2.2 - Particle/Antiparticle and Spin-Up/Spin-Down Degrees of Quantum Mechanical Freedom in Spacetime and Chirality, Gauge Invariance, and the Dirac Wavefunction

At this point, we begin to consider the Dirac wavefunction Ψ, and associated Dirac spinor $u(E^u)$, specified by: (N is the normalization constant.)

$$\Psi = N \cdot u(E^u) \cdot e^{-iE^{\sigma}x_{\sigma}} . \qquad (2.31)$$

In explicit components, the Dirac spinor $u(E^u)$ is given by:

$$u = \begin{pmatrix} u_A \\ u_B \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\ \begin{pmatrix} u_3 \\ u_4 \end{pmatrix} \end{pmatrix} , \qquad (2.32)$$

while the characteristics of this spinor are determined by the Dirac spinor equation: (Using the conventional four dimensions for the moment.)

$$(\gamma^u E_u - mc^2)u = 0 \qquad (2.33)$$

The above is simply the first-order metric equation (2.11), expressed in terms of the energy vector E^u and multiplied from the right by the Dirac spinor u. Because of the exponential in (2.31), it is possible to write: (with $\hbar = c = \hbar c = 1$)

$$id_u \Psi = E_u u e^{-iE^{\sigma}x_{\sigma}} = E_u \Psi , \qquad (2.34)$$

and hence the Dirac wave equation: (using (2.33))

$$(i\gamma^u d_u - mc^2)\Psi = 0 . \qquad (2.35)$$

Demanding the invariance of the above under local gauge (phase) transformations of the form:

$$\Psi \longrightarrow \Psi' = e^{ia(x^u)}\Psi , \qquad (2.36)$$

it is necessary to rewrite the wave equation (2.35) in the extended form:

$$\left[\gamma^u(id_u - eQA_u) - mc^2\right]\Psi = 0 , \qquad (2.37)$$

wherein

$$eQA^u \longrightarrow eQA^{u\prime} = eQA^u - d^u a . \qquad (2.38)$$

In the above, Q is the electrostatic (Coulomb) charge of the particle under consideration, $e \equiv g_Q$ is the running coupling strength of the electromagnetic Coulomb interaction ($e^2/\hbar c = 1/137.036$ at very low energy transfer), and A^u is the photon wave vector (electromagnetic potential) through which the electromagnetic interaction is mediated. It is worth noting that the electromagnetic charge Q for any given particle is not in any way supplied by the Dirac spacetime equation, but must either be input from experimental knowledge of particle phenomenology or predicted by an appropriate Grand Unified Theory of particle flavor.

To determine specific solutions of the Dirac equation, it helps to begin by considering the spinor equation (2.33) for a particle at rest, $E^k = 0$. This is given by:

$$
\begin{aligned}
&(\gamma^0 E_0 - mc^2)\, u \\
&= \begin{pmatrix} \sigma^0 E_0 - mc^2 & 0 \\ 0 & -\sigma^0 E_0 - mc^2 \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix} \\
&= \begin{pmatrix} E_0 - mc^2 & 0 & 0 & 0 \\ 0 & E_0 - mc^2 & 0 & 0 \\ 0 & 0 & -E_0 - mc^2 & 0 \\ 0 & 0 & 0 & -E_0 - mc^2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = 0
\end{aligned}
\qquad (2.39)
$$

The next step is to enforce the very important requirement that the rest mass scalar m is always to be of a positive magnitude, for both particles and antiparticles. (Zero mass particles excepted.) The energy vector E^u however, is specified so as to be positive for particles and negative for antiparticles. At rest, the magnitude $|E_0| = mc^2$ therefore, irrespective of whether E_0 itself is positive or negative. Thus, it is possible to rewrite (2.39) in the alternatives:

$$\hat{E}u = \gamma^0 u \tag{2.40)(a}$$

$$\begin{pmatrix} \hat{E} & 0 \\ 0 & \hat{E} \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix} = \begin{pmatrix} \sigma^0 & 0 \\ 0 & -\sigma^0 \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix} \tag{2.40)(b}$$

$$\begin{pmatrix} \hat{E} & 0 & 0 & 0 \\ 0 & \hat{E} & 0 & 0 \\ 0 & 0 & \hat{E} & 0 \\ 0 & 0 & 0 & \hat{E} \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}, \tag{2.40)(c}$$

where $\hat{E} \equiv E_0/|E_0| = E_0/mc^2$. The eigenvector solutions for the above, with associated $\hat{E}$ eigenvalue, are the following:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \hat{E}=1; \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \hat{E}=1; \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \hat{E}=-1; \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \hat{E}=-1 \quad . \tag{2.41}$$

