the creation and annihilation operators and all the operators which are built from these are also taken to be local functions of spacetime, so that (12.6) becomes:

\[
\Lambda^\mu (x^\sigma) = \sqrt{\frac{2\pi G}{c^3}} \left( \hat{a}^{\dagger} \mu (x^\sigma) + \hat{a}^\mu (x^\sigma) \right) = -\sqrt{2\pi} \left( \hat{a}^{\dagger} \mu (x^\sigma) + \hat{a}^\mu (x^\sigma) \right) .
\]

(12.9)

At the same time, the transformation (12.5) for the metric tensor reverts to the general transformation of (1.8).

Finally, in a general theory of quantum gravitation, the local gauge singlet \( \alpha_\rho \) of electromagnetic theory will be specified by:

\[
\alpha_\rho (x^\sigma) = p_\mu \Lambda^\mu (x^\sigma) - p_\mu \frac{\partial \Lambda^\mu}{\partial x^\nu} (x^\nu - \Lambda^\nu (x^\sigma)) + p_\mu x^\mu - p_\mu x'^\mu ,
\]

(12.10)
Because $\Lambda^\mu(x^\sigma)$ generates a metric tensor $g_{\mu\nu}(x^\sigma)$ which is locally-varying in space time and which via $-\kappa T^\mu_\nu = R^\mu_\nu - \frac{1}{2} \delta^\mu_\nu R$ yields non-linear effects, the non-linearities of gravitational theory will then be injected into electrodynamics via (12.10).

In sum, this is how to marry general relativity with quantum mechanics and electrodynamics. In the main, what is left thereafter is to effectuate non-Abelian, Yang-Mills generalizations of the foregoing, to also encompass the full electroweak, and strong interactions.
References


[9] This can be found in virtually any basic reference about gravitational theory. One such good reference is Ohanian, H.C., *Gravitation and Spacetime*, Norton (1976)


