## 2.6 - Charge Conjugation, and the Definitions and Feynman Diagrams for "Electron" and "Positron" Spinors

We begin discussion here with a more detailed consideration of the conjugate wavefunction and operation introduced in (2.126). Utilizing eq. (2.100) in the alternative forms:

$$\overline{\psi} = \psi^{\dagger}\gamma^{0} \tag{2.156)(a}$$

$$\psi = \gamma^{0}\,\overline{\psi}_{+} \quad , \tag{2.156)(b}$$

it is possible to write the conjugacy relationships:

$$\psi_c = i\gamma^2\psi_* = i\gamma^2\gamma^0\overline{\psi}_T = CP\psi_* = C\overline{\psi}_T \tag{2.157)(a}$$

$$\overline{\psi}_c = \overline{\psi}_*(-i\gamma^2) = \psi_T(-i\gamma^0\gamma^2) = \overline{\psi}^*(-P^{-1}C^{-1}) = \overline{\psi}_T(-C^{-1}). \tag{2.157)(b}$$

The above describe the wavefunctions $\psi$ , $\overline{\psi}$ . If one starts with the Dirac spinor equation (2.33) and subjects it to the same sort of hermicity analysis given in eqs. (2.93)-(2.100), one is led in the process, among other things, to define the adjoint spinor in a similar mannner to the adjoint wavefunction, in the alternative forms:

$$\overline{u} = u^{\dagger}\gamma^0 \tag{2.158)(a}$$

$$\ddot{u} = \gamma^0 u^{\dagger} \quad . \tag{2.158)(b}$$

The u spinors were derived and classified explicitly in (2.75)-(2.77). Similarly, by subjecting the spinor equation (2.33) to the conjugacy analysis of (2.121)-(2.126), one is led in the process to define conjugate spinors:

$$u_c = i\gamma^2 u_* = i\gamma^2\gamma^0\overline{u}_T = CPu_* = C\overline{u}_T \tag{2.159)(a}$$

$$\ddot{u}_c = \ddot{u}^*(-i\gamma^2) = \ddot{u}_T(-i\gamma^0\gamma^2) = \ddot{u}_*(-P^{-1}C^{-1}) = u_T(-C^{-1}). \tag{2.159)(b}$$

The connection between spinors and wavefunctions is of course given by the usual, eq. (2.31), and (2.89) in 5-space. Using (2.156) and (2.158), one may write the alternatives:

$$\Psi(E^U) = N\, u(E^U) e^{-iE^U X_U} \qquad (2.160)(a)$$

$$\overline{\Psi}(E^U) = N^{\dagger} \overline{u}(E^U) e^{\,iE^U X_U} \,. \qquad (2.160)(b)$$

Similarly, by employing (2.157) and (2.159), one may deduce the conjugate relationships:

$$\Psi_c(E^U) = N_{*} u_c(E^U) e^{\,iE^U X_U} \qquad (2.161)(a)$$

$$\overline{\Psi}_c(E^U) = N_T \overline{u}_c(E^U) e^{-iE^U X_U} \,. \qquad (2.161)(b)$$

At this point, let us return to working explicitly with the Dirac spinors, as the results derived here can readily be generalized to apply to wavefunctions as well.

First, we go back to the explicit Dirac spinor solutions $u^{(1,2,3,4)}(E^U)$ derived in (2.75). Applying the relationships (2.159) to each of these solutions, it is possible to show that: ("–" beneath superscript indicates a minus sign)

$$u_c^{(1,2,3,4)}(E^U) = i\gamma^2 u_*^{(1,2,3,4)}(E^U)$$
$$= u^{(4,\underline{3},\underline{2},1)}(-E^U) \equiv v^{(1,2,3,4)}(E^U) \,, \qquad E.(2.160)(a)$$

and that

$$\overline{u}_c^{(1,2,3,4)}(E^U) = \overline{u}_*^{(1,2,3,4)}(E^U)(-i\gamma^2)$$
$$= \overline{u}^{(4,\underline{3},\underline{2},1)}(-E^U) \equiv \overline{v}^{(1,2,3,4)}(E^U) \,. \qquad E.(2.160)(b)$$

Note that $-E^U$ includes reversal of the chiral energy component $E^5$, along with the usual $E^u$ of spacetime. Because $i\gamma^2 = CP = P^{-1}C^{-1}$, see (2.157), we note that $u_c, \overline{u}_c$ have both parity and conjugacy eigenvalues opposite to that of $u, \overline{u}$. It is the above connection which allows us, in the last term of each of E.(2.160), to define the positron spinors $v, \overline{v}$. Due to the correspondence of solutions $^{(1,2,3,4)}$ and energy $(E^U)$ between $v$ and $u_c$, and also between $\overline{v}$ and $\overline{u}_c$, the above definitions may be given in the abbreviated form: (Recall too, from (2.141) that $Q_c = -Q$.)

