2.5 - Conjugate and Transposition Symmetries of the Dirac Equation in Five Dimensions, the Finite Operators for Conjugation (C) and Time Reversal (T), and Abelian Relationships Among C, P, T and A

In the preceding section, we considered the invariance of the Dirac equation (2.88) under hermitian conjugation. In the ordinary four spacetime dimensions, this led us in eq. (2.100) to introduce both an adjoint row spinor $\overline{\Psi} = \Psi^{+}\gamma^{0}$, and the parity operation $P^{-1} = \gamma^{0}$. By further considering five-dimensional hermicity, we were led in (2.104) to the introduction of an axial wavefunction, $\Psi_A = \gamma^5 \Psi_V$ (see (2.105)(b)), along with related left and right handed chirality components as shown in (2.105)-(2.110). This in turn, led us to associate the fermion mass term $m\overline{\Psi}\Psi$ with the fifth component of the equations for $d_U\Psi$ and the adjoint $d_U\overline{\Psi}$, as shown in eqs. (2.119)(b) above. This in turn led to a geometric interpretation of the Fermion mass term, eq. (2.120).(3)

Now we shall wish to consider both the ordinary complex conjugation and the transposition symmetry of the Dirac equation, as this will lead, in particular, to definitions for the conjugacy and time-reversal operators C and T. This in turn, will provide a concrete basis upon which to justify the labelling definitions used for the Dirac spinors (2.75), (2.76); and it will also provide a basis for introducing the various "Feynman diagrams" associated with these spinors. These in turn, will form an important basis for later discussions of flavor and color symmetry, and of grand unification.

We begin once again by considering the Dirac equation in five dimensions, eq. (2.88). In eq. (2.93), we demanded the invariance of this equation under complex conjugation, and were thereby led to define both adjoint and axial spinors, and parity and axial operations, as given in (2.100) and (2.104). Now, we demand the invariance of the Dirac equation (2.88) under ordinary conjugation, which is to say, similarly to (2.93), that:

$$\left[\gamma^U\left(i\partial_U - \frac{eQ}{\hbar c} A_U\right) - \frac{mc^2}{\hbar c}\right]\Psi = \left[\gamma^U_*\left(-i\partial_U - \frac{eQ}{\hbar c} A_U\right) - \frac{mc^2}{\hbar c}\right]\Psi_* = 0 \,. \qquad (2.121)$$

Here, it is helpful, analogously to (2.94), to consider the ordinary conjugate for each of the γ^U. These are given by:

$$\gamma^0_* = \gamma^0;\ \gamma^1_* = \gamma^1;\ \gamma^2_* = -\gamma^2;\ \gamma^3_* = \gamma^3;\ \gamma^5_* = \gamma^5, \qquad (2.122)$$

and are again derived by explicit analysis of (2.8),(2.9) and (2.12). In five dimensional spacetime plus chirality, these may be summarized by:

$$\gamma^U_* = (-i\gamma^2)\,\gamma^U\,(i\gamma^2)\ . \qquad (2.123)$$

Given the above, one may explicitly substitute (2.123) into the conjugate equation (2.121) to yield:

$$\left[(-i\gamma^2)\,\gamma^U\,(i\gamma^2)\left(-i\partial_U - \frac{eQ}{\hbar c} A_U\right) - \frac{mc^2}{\hbar c}\right]\Psi_* = 0\ . \qquad (2.124)$$

Then, multiplying from the left by $i\gamma^2$, and using $g^{22} = \gamma^2\gamma^2 = -1$, this may be reduced to:

$$\left[\gamma^U\left(i\partial_U + \frac{eQ}{\hbar c} A_U\right) - \frac{mc^2}{\hbar c}\right]\Psi_c = 0, \qquad (2.125)$$

where, using $\overline{\Psi} = \Psi^\dagger\gamma^0 = \Psi^\dagger P^{-1}$, (eq. (2.100)), hence $\Psi_* = \gamma^0\overline{\Psi}_T = P\,\overline{\Psi}_T$, one may define a new (conjugate) wavefunction and a new (conjugacy) operation given by:

$$\Psi_c \equiv i\gamma^2\Psi_* = i\gamma^2\gamma^0\overline{\Psi}_T \equiv CP\,\Psi_* = C\,\overline{\Psi}_T. \qquad (2.126)$$

Noting the sign reversal of Q with respect to the original Dirac equation (2.88), it is apparant that Ψ_c has an electrostatic charge which is opposite that of the original wavefunction Ψ. In this manner, it is possible to define the conjugate wavefunction and conjugacy operation.

