SECTION 2

2.1 - A Classical Spacetime Introduction to the Dirac Equation, and the Structure of Five-Dimensional Spacetime with a Chiral Dimension

We begin discussion by considering the classical Gaussian spacetime metric equation, in <u>second order</u>: $\left(dx^u = (cdt, dx, dy, dz)\right)$

$$ds^2 = g_{uv} \cdot dx^u \cdot dx^v = dx^u \cdot dx_u \qquad (2.1)$$

where in local Lorentz (geodesic) coordinates, the metric tensor

$$g_{uv} \longrightarrow \eta_{uv} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad (2.2)$$

hence:

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 \; . \qquad (2.3)$$

The proper space interval ds is related to the proper time interval $d\tau$ by $ds = cd\tau$, while the four-vector velocity is given by:

$$u^u \equiv dx^u \, / \, d\tau \; . \qquad (2.4)$$

The energy, momentum, frequency and wave vectors E^u, P^u, $\not{f}^u$ and $\not{K}^u$ for any given mass m, are in turn related to one another, and to the velocity vector (2.4) according to:

$$E^u = cP^u = \hbar \not{f}^u = \hbar c \not{K}^u = mcu^u \; . \qquad (2.5)$$

In natural units, with $\hbar = c = 1$, this is simply:

$$E^u = P^u = \not{f}^u = \not{K}^u = mu^u \; . \qquad (2.5)(a)$$

As a consequence of $ds = cd\tau$ and (2.4) and (2.5) above, the metric equation (2.1) is easily rewritten in the alternative, by:

$$E^uE_u = c^2P^uP_u = \hbar^2 \not{f}^u \not{f}_u = \hbar^2c^2 \not{K}^u \not{K}_u = m^2c^2u^uu_u = m^2c^4 \qquad (2.6)$$

or again, with $\hbar = c = 1$,

$$E^uE_u = P^uP_u = \not{f}^u \not{f}_u = \not{K}^u \not{K}_u = m^2u^uu_u = m^2 \; . \qquad (2.6)(a)$$

It turns out, while the above is perfectly valid for discussing integer-spin Bose particles, that consideration of half-integer-spin Fermi particles requires the use of a <u>first order</u> form for the metric equation, which is more commonly known as the Dirac matrix equation. Specifically, if the Dirac matrix vector γ^u is <u>defined</u> in terms of the Gaussian metric tensor g_{uv} according to the symmetric combination:

$$\tfrac{1}{2}\left(\gamma^u\gamma^v + \gamma^v\gamma^u\right) \equiv g^{uv} , \tag{2.7}$$

then in geodesic coordinates, eq. (2.2), with the direction of motion rotated so as to point along the z-axis, a particular representation of the γ^u matrices (the Fermi-Dirac representation) is given by:

$$\gamma^u \equiv \left(\begin{pmatrix} \sigma^0 & 0 \\ 0 & -\sigma^0 \end{pmatrix}, \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix} \right), \tag{2.8}$$

with the Pauli spin matrices defined by:

$$\sigma^u \equiv \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right). \tag{2.9}$$

With these, the spacetime metric (2.1) may be easily rewritten in the first order form:

$$ds = \gamma^u \cdot dx_u . \tag{2.10}$$

Again, utilizing $ds = c\,d\tau$ and eqs. (2.4) and (2.5), the first order metric above is easily rewritten in a form corresponding to (2.6), ie.,

$$\gamma^u E_u = c\gamma^u P_u = \hbar\gamma^u f_u = \hbar c\gamma^u k_u = mc\gamma^u u_u = mc^2 , \tag{2.11}$$

or, with $\hbar = c = 1$,

$$\gamma^u E_u = \gamma^u P_u = \gamma^u f_u = \gamma^u k_u = m\gamma^u u_u = m. \tag{2.11(a)}$$

There also exists a fifth linearly independent Dirac matrix, the "chirality" matrix γ^5, which is formed out of the product of the remaining four, ie.,