$$v = u_c \qquad E.(2.161)(a)$$

$$\overline{v} = \overline{u}_c \,, \qquad E.(2.161)(b)$$

wherein the "positron spinor" is defined as the "electron spinor" conjugate. Similarly to (2.160), it is easily shown that:

$$\upsilon_c^{(1,2,3,4)}(E^U) = i\gamma^2\upsilon_*^{(1,2,3,4)}(E^U) = \upsilon^{(4,\underline{3},\underline{2},1)}(-E^U) = u^{(1,2,3,4)}(E^U) \qquad (2.162)(a)$$

and

$$\bar{\upsilon}_c^{(1,2,3,4)}(E^U) = \bar{\upsilon}_*^{(1,2,3,4)}(E^U)(-i\gamma^2) = \bar{\upsilon}^{(4,\underline{3},\underline{2},1)}(-E^U) = \bar{u}^{(1,2,3,4)}(-E^U) \ . \qquad (2.162)(b)$$

In short, and as a consequence of the earlier definitions (2.161),

$$u = \upsilon_c \qquad (2.163)(a)$$

$$\bar{u} = \bar{\upsilon}_c \ . \qquad (2.163)(b)$$

Now, the eigenvalues of the particle spin along the z-axis, for each of the solutions $u^{(1,2,3,4)}$, is given in eq. (2.68.2), $S^3 = \frac{1}{2}i\gamma^1\gamma^2$, with $i\gamma^1\gamma^2$ shown explicitly as part of (2.50). The formal association of eigenvalues with "at rest" eigenstates is made in (2.69.2), (2.70), while this association in general is summarized and tabularized in eqs. (2.75),(2.76) and Table 2.1. For $u^{(1,2,3,4)}$ respectively, we have $S^3 = (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$. Given the further antiparticle definitions and relationships in (2.160)-(2.163), this allows one to label the various distinct solutions in the following manner:

$$u\uparrow(E^U) \equiv u^{(1)}(E^U) = \upsilon^{(4)}(-E^U) = u_c^{(4)}(-E^U) = \upsilon_c^{(1)}(E^U) \qquad (2.164.1)$$

$$u\downarrow(E^U) \equiv u^{(2)}(E^U) = -\upsilon^{(3)}(-E^U) = -u_c^{(3)}(-E^U) = \upsilon_c^{(2)}(E^U) \qquad (2.164.2)$$

$$-\upsilon\uparrow(-E^U) \equiv u^{(3)}(E^U) = -\upsilon^{(2)}(-E^U) = -u_c^{(2)}(-E^U) = \upsilon_c^{(3)}(E^U) \qquad (2.164.3)$$

$$\upsilon\downarrow(-E^U) \equiv u^{(4)}(E^U) = \upsilon^{(1)}(-E^U) = u_c^{(1)}(-E^U) = \upsilon_c^{(4)}(E^U). \qquad (2.164.4)$$

It is the above set of relationships which may be used to explicitly justify the labelling of solutions carried out earlier in eqs. (2.75)-(2.76), and Table 2.1. For the adjoint spinors $\bar{u}, \bar{\upsilon}$, an identical set of relationships hold true, specifically,

$$\bar{u}\uparrow(E^U) \equiv \bar{u}^{(1)}(E^U) = \bar{\upsilon}^{(4)}(-E^U) = \bar{u}_c^{(4)}(-E^U) = \bar{\upsilon}_c^{(1)}(E^U) \quad (2.165.1)$$
$$\bar{u}\downarrow(-E^U) \equiv \bar{u}^{(2)}(E^U) = -\bar{\upsilon}^{(3)}(-E^U) = -\bar{u}_c^{(3)}(-E^U) = \bar{\upsilon}_c^{(2)}(E^U) \quad (2.165.2)$$
$$-\bar{\upsilon}\uparrow(-E^U) \equiv \bar{u}^{(3)}(E^U) = -\bar{\upsilon}^{(2)}(-E^U) = -\bar{u}_c^{(2)}(-E^U) = \bar{\upsilon}_c^{(3)}(E^U) \quad (2.165.3)$$
$$\bar{\upsilon}\downarrow(-E^U) \equiv \bar{u}^{(4)}(E^U) = \bar{\upsilon}^{(1)}(-E^U) = \bar{u}_c^{(1)}(-E^U) = \bar{\upsilon}_c^{(4)}(E^U). \quad (2.165.4)$$

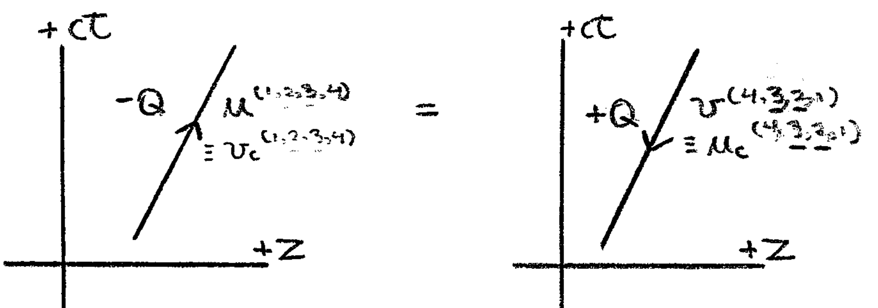
It is possible to represent (2.164) and (2.165) above in the more compact form:

$$u^{(1,\underline{2},\underline{3},4)}(E^U) \equiv \upsilon_c^{(1,\underline{2},\underline{3},4)}(E^U) = \upsilon^{(4,3,2,1)}(-E^U) \equiv u_c^{(4,3,2,1)}(-E^U) \quad (2.166)(a)$$