The quantity $\hat{E} = \gamma^0$ is the first of many examples we shall encounter of a "quantized degree of freedom." In a fundamental sense, this quantity is a natural consequence of the quantum mechanical description of classical spacetime, eq. (2.1), via the first order Dirac metric equation, eq. (2.10). Due to this very close connection with the spacetime metric, we refer to this specifically as a "spacetime" degree of freedom. While there are certain interrelationships, this must be carefully distinguished, for example, from "flavor," "color" and "generational" (family) degrees of freedom. The electromagnetic charge Q, which was already encountered briefly in eq. (2.37), is perhaps the best known example of a particular "flavor" degree of freedom. The uniquely defining feature of <u>any</u> quantum degree of freedom is that it is the definitive mechanism whereby eigensolutions for various permissable particles and states of particle may be uniquely <u>labelled</u>, so as to <u>distinguish</u> any one particle in any one state from any other particle in any other state. This is particularly important in situations involving the Pauli exclusion principle, to determine which particles may coexist with which other particles in the same physical system. Consequently, it is very important to enumerate as completely as possible, all of the different degrees of freedom which nature may make available to the elementary particles. Some of these degrees of freedom, from the flavor and color groups, were shown back in Table 1.1.

Because $\hat{E} = E_0/|E_0| = \gamma^0$ distinguishes particle from antiparticle depending upon whether it is equal to +1 or -1, we shall refer to $\hat{E}$ specifically as the spacetime "particle" degree of freedom, with permissable eigenvalues ± 1. To this point, we have considered only particles at rest, $E^k=0$. For $E^k \neq 0$, $\hat{E}$ generalizes to:

$$\hat{E}^u \equiv \left(E_0/|E_0|, E_1/|E_1|, E_2/|E_2|, E_3/|E_3|\right) = \left(\gamma^0, \gamma^0, \gamma^0, \gamma^0\right) = \pm 1 \quad (2.42)$$

Because $E^u = mcu^u$ (eq. (2.5)) and m is always positive, $\hat{E}^u = -1$ antiparticle solutions clearly involve a velocity vector with negative magnitude, $\hat{u}^u \equiv u^u / |u^u| = -1$. This is why it is possible to interpret an antiparticle simply as a particle which follows a reversed path through spacetime.[2.2] The actual operation whereby a particle with $\hat{E}^u = 1$ is transformed into its own antiparticle with $\hat{E}^u = -1$ is known as the "conjugation" operation C. Related operations, which seperately involve the space and time components P^k and P^0 of P^u are, respectively, the "parity" operation P and the "time reversal" operation T. Specifically, these operators are defined so as to produce the following transformations on a given Dirac wavefunction:

$$C: \quad \hat{E}^u \longrightarrow -\hat{E}^u \qquad (2.43)(a)$$

$$P: \quad \hat{E}^0 \longrightarrow \hat{E}^0 \; ; \; \hat{E}^k \longrightarrow -\hat{E}^k \qquad (2.43)(b)$$

$$T: \quad \hat{E}^0 \longrightarrow -\hat{E}^0 \; ; \; \hat{E}^k \longrightarrow \hat{E}^k \, , \qquad (2.43)(c)$$

and hence the combined transformation:

$$\text{CPT:} \quad E^u \longrightarrow E^u \tag{2.44}$$

which produces no net change, and in some sense serves as the definition of an antiparticle, as a particle following a reversed path through space and time.