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & \sigma^5 \\ \sigma^5 & 0 \end{pmatrix} \neq \gamma^0 \neq \gamma^0\gamma^0 = \gamma^5\gamma^5 \tag{2.12(a)}$$

$$\sigma^5 \equiv -i\sigma^0\sigma^1\sigma^2\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \sigma^0 = \sigma^0\sigma^0 = \sigma^5\sigma^5. \tag{2.12(b)}$$

The fact that $\gamma^5 \neq \gamma^0$, but that $\sigma^5 = \sigma^0$, is the source of important structural differences between the Dirac and Pauli matrices; and is related to the fact that the symmetric combination $\frac{1}{2}(\sigma^u\sigma^v + \sigma^v\sigma^u) = \delta^{uv}$ (δ^{uv} is the Kronecker delta) fails by itself to produce the necessary metric signature diag (g^{uv}) = $(1,-1,-1,-1)$, while the combination $\frac{1}{2}(\gamma^u\gamma^v + \gamma^v\gamma^u) = g^{uv}$ does indeed produce the correct signature. (In fact, the γ^u are defined, in part, so as to achieve precisely this signature.) Because none of the γ^u including γ^5 is itself identified with the unit (identity) matrix, it is necessary to create this matrix out of a second order product, for example, $\gamma^5\gamma^5$. On the other hand, $\sigma^5 = \sigma^0$ is also itself the unit matrix, hence no additional steps are necessary to create this matrix. This is to say that:

$$\gamma^0 \neq \gamma^5\gamma^5 = \begin{pmatrix} \sigma^5\sigma^5 & 0 \\ 0 & \sigma^5\sigma^5 \end{pmatrix} = \begin{pmatrix} \sigma^0 & 0 \\ 0 & \sigma^0 \end{pmatrix} = \gamma^0\gamma^0 = -\gamma^1\gamma^1 = -\gamma^2\gamma^2 = -\gamma^3\gamma^3 \quad (2.13)(a)$$

but that

$$\sigma^0 = \sigma^5\sigma^5 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \sigma^0\sigma^0 = \sigma^1\sigma^1 = \sigma^2\sigma^2 = \sigma^3\sigma^3 \; . \quad (2.13)(b)$$

These structural differences are best illustrated by examining the symmetric and antisymmetric structure constants, "s" and "a" respectively, for each of the Dirac and Pauli matrices. This current discussion of spacetime structure will also form the basis for later discussion of flavor and color structure relationships; which are in turn vital elements in the prediction of Grand Unification mass scales via the renormalization group equations.

For example, examining the Dirac matrices γ^u, one may begin by rewriting the γ^5 definition (2.12)(a) in the form:

$$\frac{1}{2}\left[\gamma^0\gamma^5 - \gamma^5\gamma^0\right] = \gamma^0\gamma^5 = i\gamma^1\gamma^2\gamma^3 \equiv a^{05}{}_{123}\, \gamma^1\gamma^2\gamma^3 \; . \quad (2.14)$$

Considering the well known properties of the γ^u matrices, the above is readily generalized to the antisymmetric structure definition:

$$-i\sigma^{UV} \equiv \frac{1}{2}\left[\gamma^U\gamma^V - \gamma^V\gamma^U\right] \equiv (i/3!)\, a^{UV}{}_{ABC}\, \gamma^A\gamma^B\gamma^C \; , \quad (2.15)$$

where:

$$a^{01235} = a^{12350} = a^{23501} = a^{35012} = a^{50123} = 1 \qquad (2.16)(a)$$

$$a^{UVABC} = -a^{VUABC} \, , \qquad (2.16)(b)$$

and where by convention, we shall establish the upper-case spacetime index $U = 0,1,2,3,5$; while the lower case indexes are the usual $u = 0,1,2,3$. Utilizing upper case indices, one may designate the symmetric structure constants by:

$$g^{UV} \equiv \tfrac{1}{2}(\gamma^U\gamma^V + \gamma^V\gamma^U) \equiv s^{UV}{}_{AB}\,\gamma^A\gamma^B , \qquad (2.17)$$

where: (note the origin of this signature in (2.13)(a))

$$g^{UV} \equiv s^{UV}{}_{55} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad (2.18)(a)$$