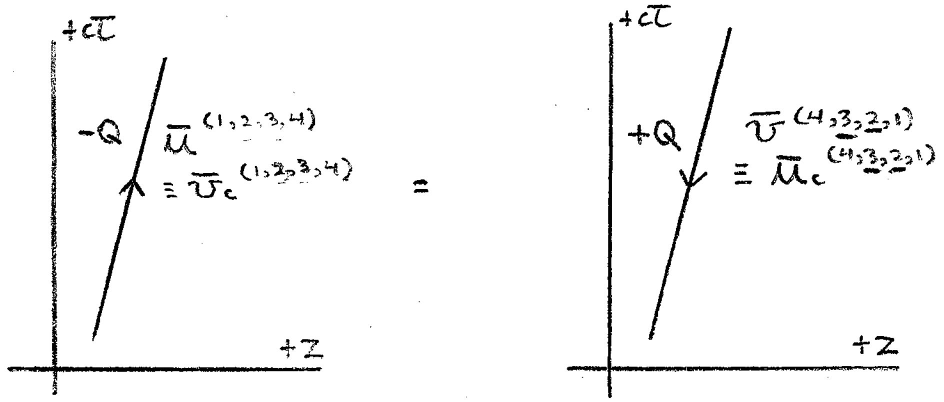
and

$$\bar{u}^{(1,\underline{2},\underline{3},4)}(E^U) \equiv \bar{\upsilon}_c^{(1,\underline{2},\underline{3},4)}(E^U) = \bar{\upsilon}^{(4,3,2,1)}(-E^U) \equiv \bar{u}_c^{(4,3,2,1)}(-E^U)\ . \quad (2.166)(b)$$

From here, rather than to explicitly show $(E^U)$ of $(-E^U)$ for each of $\bar{u}$, $\bar{u}$ , it is often simpler and more revealing to draw a so-called "Feynman diagram" depicting the relationships (2.166). Particularly, the worldline of a $+E^U$ spinor (or wavefunction) is drawn so as to point toward both the +ct and +z axes on an ordinary spacetime diagram. If chirality is included, this worldline is also positively oriented with respect to the chirality ($X^5$) axis. For a $-E^U$ spinor, the worldline points negatively along $X^0$=ct, $X^3$=z and $X^5$. These worldlines are presumed to be stationary with respect to $X^1$=x , $X^2$=y. Consequently, the digrams for (2.166)(a) and (b) respectively, which provide exactly the same information as (2.166), are the following: (with $Q_c = -Q$, see discussion following (2.100), (2.126), and $u$ chosen by convention, for these diagrams to have $-Q$.)

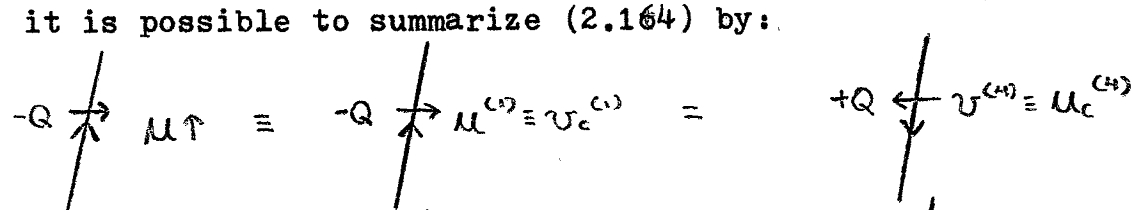



(2.167)(a)

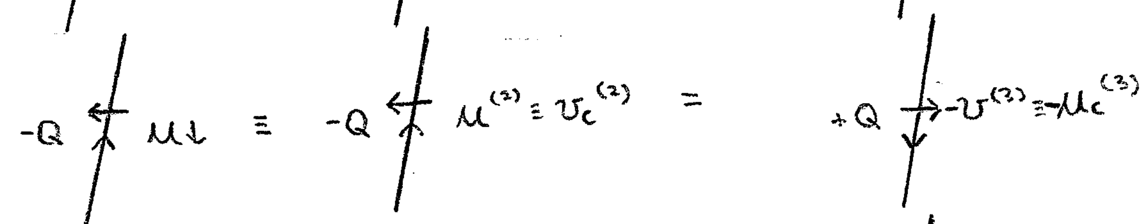



(2.167)(b) .