The spacetime quantum number $\hat{E}^u = \gamma^0 = \pm 1$ is not the only spacetime degree of freedom associated with the Dirac equation. There is an additional degree of freedom:

$$S^3 = \tfrac{1}{2}\Sigma^3 = \tfrac{1}{2}i\gamma^1\gamma^2 = \tfrac{1}{2}\gamma^0\gamma^3\gamma^5 = \pm\tfrac{1}{2} \tag{2.45}$$

associated with intrinsic spin. To see how this comes about, it is easiest to begin with the Dirac metric in the form of (2.11), $(\gamma^u E_u - mc^2) = 0$, but rewritten as:

$$\gamma^0 E_0 = \gamma^k E_k + mc^2 \tag{2.46}$$

Multiplying from the left by γ^0, and accounting for (2.15) and (2.16), the above is readily rewritten:

$$\begin{aligned} E^0 &= \gamma^0\gamma^k E_k + \gamma^0 mc^2 = \\ &= \tfrac{1}{2}\left[\gamma^0\gamma^k - \gamma^k\gamma^0\right] E_k + \gamma^0 mc^2 \\ &= -i\sigma^{0k}E_k + \gamma^0 mc^2 \; . \end{aligned} \tag{2.47}$$

Generalizing to a five-dimensional manifold (four spacetime dimensions, one chirality dimension), with metric signature given in (2.18)(a), the above takes the form:

$$E^U = -i\sigma^{UV}E_V + \gamma^U mc^2 \; . \tag{2.48}$$

It is easy here to see, with $E^k = E^5 = 0$, that:

$$\hat{E}^0 = E^0/mc^2 = E^0/|E^0| = \gamma^0 = \pm 1 \; , \tag{2.49}$$

which describes clearly the antiparticle degree of freedom. Showing explicit components, the σ^{UV} matrices of (2.15) are given by:

$$
\sigma^{UV} \equiv \tfrac{1}{2} i\left[\gamma^U\gamma^V - \gamma^V\gamma^U\right] = -(1/3!)\,a^{UV}{}_{ABC}\gamma^A\gamma^B\gamma^C
$$

$$
= \begin{pmatrix}
0 & i\gamma^0\gamma^1 & i\gamma^0\gamma^2 & i\gamma^0\gamma^3 & i\gamma^0\gamma^5 \\
-i\gamma^0\gamma^1 & 0 & i\gamma^1\gamma^2 & i\gamma^1\gamma^3 & i\gamma^1\gamma^5 \\
-i\gamma^0\gamma^2 & -i\gamma^1\gamma^2 & 0 & i\gamma^2\gamma^3 & i\gamma^2\gamma^5 \\
-i\gamma^0\gamma^3 & -i\gamma^1\gamma^3 & -i\gamma^2\gamma^3 & 0 & i\gamma^3\gamma^5 \\
-i\gamma^0\gamma^5 & -i\gamma^1\gamma^5 & -i\gamma^2\gamma^5 & -i\gamma^3\gamma^5 & 0
\end{pmatrix}
$$

$$
= \begin{pmatrix}
\begin{pmatrix}0&0\\0&0\end{pmatrix} & i\begin{pmatrix}0&\sigma^1\\\sigma^1&0\end{pmatrix} & i\begin{pmatrix}0&\sigma^2\\\sigma^2&0\end{pmatrix} & i\begin{pmatrix}0&\sigma^3\\\sigma^3&0\end{pmatrix} & i\begin{pmatrix}0&\sigma^0\\-\sigma^0&0\end{pmatrix} \\
-i\begin{pmatrix}0&\sigma^1\\\sigma^1&0\end{pmatrix} & \begin{pmatrix}0&0\\0&0\end{pmatrix} & \begin{pmatrix}\sigma^3&0\\0&\sigma^3\end{pmatrix} & -\begin{pmatrix}\sigma^2&0\\0&\sigma^2\end{pmatrix} & i\begin{pmatrix}\sigma^1&0\\0&-\sigma^1\end{pmatrix} \\
-i\begin{pmatrix}0&\sigma^2\\\sigma^2&0\end{pmatrix} & -\begin{pmatrix}\sigma^3&0\\0&\sigma^3\end{pmatrix} & \begin{pmatrix}0&0\\0&0\end{pmatrix} & \begin{pmatrix}\sigma^1&0\\0&\sigma^1\end{pmatrix} & i\begin{pmatrix}\sigma^2&0\\0&-\sigma^2\end{pmatrix} \\
-i\begin{pmatrix}0&\sigma^3\\\sigma^3&0\end{pmatrix} & \begin{pmatrix}\sigma^2&0\\0&\sigma^2\end{pmatrix} & -\begin{pmatrix}\sigma^1&0\\0&\sigma^1\end{pmatrix} & \begin{pmatrix}0&0\\0&0\end{pmatrix} & i\begin{pmatrix}\sigma^3&0\\0&-\sigma^3\end{pmatrix} \\
-i\begin{pmatrix}0&\sigma^0\\-\sigma^0&0\end{pmatrix} & -i\begin{pmatrix}\sigma^1&0\\0&-\sigma^1\end{pmatrix} & -i\begin{pmatrix}\sigma^2&0\\0&-\sigma^2\end{pmatrix} & -i\begin{pmatrix}\sigma^3&0\\0&-\sigma^3\end{pmatrix} & \begin{pmatrix}0&0\\0&0\end{pmatrix}
\end{pmatrix} \qquad (2.50)
$$