At this point, we consider the invariance of the Dirac equation under ordinary transposition. Similarly to (2.93) and (2.121), this is to say that:

$$\left[\gamma^U\left(id_U - \frac{eQ}{\hbar c} A_U\right) - \frac{mc^2}{\hbar c}\right]\Psi = \Psi_T\left[\gamma_T^U\left(id_U - \frac{eQ}{\hbar c} A_U\right) - \frac{mc^2}{\hbar c}\right] = 0 \ . \tag{2.127}$$

Here, analogously to (2.94) and (2.122), one considers the ordinary transpose for each of the γ^U. These are given by:

$$\gamma_T^0 = \gamma^0;\ \gamma_T^1 = -\gamma^1;\ \gamma_T^2 = \gamma^2;\ \gamma_T^3 = -\gamma^3;\ \gamma_T^5 = \gamma^5 \ , \tag{2.128}$$

which, again may be summarized in five-spacetime/chirality by:

$$\gamma_T^U = (-i\gamma^3\gamma^1)\,\gamma^U\,(i\gamma^1\gamma^3) \tag{2.129}$$

Substituting the above into (2.127) one may write:

$$\Psi_T\left[(-i\gamma^3\gamma^1)\gamma^U\,(i\gamma^1\gamma^3)\left(id_U - \frac{eQ}{\hbar c} A_U\right) - \frac{mc^2}{\hbar c}\right] = 0 \ . \tag{2.130}$$

Here, we multiply from the right by $-i\gamma^3\gamma^1$ and utilize $g^{11} = \gamma^1\gamma^1 = -1$, $g^{33} = \gamma^3\gamma^3 = -1$, to yield:

$$\Psi_t^\dagger\left[\gamma^U\left(id_U - \frac{eQ}{\hbar c} A_U\right) - \frac{mc^2}{\hbar c}\right] = 0 \tag{2.131}$$

where the newly defined (time reversal) wavefunction and (time reversal) operator for the transposition symmetry is given by:

$$\Psi_t^\dagger \equiv \Psi_T\left[-i\gamma^3\gamma^1\right] \equiv \Psi_T\left[-T^{-1}\right] . \tag{2.132}$$

Note that we shall utilize a lower case "t" to denote the time reversed wavefunction, so that it may be distingished from the transposed wavefunction which is denoted by an upper case "T".

At this point, it is helpful to examine the effect of each of the P, C and T operators as defined in the last two sections, to make sure that they do indeed produce the sort of transformations that were discussed briefly back in (2.43). We start with parity.

The parity operator $P^{-1}=\gamma^0$ was initially defined in (2.100). This operator is unitary, hence:

$$P^{-1} = \gamma^0 \; ; \; P = \gamma^0 \; ; \; P^{-1}P = 1 \; (= \gamma^5\gamma^5). \tag{2.133}$$

where, also from (2.100),

$$\overline{\Psi} = \psi^\dagger\gamma^0 = \psi^\dagger P^{-1} \tag{2.134(a)}$$

$$\psi = \gamma^0\overline{\psi}^\dagger = P\,\overline{\psi}^\dagger \quad . \tag{2.134(b)}$$

We may define a parity inversion matrix which inverts the components of any four-vector in the manner required by (2.43)(b).

Such a matrix may be given by: (Note that the chiral dimension is also inverted by parity)

$$P^U{}_V = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \quad . \tag{2.135}$$

The effects of a parity transformation on an electromagnetic current of the form $J^U_{(em)} = eQ\overline{\Psi}^U\psi \equiv eQ\,J^U$, with J^U given in (2.118), may be summarized as follows:

$$eQ\overline{\Psi}P^{-1}\gamma^U P\Psi = eQ\overline{\Psi}\gamma^0\gamma^U\gamma^0\Psi = P^U{}_V \; eQ\overline{\Psi}\gamma^V\Psi \quad , \tag{2.136}$$

see (2.95), (2.96). Specifically, using (2.94), (2.96) the above holds because

$$eQ\overline{\Psi}\gamma^0\gamma^U\gamma^0\Psi = \begin{cases} eQ\overline{\Psi}\gamma^0\Psi \\ -eQ\overline{\Psi}\gamma^k\Psi \\ -eQ\overline{\Psi}\gamma^5\Psi \end{cases} \tag{2.137}$$