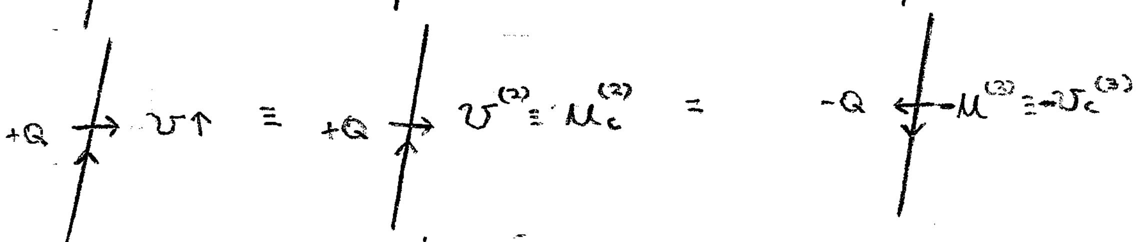
Henceforth, it will be simpler to not explicitly draw the ct and z axes. Unless otherwise noted, a worldline directed toward the upper right corner of the page is presumed to be that of a $+E^U$ spinor, with axes oriented as shown in (2.167). In the same way that (2.166) is a summarization of (2.164) and (2.165), the diagrams (2.167) are a summarization of a more detailed set of diagrams which explicitly depict spin states. In particular, using the rules outlined above, it is possible to summarize (2.164) by:



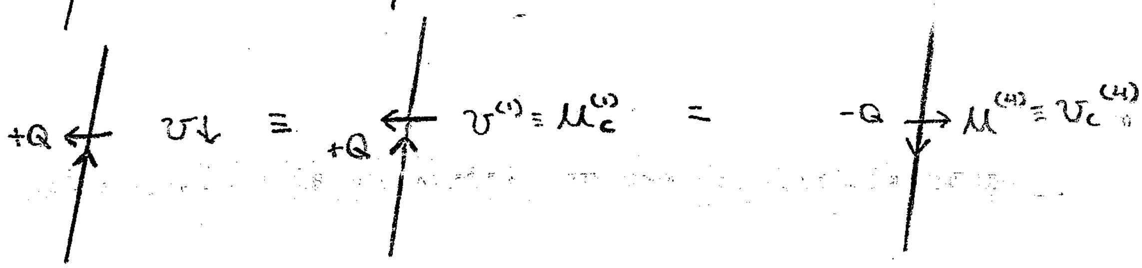

(2.168.1)




(2.168.2)