$$
\equiv \begin{pmatrix}
0 & ia^1 & ia^2 & ia^3 & i\gamma^0\gamma^5 \\
-ia^1 & 0 & \Sigma^3 & -\Sigma^2 & i\hat{\Sigma}^1 \\
-ia^2 & -\Sigma^3 & 0 & \Sigma^1 & i\hat{\Sigma}^2 \\
-ia^3 & \Sigma^2 & -\Sigma^1 & 0 & i\hat{\Sigma}^3 \\
-i\gamma^0\gamma^5 & -i\hat{\Sigma}^1 & -i\hat{\Sigma}^2 & -i\hat{\Sigma}^3 & 0
\end{pmatrix} .
$$

Included in the above are the general relationships and definitions:

$$a^k \equiv -i\sigma^{0k} = \gamma^0\gamma^k = (i/3!)\,a^{0k}{}_{ABC}\gamma^A\gamma^B\gamma^C \qquad (2.51)(a)$$

$$\Sigma^k \equiv (1/2!)\,a^k{}_{ij}\sigma^{ij} = \gamma^0\gamma^k\gamma^5 = (i/2!)\,a^k{}_{ij}\gamma^i\gamma^j \qquad (2.51)(b)$$

$$\hat{\Sigma}^k \equiv -i\sigma^{k5} = \gamma^k\gamma^5 = (i/3!)\,a^{k5}{}_{ABC}\gamma^A\gamma^B\gamma^C \; . \qquad (2.51)(c)$$

with the further relationships:

$$\gamma^k = \gamma^0 a^k = \gamma^0\gamma^5\Sigma^k = -\gamma^5\hat{\Sigma}^k \qquad (2.52)(a)$$

$$\hat{\Sigma}^k = \gamma^0\Sigma^k = \hat{E}^U\Sigma^k \; , \qquad (2.52)(b)$$

and where $a^{ijk}=a^{jki}=a^{kij}=1$ and $a^{ijk}=-a^{jki}$ for $i,j,k = 1,2,3$ are the antisymmetric structure constants for the SU(2) space (x,y,z) subgroup. In the above, use has also been made of:

$$\sigma^{UV} = \begin{cases} i\gamma^U\gamma^V & U \neq V \\ 0 & U = V \end{cases} \qquad (2.53)(a)$$

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \qquad (2.53)(b)$$

$$\hat{E}^U = \gamma^0 \; . \qquad (2.53)(c)$$

To properly identify the "spin" degree of freedom (2.45), it is first necessary to consider the conservation of angular momentum. If the antisymmetric 5-tensor (4 spacetime dimensions, 1 chiral dimension) J^{AB} denotes total angular momentum, then by definition, this angular momentum must be conserved. Utilizing the 5-vector energy Hamiltonian $H^U = E^U$ as given in (2.48), another way to put this is to say that the total angular momentum must commute with the Hamiltonian, ie.,

$$[E^U, J^{AB}] = [E^U J^{AB} - J^{AB} E^U] = 0 \tag{2.54}$$

First, consideration of the orbital angular momentum: (including chirality

$$L^{AB} = X^A E^B - X^B E^A \tag{2.55}$$

with:

$$i\delta^{AB} = [X^A E^B - E^B X^A] = [X^A, E^B] \tag{2.56}$$

leads to the conclusion that:

$$[E^U, L^{AB}] = -[\sigma^{UA} E^B - \sigma^{UB} E^A] + a^{UAB}{}_{CD} \gamma^C \gamma^D , \tag{2.57}$$

hence, that orbital angular momentum, by itself, is not conserved. However, upon considering the commutation of the σ^{AB} matrices with the Hamiltonian, it turns out that:

$$[E^U, \tfrac{1}{2}\sigma^{AB}] = [\sigma^{UA} E^B - \sigma^{UB} E^A] - a^{UAB}{}_{CD} \gamma^C \gamma^D . \tag{2.58}$$

