2.3 - Determination and Labelling of the Spinor Eigensolutions to the Five-Dimensional Dirac Equation, and the High and Low Energy Approximations

As a consequence of the above discussion, it is now possible to label the four distinct solutions of the Dirac equation in terms of four linearly independent, though multiplicatively interrelated "quantized degrees of freedom." Specifically, from eqs. (2.40), (2.63), (2.65) and (2.67) respectively, and utilizing $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, these spacetime degrees of freedom are the following:

$$\hat{E}^U = \gamma^0 = i\gamma^1\gamma^2\gamma^3\gamma^5 = \pm 1 \tag{2.68.1}$$

$$S^3 = \tfrac{1}{2}\Sigma^3 = \tfrac{1}{2}\gamma^0\gamma^3\gamma^5 = \tfrac{1}{2}i\gamma^1\gamma^2 = \pm\tfrac{1}{2} \tag{2.68.2}$$

$$\hat{S}^3 = \tfrac{1}{2}\hat{\Sigma}^3 = \tfrac{1}{2}\gamma^3\gamma^5 = \tfrac{1}{2}i\gamma^0\gamma^1\gamma^2 = \pm\tfrac{1}{2} \tag{2.68.3}$$

$$S(S+1) = (S^{1^2} + S^{2^2} + S^{3^2}) = \tfrac{3}{4}\gamma^5\gamma^5 \Longrightarrow S=\tfrac{1}{2}\ . \tag{2.68.4}$$

(it is convenient, using $S+1 = 3/2$ to rewrite (2.68.4) as $S = (2/3)(S^{1^2} + S^{2^2} + S^{3^2})$.) The existence of four related degrees of freedom in the above is tied directly to the fact, out of the total of sixteen linearly independent Dirac matrices (five γ^U, ten σ^{UV} and one unit matrix $\gamma^5\gamma^5$) that exactly four of these matrices can be simultaneously diagonalized, specifically, γ^0, $\gamma^0\gamma^3\gamma^5$, $\gamma^3\gamma^5$ and the unit matrix $I=\gamma^5\gamma^5$. (recall (2.13) and the ensuing discussion.) These diagonalized matrices are particularly important, because the eigenvalues of these matrices, which are simply the quantized degrees of freedom $\hat{E}^U, S^3, \hat{S}^3, S$ from (2.68), may be used to label the various eigenvector solutions of the Dirac equation. Mathematically speaking, these eigenvalues and their associated Dirac eigenvectors u, for each of (2.68) respectively, are determined by the following:

$$\det\,(\gamma^0 - \hat{E}^U\gamma^5\gamma^5) \;\;= 0 \qquad (2.69.1)$$
$$(\gamma^0 - \hat{E}^U\gamma^5\gamma^5)u = 0$$

$$\det\,(\tfrac{1}{2}\gamma^0\gamma^3\gamma^5 - S^3\gamma^5\gamma^5) \;\;= 0 \qquad (2.69.2)$$
$$(\tfrac{1}{2}\gamma^0\gamma^3\gamma^5 - S^3\gamma^5\gamma^5)u = 0$$

$$\det\,(\tfrac{1}{2}\gamma^3\gamma^5 - \hat{S}^3\gamma^5\gamma^5) \;\;= 0 \qquad (2.69.3)$$
$$(\tfrac{1}{2}\gamma^3\gamma^5 - \hat{S}^3\gamma^5\gamma^5)u = 0$$

$$\det\,((2/3)({S^1}^2 + {S^2}^2 + {S^3}^2) - S\gamma^5\gamma^5) \;\;= 0 \qquad (2.69.4)$$
$$((2/3)({S^1}^2 + {S^2}^2 + {S^3}^2) - S\gamma^5\gamma^5)u = 0 \,.$$

Recall again, that $\hat{S}^3 = \hat{E}^U S^3$, which means that as among $\hat{E}^U$, S^3 and $\hat{S}^3$, that only two of these three are truly independent. The eigenvector solutions to the above, and associated eigenvalues, are the following:

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\begin{matrix}\hat{E}^U=1\\ S^3=\frac{1}{2}\\ \hat{S}^3=\frac{1}{2}\\ S=\frac{1}{2}\end{matrix}\,,\quad \begin{pmatrix}0\\1\\0\\0\end{pmatrix}\begin{matrix}\hat{E}^U=1\\ S^3=-\frac{1}{2}\\ \hat{S}^3=-\frac{1}{2}\\ S=\frac{1}{2}\end{matrix}\,,\quad \begin{pmatrix}0\\0\\1\\0\end{pmatrix}\begin{matrix}\hat{E}^U=-1\\ S^3=\frac{1}{2}\\ \hat{S}^3=-\frac{1}{2}\\ S=\frac{1}{2}\end{matrix}\,,\quad \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\begin{matrix}\hat{E}^U=-1\\ S^3=-\frac{1}{2}\\ \hat{S}^3=\frac{1}{2}\\ S=\frac{1}{2}\end{matrix}\,. \qquad (2.70)$$

Referring back to (2.41), we see that these are simply the eigenvector solutions to the Dirac equation obtained for a particle at rest.