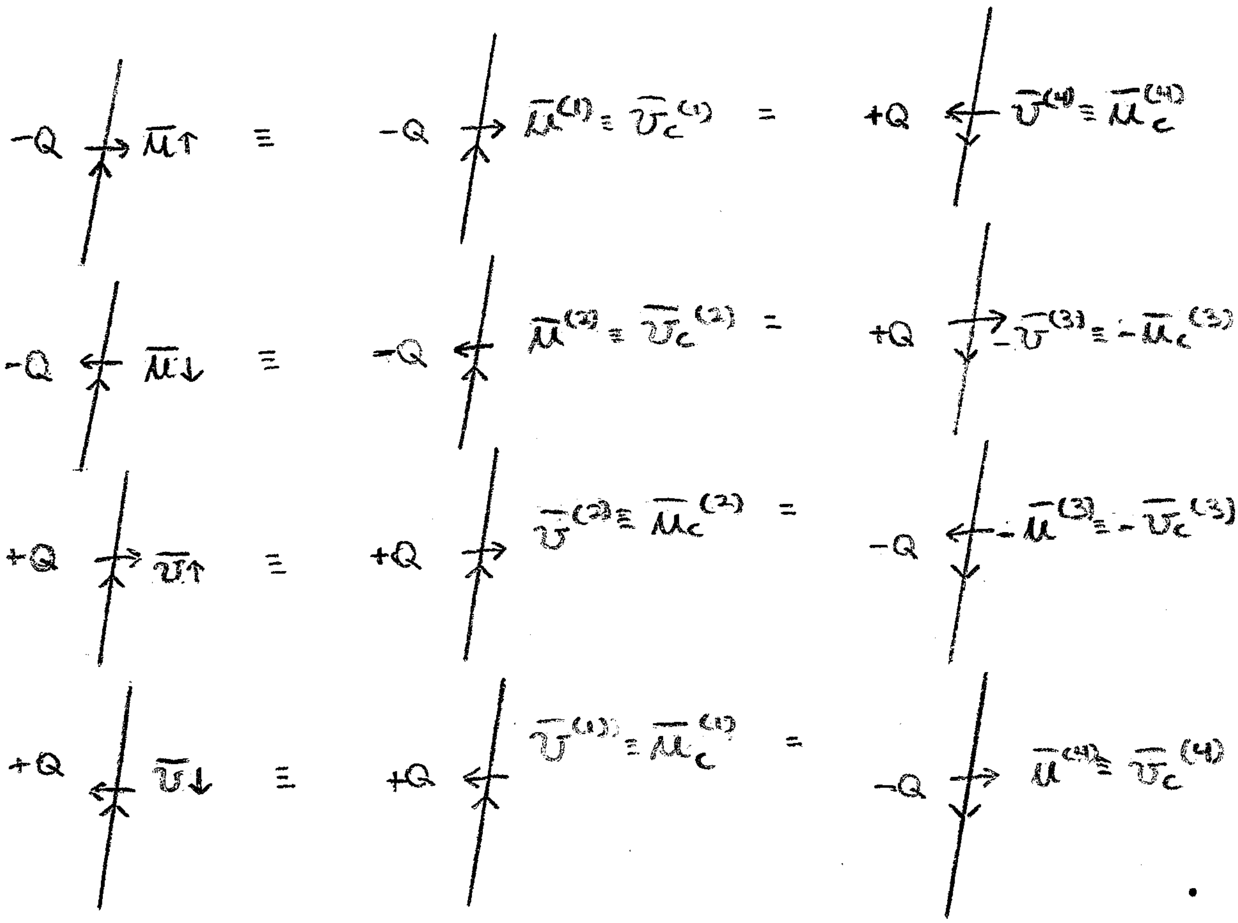



(2.168.3)




(2.168.4) ,

while (2.165) is summarized by the adjoint diagrams:



(2.169.1)

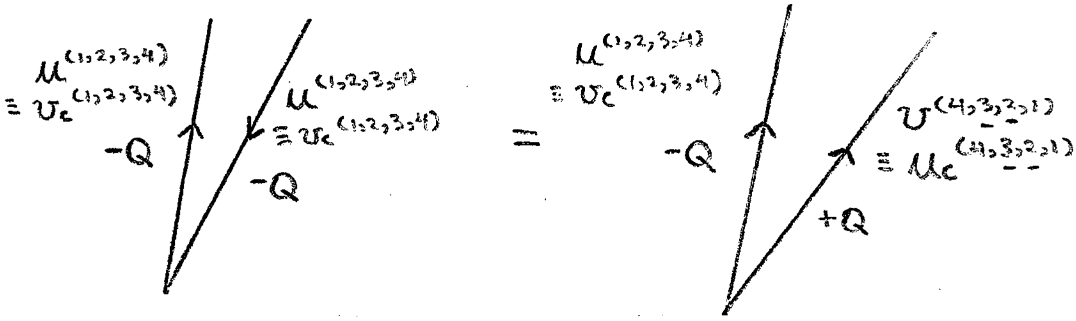
(2.169.2)

(2.169.3)

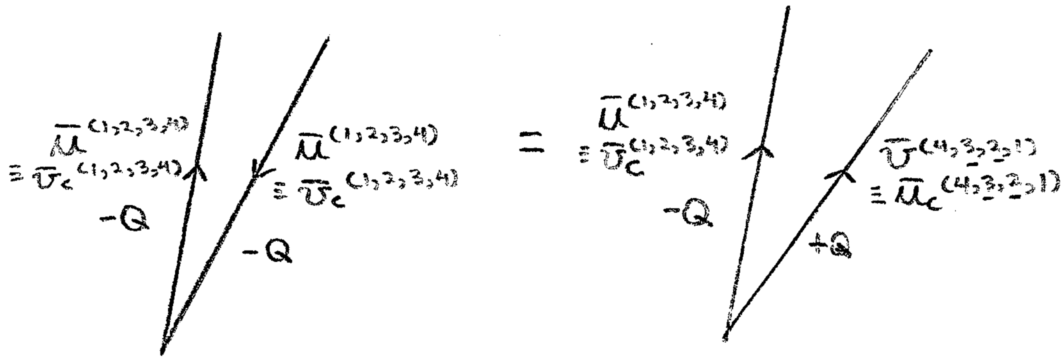
(2.169.4)

Note, in (2.168) and (2.169) above, that we have drawn the diagrams so as to depict all spinors as positive energy $+E^U$ spinors. This is to say that the above diagrams contain a reversal of the $-E^U$ and $+E^U$ terms in (2.164.3 and 4) and (2.165.1 and 2). In general, it is easier to have both positron and electron spinors $v$ and u represented in a positive energy state. One should also note the behavior of the electrostatic Coulomb charge Q in the above diagrams.

From here, the next step is to consider electron/positron spinor pair production. This is done most readily by considering a Dirac spinor travelling backwards through spacetime and chirality ($-E^U$) which, at some specified event, experiences a collision which reverses the path of the spinor so as to begin travelling forward through spacetime and chirality, with ($+E^U$). Making further use of the equalities in (2.167), one may summarize all components of spinor pair production, and adjoint pair production, as follows:

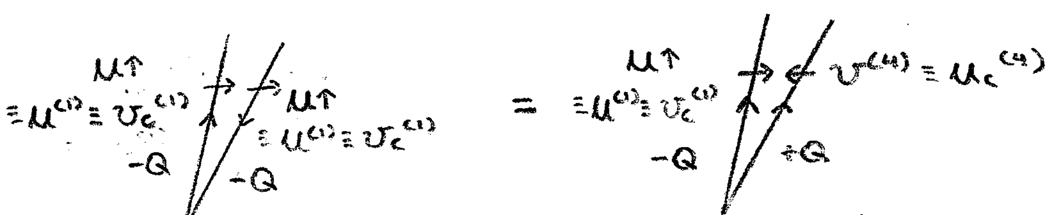



(2.170)(a)