$$s^{UV}{}_{uv} = s^{UV}{}_{u5} = s^{UV}{}_{5v} = 0 \, , \qquad (2.18)(b)$$

and that the usual metric tensor

$$g^{uv} \equiv s^{uv}{}_{55} \, . \qquad (2.19)$$

Note that $s^{UV}{}_{AB}$ is itself used to raise and lower indices, and that this tensor is <u>not</u> symmetric as between the first and second pairs of indices, ie., $s^{0055} = 1 \neq s^{5500} = 0$. It is intriguing that chirality, from a geometric standpoint, appears naturally to be associated with a fifth dimension of spacetime, as in (2.18)(a).

For the Pauli spin matrices σ^u, a similar exercise leads to a somewhat modified result. For example, using the σ^5 definition of (2.12)(b), one may write an analog to (2.14), namely:

$$0 = \tfrac{1}{2}\left[\sigma^0\sigma^5 - \sigma^5\sigma^0\right] \neq \sigma^0\sigma^5 = -i\sigma^1\sigma^2\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \qquad (2.20)$$

Because $\sigma^0\sigma^5 = \sigma^0\sigma^0 = \sigma^0$ whereas $\gamma^0\gamma^5 \neq \gamma^0\gamma^0 \neq \gamma^0$, it is possible to simplify the above somewhat. Specifically, using $1 = -i\sigma^1\sigma^2\sigma^3$, it is possible to write the following simplified analog to (2.14):

$$\tfrac{1}{2}\left[\sigma^1\sigma^2 - \sigma^2\sigma^1\right] = \sigma^1\sigma^2 = i\sigma^3 = ia^{12}{}_3\sigma^3 \,, \tag{2.21}$$

and this is readily generalized to:

$$\tfrac{1}{2}\left[\sigma^u\sigma^v - \sigma^v\sigma^u\right] = ia^{uv}{}_\gamma\sigma^\gamma, \tag{2.22}$$

where:

$$a^{123} = a^{231} = a^{312} = 1 \qquad (2.23)(a)$$

$$a^{0uv} = a^{uv0} = a^{v0u} = 0 \qquad (2.23)(b)$$

$$a^{uv} = -a^{vu} \quad . \qquad (2.23)(c)$$

Thus, the antisymmetric structure constants for the Pauli matrices require only three, as opposed to five indices. Further, it is not necessary here to introduce a chiral index, ie., u = 0,1,2,3 but does not range over to the index 5. For the symmetric constants, it is not necessary to form the unit matrix from a combination such as $\sigma^5\sigma^5$, as this matrix is already supplied as σ^0. Thus, the slightly modified analog of (2.17) for σ^u is given by:

$$\tfrac{1}{2}(\sigma^u\sigma^v + \sigma^v\sigma^u) = s^{uv}{}_\gamma\sigma^\gamma, \tag{2.24}$$

with: (here, $s^{uv}{}_\gamma$ is used to raise and lower indices)

$$\delta^{uv} \equiv s^{uv}{}_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (2.25)(a)$$

$$s^{uv0} = s^{v0u} = s^{0uv} = 1 \qquad (2.25)(b)$$

$$s^{jkl} = 0 \quad ; \; j,k,l = 1,2,3 \text{ only} \qquad (2.25)(c)$$

$$s^{uv\gamma} = s^{vu\gamma} \, . \qquad (2.25)(d)$$

In this case, the symmetric constants need only three, rather than four indices, due to the fact that $\sigma^5\sigma^5 = \sigma^0$.