The conjugation operator $C = i\gamma^2\gamma^0$, defined in (2.126), is also unitary. Therefore,

$$C^{-1} = i\gamma^0\gamma^2 \; ; \; C = i\gamma^2\gamma^0 \; ; \; C^{-1}C = 1 \; (= \gamma^5\gamma^5). \tag{2.138}$$

It is also useful to define the combined CP operators:

$$P^{-1}C^{-1} = i\gamma^2 \; ; \; CP = i\gamma^2 \; ; \; P^{-1}C^{-1}CP = 1 \; (= \gamma^5\gamma^5) \; . \tag{2.139}$$

The conjugate wavefunction, also defined in (2.126), and adjoint conjugate wavefunction derived from (2.126) by applying $\overline{\Psi} = \psi^\dagger\gamma^0$, eq. (2.100), are given by:

$$\Psi_c = i\gamma^2\Psi_* = i\gamma^2\gamma^0\,\overline{\Psi}_T = CP\,\Psi_* = C\,\overline{\Psi}_T \tag{2.140)(a}$$

$$\overline{\Psi}_c = \overline{\Psi}^*(-i\gamma^2) = \Psi_T(-i\gamma^0\gamma^2) = \overline{\Psi}^*(-P^{-1}C^{-1}) = \Psi_T(-C^{-1}). \tag{2.140)(b}$$

Noting also from the sign reversal of Q in the conjugate Dirac equation (2.125), one may deduce that the electrostatic charge of the conjugate wavefunction Ψ_c is related to that of Ψ by:

$$Q_c = -Q\ . \tag{2.141}$$

Similarly to (2.135), one may define a conjugacy matrix so as to invert the components of a four vector according to (2.43)(a), while leaving the fifth chirality component unaffected. This is given by:

$$C^U{}_V = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.142}$$

The effects of a conjugacy operation on an electromagnetic current $J^U_{(em)}$ may then be summarized by: (using (2.138)-(2.141))

$$\begin{aligned} eQ_c\overline{\Psi}_c\gamma^U\Psi_c &= eQ_c\Psi_T(-C^{-1})\gamma^U C\overline{\Psi}_T = eQ_c\Psi_T(-i\gamma^0\gamma^2)\gamma^U(i\gamma^2\gamma^0)\overline{\Psi}_T \\ &= C^U{}_V eQ\,\overline{\Psi}\gamma^V\Psi\ . \end{aligned} \tag{2.143}$$

This holds, in particular, because:((2.128) is useful here) (Note, $\Psi_T\gamma^U_T\overline{\Psi}_T = \overline{\Psi}\gamma^U\Psi$)

$$eQ_c\Psi_T(-i\gamma^0\gamma^2)\gamma^U(i\gamma^2\gamma^0)\overline{\Psi}_T = \begin{cases} -eQ\Psi_T\gamma^u_T\overline{\Psi}_T = -eQ\overline{\Psi}\gamma^u\Psi \\ eQ\Psi_T\gamma^5_T\overline{\Psi}_T = eQ\overline{\Psi}\gamma^5\Psi. \end{cases} \tag{2.144}$$

The time reversal operator, which is antiunitary, was defined in (2.132). This operator is summarized by:

$$T^{-1} = i\gamma^3\gamma^1\ ;\ T = i\gamma^1\gamma^3\ ;\ T^{-1}T = -1\ (= -\gamma^5\gamma^5). \tag{2.145}$$

The time reversed wavefunction and adjoint wavefunction, from (2.132) using $\overline{\Psi} = \Psi^\dagger\gamma^0$, are given by:

$$\Psi_t = i\gamma^1\gamma^3\Psi_* = T\,\Psi_* \tag{2.146)(a}$$

$$\overline{\Psi}_t = \overline{\Psi}_*(i\gamma^3\gamma^1) = \overline{\Psi}_*\,T^{-1}\ . \tag{2.146)(b}$$

The time reversal matrix, which has the effect mandated by (2.43)(c), but which leaves the chiral components unaffected, is given by:

$$T^U{}_V = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} . \tag{2.147}$$