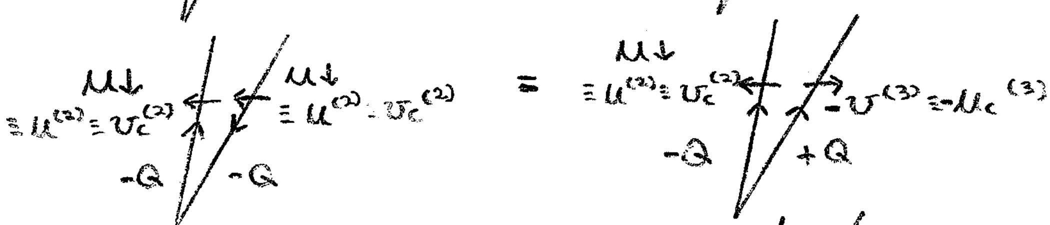



(2.170)(b)

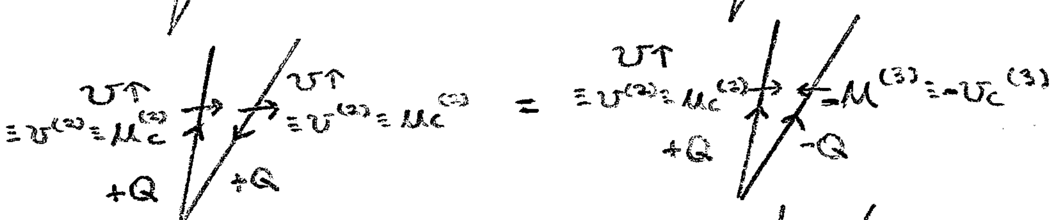
To explicitly examine distinct spin states, one merely writes out each of the component diagrams in (2.170), making use of the more detailed diagrams (2.168), (2.169). For u, $v$, these are given by:



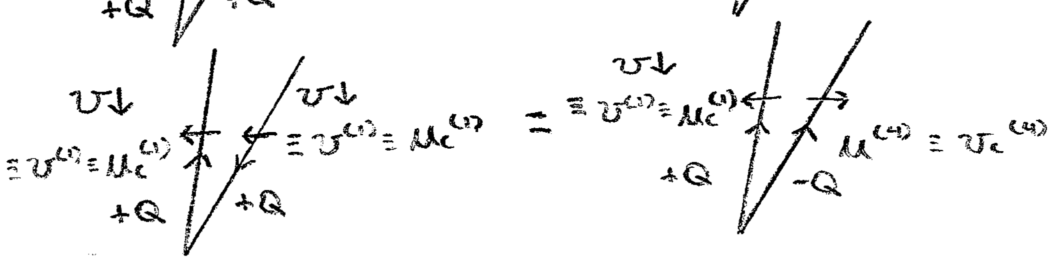

(2.171.1)




(2.171.2)

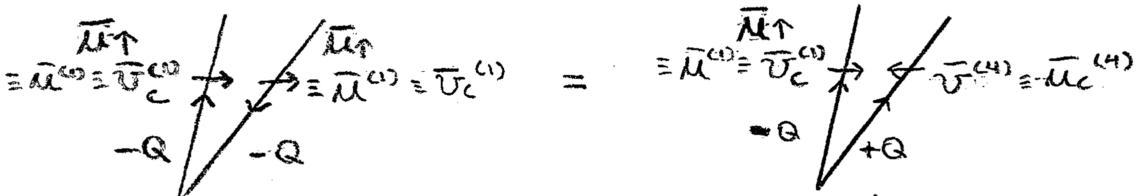



(2.171.3)

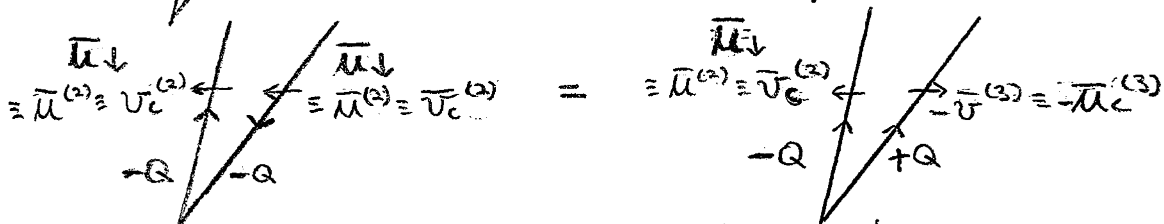



(2.171.4)