To generalize the above for a particle in motion, ie., to Lorentz transform (2.70) above out of the rest frame, it is helpful to begin with eq. (2.48), written as:

$$(E^U - \gamma^U mc^2)u = -i\sigma^{UV}E_V \,, \qquad (2.71)$$

and then to obtain the explicit solution for E^0, ie.,

$$(E^0 - \gamma^0 mc^2)u = -i\sigma^{0V}E_V$$
$$= \begin{pmatrix}\sigma^0(E^0-mc^2) & 0\\ 0 & \sigma^0(E^0+mc^2)\end{pmatrix}\begin{pmatrix}u_A\\u_B\end{pmatrix} = \begin{pmatrix}0 & \sigma^kE_k+\sigma^5E_5\\ \sigma^kE_k-\sigma^5E_5 & 0\end{pmatrix}\begin{pmatrix}u_A\\u_B\end{pmatrix} \qquad (2.72)$$
$$= \begin{pmatrix}E^0-mc^2 & 0 & 0 & 0\\ 0 & E^0-mc^2 & 0 & 0\\ 0 & 0 & E^0+mc^2 & 0\\ 0 & 0 & 0 & E^0+mc^2\end{pmatrix}\begin{pmatrix}u_1\\u_2\\u_3\\u_4\end{pmatrix} = \begin{pmatrix}0 & 0 & E_5+E_3 & E_1-iE_2\\ 0 & 0 & E_1+iE_2 & E_5-E_3\\ -E_5+E_3 & E_1-iE_2 & 0 & 0\\ E_1+iE_2 & -E_5-E_3 & 0 & 0\end{pmatrix}\begin{pmatrix}u_1\\u_2\\u_3\\u_4\end{pmatrix}.$$

This yields the simultaneous equations: (recall that $\sigma^5 = \sigma^0$)

$$\sigma^0(E^0-mc^2)u_A = (\sigma^kE_k + \sigma^5E_5)u_B \qquad (2.73)(a)$$
$$\sigma^0(E^0+mc^2)u_B = (\sigma^kE_k - \sigma^5E_5)u_A \,, \qquad (2.73)(b)$$

or alternatively, in expanded form:

$$(E^0-mc^2)u_1 = (E_5+E_3)u_3 + (E_1-iE_2)u_4 \tag{2.74)(a}$$

$$(E^0-mc^2)u_2 = (E_1+iE_2)u_3 + (E_5-E_3)u_4 \tag{2.74)(b}$$

$$(E^0+mc^2)u_3 = (-E_5+E_3)u_1 + (E_1-iE_2)u_2 \tag{2.74)(c}$$

$$(E^0+mc^2)u_4 = (E_1+iE_2)u_1 + (-E_5-E_3)u_2 \;, \tag{2.74)(d}$$

where E_5, newly introduced, is associated with the chirality dimension. (Ordinarily, E^5 is taken to be equal to zero. In the event that $E^5\neq 0$, the spacetime interval ds^2 of (2.1) would acquire an additional term of the form $dx^5dx_5 \neq 0$; and the ordinary Lorentz transformation involving dx^0 and dx^3 (motion assumed along the z-axis) would have to be extended so as to include dx^5.)

Then, transforming the "at rest" solutions (2.70) out of the rest frame with the help of (2.73) and (2.74), and utilizing the fact that $\hat{E}^0 = E^0/|E^0| = 1$ for the first two solutions in (2.70) and $\hat{E}^0 = E^0/|E^0| = -1$ for the latter two, it becomes possible to obtain the general solutions for a particle in motion, along with appropriate labels. Specifically, one obtains the positive energy solutions:

$$\begin{array}{l} u\uparrow(ct,z) \equiv \\ u^{(1)}(ct,z) = \end{array} \begin{pmatrix} 1 \\ 0 \\ \dfrac{\sigma^k E_k - \sigma^5 E_5}{\sigma^0(E^0 + mc^2)} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \dfrac{-E_5 + E_3}{E^0 + mc^2} \\ \dfrac{E_1 + iE_2}{E^0 + mc^2} \end{pmatrix} \tag{2.75.1}$$