Combining (2.15) with (2.17) for γ^u, and (2.22) with (2.24) for σ^u, one may write the product of any two Dirac matrices in the following manner:

$$\gamma^U\gamma^V = s^{UV}{}_{AB}\,\gamma^A\gamma^B + (i/3!)a^{UV}{}_{ABC}\,\gamma^A\gamma^B\gamma^C \,, \tag{2.26}$$

while the product of any two Pauli matrices is written as:

$$\sigma^u\sigma^v = (\, s^{uv}{}_\gamma + i a^{uv}{}_\gamma \,)\,\sigma^\gamma. \tag{2.27}$$

Obviously, the form for $\sigma^u\sigma^v$ is somewhat simpler than that for $\gamma^U\gamma^V$. While the product $\gamma^U\gamma^V$ must be constructed out of square and cubic combinations of the γ^U matrices, with U = 0,1,2,3,5: the product $\sigma^u\sigma^v$ may be constructed strictly out of <u>linear</u> combinations of the σ^u matrices, with u = 0,1,2,3. It is to be noted that the <u>flavor</u> and <u>color</u> structure relationships, which will be a topic of attention in the later discussion, are quite similar to the <u>spacetime</u> structure relationships for the σ^u, insofar as the construction of $\sigma^u\sigma^v$ out of strictly linear σ^u combinations is concerned.

Finally, it is instructive to form certain second order (Casimir) operators out of these structure constants. Particularly, for the Dirac matrices:

$$a^{UABCD}a_{VABCD} = \begin{pmatrix} -4! & 0 & 0 & 0 & 0 \\ 0 & -4! & 0 & 0 & 0 \\ 0 & 0 & -4! & 0 & 0 \\ 0 & 0 & 0 & -4! & 0 \\ 0 & 0 & 0 & 0 & -4! \end{pmatrix} \tag{2.28)(a}$$

$$s^{UABC}\;s_{VABC} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.28)(b}$$

where:

$$s^{UA55}s_{VA55} = g^{UA}g_{VA} = \delta^U{}_V \tag{2.29}$$

is the usual condition for normalization of the metric tensor when examined in four dimensions. For the Pauli matrices, these matrix products are somewhat different:

$$a^{u\sigma_T} a_{v\sigma_T} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \qquad (2.30)(a)$$

$$s^{u\sigma_T} s_{v\sigma_T} = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}. \qquad (2.30)(b)$$

The above are related to the fact that the three SU(2) σ^k (space) matrices when taken together with the unit U(1) σ^0 (time) matrix form an SU(2)xU(1) symmetry group. The 2 which appears in the diagonal elements in $a^{j\sigma_T} a_{k\sigma_T}$ is directly related to the 2 appearing in SU(2); while for any subgroup SU(N)xU(1), the product $a^{j\sigma_T} a_{k\sigma_T} = N\delta^j{}_k$, $j,k = 1,2,3 \ldots N^2-1$, −2.1 while $a^{0\sigma_T} a_{0\sigma_T} = 0$. The fact that $a^{0\sigma_T} a_{0\sigma_T} = 0$, and that $s^{0\sigma_T} s_{0\sigma_T} = 4$ while $s^{j\sigma_T} s_{k\sigma_T} = 2$ reflects the fact that σ^0 commutes symmetrically with all of the remaining σ^u, while each of the σ^k commutes antisymmetrically with the others, though symmetrically with the unit matrix σ^0. In a fundamental sense, this provides a "weight" factor which determines the degree to which any given σ^u matrix commutes and anticommutes with all of the others.

Currently, our discussion is about <u>spacetime</u> structure. In the context of later discussions about <u>flavor</u> structure, the flavor analogues for (2.30) provide a way of accounting for the so-called "abelian" ($s^{u\sigma_T} s_{v\sigma_T}$) and "non-abelian" ($a^{u\sigma_T} a_{v\sigma_T}$) characteristics of any specific interaction. By properly utilizing these operators in the renormalization group equations, one can determine how quickly the running coupling for a particular interaction will either increase (abelian) or decrease (non-abelian) with a corresponding increase in energy transfer. In short, this helps to determine the "charge screening" characteristics of the various interactions. This, in turn, is used directly to determine the mass scale M_G at which grand unification takes place, on the basis of low-energy particle phenomenology and running coupling data. By the arguments presented in the introductory section, this scale must turn out to be given by $M_G = 1.22 \times 10^{19}$ (GeV/c^2), for a grand unified theory which properly incorporates quantum gravitation, and in which higher order renormalization terms are correctly accounted for. This is one reason that particular emphasis has been placed upon determination of the Dirac and Pauli spacetime structure constants in the above discussion.