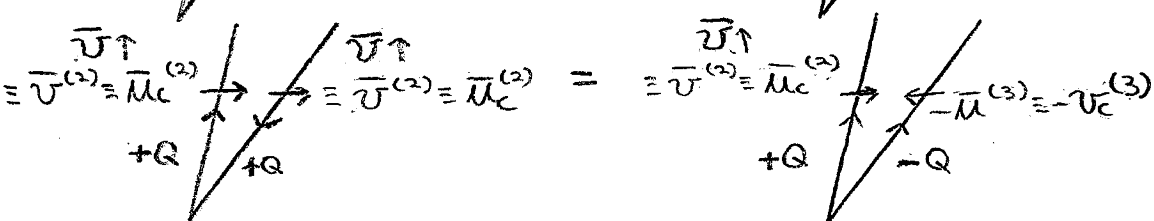
while the adjoint diagrams for $\bar{u}, \bar{v}$ are given by:



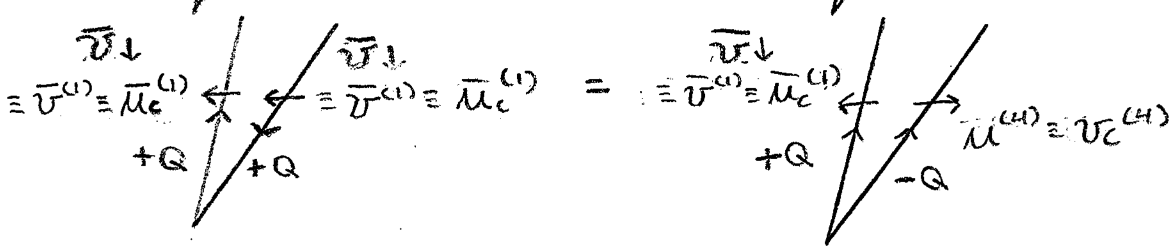

(2.172.1)




(2.172.2)

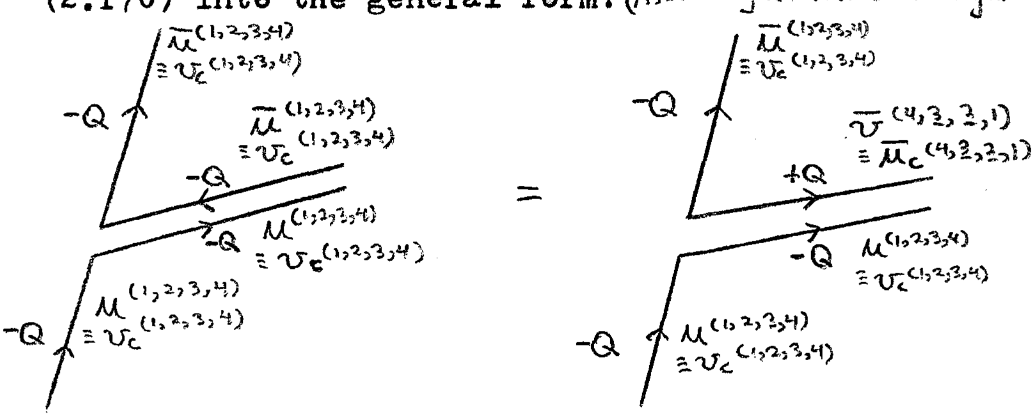



(2.172.3)



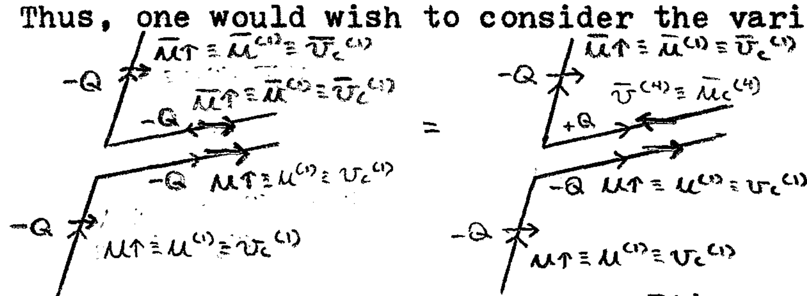

(2.172.4)

At this point, we can begin to consider transition currents between particles in various spin states. If the adjoint spinors are used to designate outgoing particle states, while the column spinors designate incoming states, Then it is possible to combine the two diagrams in (2.170) into the general form: (Assuming all lines are light-like, even if pictorial angle >45°.)

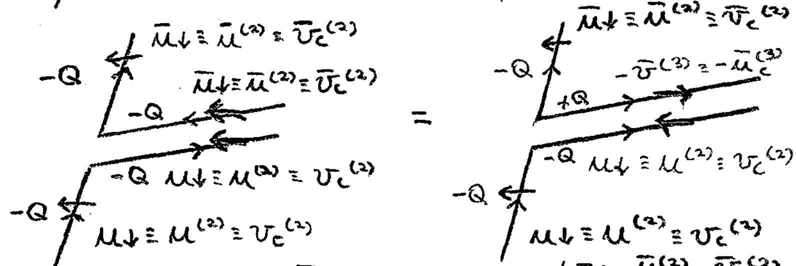



(2.173)