$$\begin{array}{l} u\downarrow(ct,z) \equiv \\ u^{(2)}(ct,z) = \end{array} \begin{pmatrix} 0 \\ 1 \\ 0 \\ \dfrac{\sigma^k E_k - \sigma^5 E_5}{\sigma^0(E^0 + mc^2)} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ \dfrac{E_1 - iE_2}{E^0 + mc^2} \\ \dfrac{-E_5 - E_3}{E^0 + mc^2} \end{pmatrix} , \tag{2.75.2}$$

along with the negative energy solutions:

$$v\uparrow(-ct,-z) \equiv u^{(3)}(ct,z) = \begin{pmatrix} \dfrac{-\sigma^k E_k + \sigma^5 E_5}{\sigma^0(E^0 - mc^2)} \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \dfrac{-E_5 + E_3}{E^0 - mc^2} \\ \dfrac{-E_1 + iE_2}{E^0 - mc^2} \\ 1 \\ 0 \end{pmatrix} \qquad (2.75.3)$$

$$v\downarrow(-ct,-z) \equiv u^{(4)}(ct,z) = \begin{pmatrix} 0 \\ \dfrac{-\sigma^k E_k + \sigma^5 E_5}{\sigma^0(E^0 - mc^2)} \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \dfrac{-E_1 - iE_2}{E^0 - mc^2} \\ \dfrac{-E_5 - E_3}{E^0 - mc^2} \\ 0 \\ 1 \end{pmatrix} . \qquad (2.75.4)$$

where:

$$\begin{aligned} u\uparrow(ct,z) &\equiv u^{(1)}(ct, z) \equiv |E^U = 1,\ S^3 = \tfrac{1}{2},\ \hat{S}^3 = \tfrac{1}{2},\ S = \tfrac{1}{2}\rangle \\ u\downarrow(ct,z) &\equiv u^{(2)}(ct, z) = |E^U = 1,\ S^3 = -\tfrac{1}{2},\ \hat{S}^3 = -\tfrac{1}{2},\ S = \tfrac{1}{2}\rangle \\ v\uparrow(-ct,-z) &\equiv u^{(3)}(ct, z) = |E^U = -1,\ S^3 = \tfrac{1}{2},\ \hat{S}^3 = -\tfrac{1}{2},\ S = \tfrac{1}{2}\rangle \\ v\downarrow(-ct,-z) &\equiv u^{(4)}(ct, z) = |E^u = -1,\ S^3 = -\tfrac{1}{2},\ \hat{S}^3 = \tfrac{1}{2},\ S = \tfrac{1}{2}\rangle \end{aligned} \qquad (2.76)$$

designate the labelling definitions of the four solutions (2.75), in terms of the four quantized degrees of freedom (2.68). The $\uparrow$ and $\downarrow$ labels, respectively, denote "spin up" and "spin down" with respect to the z-axis, and are determined according to (2.69.2), by reading off each of the diagonal elements from the Dirac matrix $S^3 = \frac{1}{2}\sigma^{12}$. The use of (ct,z) and (-ct,-z) follows directly from the eigenvalues for $E^U = \gamma^0$. This notation is particularly useful for examining the C, P and T transformations, as summarized in (2.43). The helicity, $\hat{S}^3$, is readily determined by noting the relative orientation of S^3 and $\hat{E}^3$ (hence $\uparrow$ or $\downarrow$ and $\pm z$) in the general case. The casimir spin operator $S = \frac{1}{2}$ for all Dirac particles, regardless of spin and motion orientation. The spinor labels u and v are used respectively to denote "electron" and "positron" spinors, which may be regarded as each other's antiparticle, each pursuing a mutually reversed course through spacetime. This assignment of labels will be rigorously justified in Section 2.6. The "electron" and "positron" flavors of particle, involve a somewhat different use of the same terminology. Particularly, the "electron" and "positron" particles, defined as the fundamental real spin $\frac{1}{2}$ fermions with electrostatic charge $Q = \mp 1$, are but one pair out of four real spin $\frac{1}{2}$ fermionic electron-positron spinor pairs, for which the electrostatic charge is given by $Q = \mp 1, \pm 2/3, \mp 1/3, \pm 0$, for the electron, up, down and neutrino pairs respectively. (See Table 1.1). It is to be again noted, that we must currently regard these charges as experimentally observed quantities, as the Dirac equation for abelian Quantum Electrodynamics U(1) is not able to predict these. The theoretical prediction why it is that these particular charge magnitudes exist in nature, can not be addressed without an appropriate Grand Unified Theory of particle flavor. The particle-antiparticle relationship between electron and positron spinors, regardless of the particular flavor of particle, is developed in detail upon consideration of the charge conjugacy "$\mathcal{C}$" operation, which will be discussed shortly.