Therefore, combining (2.57) and (2.58) with (2.54), it is apparant that:

$$[E^U, J^{AB}] = [E^U, L^{AB} + \tfrac{1}{2}\sigma^{AB}] = 0 , \tag{2.59}$$

which is to say that:

$$J^{AB} = L^{AB} + S^{AB} = L^{AB} + \tfrac{1}{2}\sigma^{AB} \tag{2.60}$$

So it is that the intrinsic spin angular momentum S^{AB} turns out to be given by:

$$S^{AB} = \tfrac{1}{2}\sigma^{AB} . \tag{2.61}$$

If one considers strictly the space coordinates x,y,z, the spin about any given coordinate axis may be given by: (see (2.51)(b))

$$S^i \equiv (1/2!)a^i{}_{jk}S^{jk} = \tfrac{1}{2}\Sigma^i = \tfrac{1}{2}(1/2!)a^i{}_{jk}\sigma^{jk} \quad . \qquad (2.62)$$

In the particular representation (2.8),(2.9) of the Dirac matrices, it is appropriate to align the direction of intrinsic spin to point along the z-axis. Thus, we are led specifically to consider S^3. Pulling together all of (2.50),(2.51)(b),(2.61) and (2.62), we find that: (also, of course, (2.9))

$$S^3 = S^{12} = \tfrac{1}{2}\sigma^{12} = \tfrac{1}{2}\Sigma^3 = \tfrac{1}{2}i\gamma^1\gamma^2 = \tfrac{1}{2}\gamma^0\gamma^3\gamma^5$$
$$= \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix} = \pm\tfrac{1}{2} \quad , \qquad (2.63)$$

which of course, is the spin degree of freedom (2.45).

While S^3 gives the direction and magnitude of intrinsic spin in the direction of the positive z-axis, there is an additional "helicity" degree of freedom which describes the spin orientation with respect to the direction of motion. For motion aligned as well along the z-axis, this operator may be defined as: ($\hat{E}^U = \gamma^0$)

$$\hat{S}^k \equiv \hat{E}^u S^k = \gamma^0 S^k \quad . \qquad (2.64)$$

Considering the above in conjunction with (2.50) and (2.63), this allows one to explicitly write, for the z-axis:

$$\hat{S}^3 = -iS^{35} = -\tfrac{1}{2}i\sigma^{35} = \tfrac{1}{2}\hat{\Sigma}^3 = \tfrac{1}{2}\gamma^3\gamma^5 = \tfrac{1}{2}i\gamma^0\gamma^1\gamma^2$$
$$= \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} = \pm\tfrac{1}{2} \quad . \qquad (2.65)$$

It is instructive to note that the helicity operator viewed in this manner, is identically equivalent to the third component $\frac{1}{2}\gamma^3\gamma^5$ of the axial vector $\frac{1}{2}\gamma^U\gamma^5$.

Finally, there is the Casimir spin operator S. Utilizing (2.50) this operator may be defined by:

$$S(S+1) = S_1^2 + S_2^2 + S_3^2 = \tfrac{1}{4}(\Sigma_1^2 + \Sigma_2^2 + \Sigma_3^2)$$

$$= \begin{pmatrix} 3/4 & 0 & 0 & 0 \\ 0 & 3/4 & 0 & 0 \\ 0 & 0 & 3/4 & 0 \\ 0 & 0 & 0 & 3/4 \end{pmatrix} , \qquad (2.66)$$

or, more simply:

$$S = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} = \tfrac{1}{2} . \qquad (2.67)$$

It is this operator, which is independent of the orientation of spin and motion, that is often referred to when one speaks generally about spin $\frac{1}{2}$ particles.