The effects of time reversal, which involves the interchange of initial and final wavefunctions, may then be summarized by:

$$\begin{aligned} eQ\bar\Psi^i_t\gamma^U\Psi^f_t &= eQ\bar\Psi^i_* T^{-1}\gamma^U T\Psi^f_* = eQ\bar\Psi^i_*(i\gamma^3\gamma^1)\gamma^U(i\gamma^1\gamma^3)\Psi^f_* \\ &= -eQ\Psi^i_T\gamma^0\gamma^3\gamma^1\gamma^U\gamma^1\gamma^3\gamma^0\bar\Psi^f_T = T^U{}_V\; eQ\bar\Psi^f\gamma^V\Psi^i \quad . \end{aligned} \tag{2.148}$$

Making particular use of (2.128), the above holds because: (Again, $\Psi_T\gamma^U_T\bar\Psi_T = \bar\Psi\gamma^U\Psi$)

$$-eQ\Psi^i_T\gamma^0\gamma^3\gamma^1\gamma^U\gamma^1\gamma^3\gamma^0\bar\Psi^f_T = \begin{cases} -eQ\Psi^i_T\gamma^0_T\bar\Psi^f_T = -eQ\bar\Psi^f\gamma^0\Psi^i \\ eQ\Psi^i_T\gamma^k_T\bar\Psi^f_T = eQ\bar\Psi^f\gamma^k\Psi^i \\ eQ\Psi^i_T\gamma^5_T\bar\Psi^f_T = eQ\bar\Psi^f\gamma^5\Psi^i \ . \end{cases} \tag{2.149}$$

As we should expect by virtue of (2.44), the combined CPT matrix, using (2.135), (2.142) and (2.147), is given by:

$$C^U{}_V\; P^V{}_W\; T^W{}_X = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} . \tag{2.150}$$

Hence, this produces no net change in any spacetime component of a vector upon which it operates, though from (2.135), PT does invert the chiral dimension.

Finally, having discussed the C, P and T symmetry group operators, it is important for completeness that the axial operator $A=\gamma^5$ also be examined. This operator, recall, was first introduced in eq. (2.104), in order to ensure the hermicity of the Dirac equation in <u>five</u> dimensions, and it was also utilized in eqs. (2.105)(b) and (2.107)(b). This operator as it happens, like the T operator, is antiunitary. Particularly, using the equations just cited, one may write:

$$A^{-1} = -\gamma^5 \ ; \ A = \gamma^5 \ ; \ A^{-1}A = -1 \ . \tag{2.151}$$

The axial wavefunction and adjoint wavefunction are of course given by:

$$\Psi_A = \gamma^5\Psi = A\Psi \quad (2.152)(a)$$
$$\overline{\Psi}_A = \overline{\Psi}(-\gamma^5) = \overline{\Psi}A^{-1} \,. \quad (2.152)(b)$$

Recall that the C and T matrices both left the chiral γ^5 components of a five-vector unaltered, but that P reverses chirality. The axial matrix reverses the chiral component of a vector, but leaves the ordinary four dimensional spacetime components intact. Specifically,

$$A^U{}_V = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}. \quad (2.153)$$

Consequently, using the above with (2.150), the combined operation CPTA = 1.

The effects of an axial transformation on $J^U_{(em)}$ may then be summarized by:

$$eQ\overline{\Psi}A^{-1}\gamma^U A\Psi = eQ\overline{\Psi}(-\gamma^5)\gamma^U\gamma^5\Psi = A^U{}_V\, eQ\overline{\Psi}\gamma^V\Psi. \quad (2.154)$$

This holds true because:

$$-eQ\overline{\Psi}\gamma^5\gamma^U\gamma^5\Psi = \begin{cases} eQ\overline{\Psi}\gamma^u\Psi \\ -eQ\overline{\Psi}\gamma^5\Psi. \end{cases} \quad (2.155)$$

This illustrates why chiral asymmetry is so closely connected with parity non-conservation.

For the most part, the C, P, T and A operations developed here will be useful for later reference, particularly when we begin to consider certain flavor interactions which do not conserve one or more of these seperate symmetries. There is however, a more immediate concern with the conjugation C operation, as this operation furnishes the basis upon which the spinor labelling of (2.75) and (2.76) is based; and it also provides us with the means to introduce the various Feynman diagrams upon which a significant share of our later discussions will be based.