It is often useful to denote the assignment of states (2.76) in a tabular form, designating the various spacetime quantized degrees of freedom, along with associated eigenstate solutions, which are obtained when solving the Dirac equation. This is depicted in the table below:

	$\hat{E}^u=\gamma^0$	$S^3=i\gamma^1\gamma^2$	$\hat{S}^3=\hat{E}^u S^3$	$S=(2/3)(S^{1\,2}+S^{2\,2}+S^{3\,2})$
$u\uparrow(E^u) \equiv u^{(1)}(E^u)$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$u\downarrow(E^u) \equiv u^{(2)}(E^u)$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
$v\uparrow(-E^u) \equiv u^{(3)}(E^u)$	-1	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
$v\downarrow(-E^u) \equiv u^{(4)}(E^u)$	-1	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

Table 2.1 - Spacetime Degrees of Freedom (Quantum Numbers) and Associated Spinor Eigenstates

It is also useful to know the high and low energy approximations of the four eigenvector solutions (2.75). To do this, it is necessary to know the explicit components of the energy vector E^u. (We shall assume currently that the chiral $E^5=0$.) These are easily deduced starting with the second order spacetime metric (2.1)-(2.3):

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 , \tag{2.77}$$

though rewritten in the form:

$$\begin{aligned} 1 &= (dt^2/d\tau^2)(1 - (dx^2/c^2dt^2) - (dy^2/c^2dt^2) - (dz^2/c^2dt^2)) \\ &= (dt^2/d\tau^2)(1 - v^2/c^2) , \end{aligned} \tag{2.78}$$

or, more simply,

$$dt/d\tau = 1/\sqrt{1 - v^2/c^2} . \tag{2.79}$$

Utilizing the above, with the velocity vector definition (2.4), one may deduce explicit components:

$$\begin{aligned} u^u \equiv \frac{dx^u}{d\tau} &= \left(\frac{cdt}{d\tau} , \frac{dx}{d\tau} , \frac{dy}{d\tau} , \frac{dz}{d\tau} \right) \\ &= \left(\frac{cdt}{d\tau} , \frac{dx}{dt}\frac{dt}{d\tau} , \frac{dy}{dt}\frac{dt}{d\tau} , \frac{dz}{dt}\frac{dt}{d\tau} \right) \\ &= \left(\frac{c}{\sqrt{1-\frac{v^2}{c^2}}} \; \frac{v_x}{\sqrt{1-\frac{v^2}{c^2}}} \; \frac{v_y}{\sqrt{1-\frac{v^2}{c^2}}} \; \frac{v_z}{\sqrt{1-\frac{v^2}{c^2}}} \right) \end{aligned} \tag{2.80}$$

and, utilizing the energy vector (2.5), this becomes:

$$E^u = mcu^u = mc^2 \left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} , \frac{v_x/c}{\sqrt{1-\frac{v^2}{c^2}}} , \frac{v_y/c}{\sqrt{1-\frac{v^2}{c^2}}} , \frac{v_z/c}{\sqrt{1-\frac{v^2}{c^2}}} \right) . \tag{2.81}$$

For motion along the z-axis, $E^1=E^2=0$, and $v_z=v$.

With these explicit components, one returns then to the Dirac spinor solutions (2.75), to examine the $E^1=E^2=E^5=0$ solution. Utilizing the explicit components obtained above, one notes that the ratio:

$$\frac{E_3}{E^0 + mc^2} = \frac{\sqrt{1-\frac{v^2}{c^2}}}{\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}+1} = \frac{\frac{v}{c}}{1+\sqrt{1-\frac{v^2}{c^2}}} \tag{2.82}$$
$$\simeq \frac{\frac{1}{2}\frac{v}{c}}{1-\frac{1}{4}\frac{v^2}{c^2}} \simeq \frac{1}{2}\frac{v}{c} + \frac{1}{8}\frac{v^3}{c^3} \simeq \frac{1}{2}\left(\frac{v}{c}\right)$$

for low velocity $v/c \ll 1$, using the approximation $\sqrt{1 - v^2/c^2} \simeq 1 - \frac{1}{2}v^2/c^2$. For large velocity, $v/c \rightarrow 1$, $\sqrt{1 - v^2/c^2} \rightarrow 0$, and the above becomes:

$$\frac{E_3}{E^0 + mc^2} = \frac{\sqrt{1-\frac{v^2}{c^2}}}{\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}+1} \simeq \frac{v}{c} \rightarrow 1\ . \tag{2.83}$$

Substituting the above back into the Dirac spinors (2.75), and reversing the spacetime orientation of the positron spinors $\mathcal{V}$, one arrives at the low and high energy approximations:

$$\underset{(ct,z)}{u\uparrow} = \begin{pmatrix} 1 \\ 0 \\ \frac{E_3}{E^0+mc^2} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \frac{\frac{v}{c}}{1+\sqrt{1-\frac{v^2}{c^2}}} \\ 0 \end{pmatrix} \underset{\text{low}}{\simeq} \begin{pmatrix} 1 \\ 0 \\ \frac{1}{2}\frac{v}{c} \\ 0 \end{pmatrix} \underset{\text{high}}{\simeq} \begin{pmatrix} 1 \\ 0 \\ \frac{v}{c} \rightarrow 1 \\ 0 \end{pmatrix} \tag{2.84.1}$$

$$\underset{(ct,z)}{u\downarrow} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{-E_3}{E^0+mc^2} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{-\frac{v}{c}}{1+\sqrt{1-\frac{v^2}{c^2}}} \end{pmatrix} \underset{\text{low}}{\simeq} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -\frac{1}{2}\frac{v}{c} \end{pmatrix} \underset{\text{high}}{\simeq} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -\frac{v}{c} \rightarrow -1 \end{pmatrix} \tag{2.84.2}$$

$$\underset{(ct,z)}{\mathcal{V}\uparrow} = \begin{pmatrix} \frac{E_3}{E^0+mc^2} \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\frac{v}{c}}{1+\sqrt{1-\frac{v^2}{c^2}}} \\ 0 \\ 1 \\ 0 \end{pmatrix} \underset{\text{low}}{\simeq} \begin{pmatrix} \frac{1}{2}\frac{v}{c} \\ 0 \\ 1 \\ 0 \end{pmatrix} \underset{\text{high}}{\simeq} \begin{pmatrix} \frac{v}{c} \rightarrow 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \tag{2.84.3}$$

$$\underset{(ct,z)}{v_{\downarrow}} = \begin{pmatrix} 0 \\ \dfrac{-E_3}{E^0+mc^2} \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \dfrac{-\frac{v}{c}}{1+\sqrt{1-\frac{v^2}{c^2}}} \\ 0 \\ 1 \end{pmatrix} \underset{\text{low}}{\simeq} \begin{pmatrix} 0 \\ -\frac{1}{2}\frac{v}{c} \\ 0 \\ 1 \end{pmatrix} \underset{\text{high}}{\simeq} \begin{pmatrix} 0 \\ -\frac{v}{c} \rightarrow -1 \\ 0 \\ 1 \end{pmatrix} . \tag{2.84.4}$$

Utilizing the above, in conjunction with the chirality operator γ^5, one notes the standard result, at high energies, that the chirality operator γ^5 becomes equivalent with the helicity operator insofar as its projection properties are concerned. Specifically, utilizing the quantum numbers in Table 2.1, but being careful to note that the positron spinors v in (2.84) are represented in a positive (ct,z) basis, and hence that $\hat{E}^u = E^u/|E^u| = 1$ for all of the above spinors in (2.84), one notes first that the helicity: (See (2.64),(2.65))

$$\hat{S}^3 = E^u S^3 = S^3 = \tfrac{1}{2}\hat{\Sigma}^3 \tag{2.85}$$

for all of the spinors (2.84). One then notes that γ^5, as defined in (2.12)(a), always reverses the upper and lower components of any spinor to which it is applied, ie., ($\sigma^5 = \sigma^0$)

$$\gamma^5 u = \begin{pmatrix} 0 & \sigma^5 \\ \sigma^5 & 0 \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix} = \begin{pmatrix} u_B \\ u_A \end{pmatrix} . \tag{2.86}$$

For the high energy solutions (2.84.4), particularly when v/c=1, the helicity operator (2.85) will also reverse the upper and lower components in precisely the same manner, ie.,

$$\hat{\Sigma}^3 u \underset{\text{high}}{\simeq} \gamma^5 u \tag{2.87}$$

This allows one to associate chirality with helicity, but only at extreme relativistic energies. It is for this reason, in part, that conservation or non-conservation of parity is linked so closely with that of chirality.