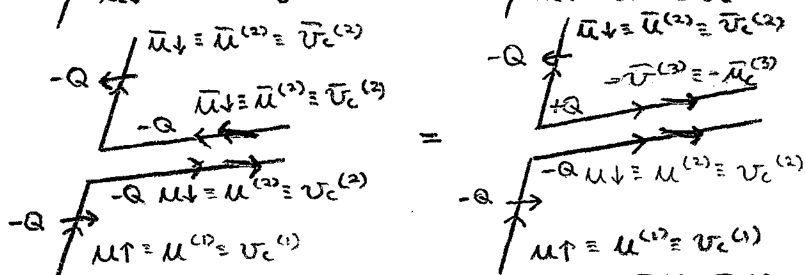
from which one can consider various forms of transition between particle states. Specifically, it is desirable to consider the four distinct combinations of spin transition $\uparrow\to\uparrow$, $\downarrow\to\downarrow$, $\uparrow\to\downarrow$, $\downarrow\to\uparrow$ ; for each of the electron and positron spinors. For the electron spinors, the respective transitions are $u^{(1)} \to u^{(1)}$, $u^{(2)} \to u^{(2)}$, $u^{(1)} \to u^{(2)}$ and $u^{(2)} \to u^{(1)}$, see, eg., eqs. (6.164) and (2.165). Thus, one would wish to consider the various transitions: (This is the explicit expansion of the (2.173) above.)



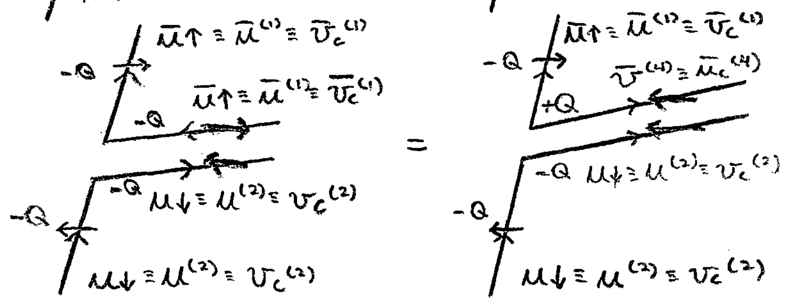

(2.174)(a)




(2.174)(b)



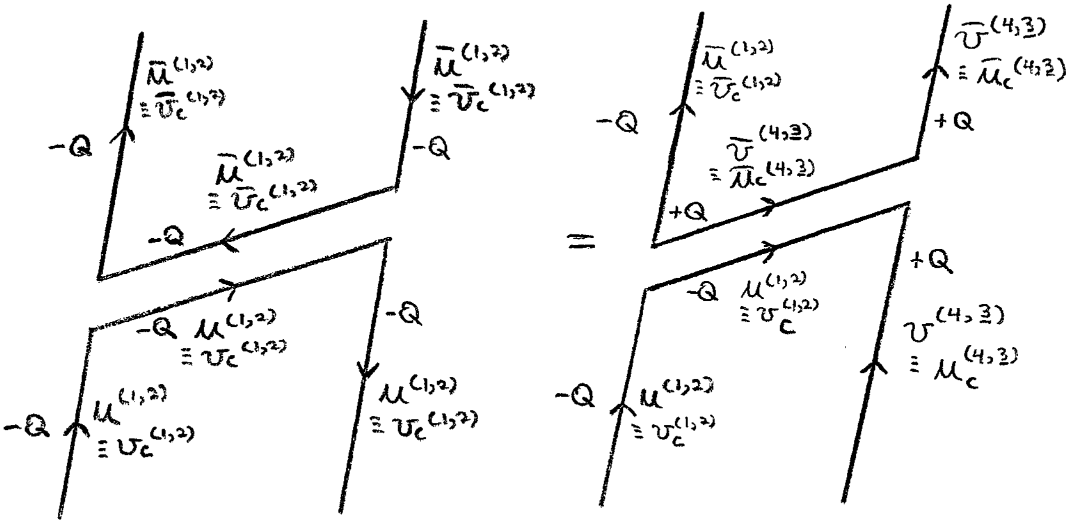

(2.174)(c)




(2.174)(d)

Note in particular that the double lines, when each is directed so as to move foreward in space time ($+E^U$), can be used to examine the total spin orientation of the particle emitted during transition. Thus, in (a) and (b) above, we see that the emitted particle (which is associated with a photon, or a vector Boson more generally) has a net spin along the z-axis of $S^3=0$. In (c) and (d) above, one finds that the z-spin $S^3=+1$ and $S^3=-1$ respectively. Note in all cases that the net photon charge $Q=0$. As we shall see in more detail in the next two sections, these spins are closely related to the various permitted covariant (real and virtual) polarization states of photons and vector Bosons. For "positron" spinors one can form a set of diagrams similar to (2.173) above. However, aside from the fact that the electrostatic charge Q reverses sign, the diagrams depicting spin transitions are identical. That is, insofar as spin is concerned, the transitions allowed for both electron and positron spinors are identical. As it is spin which most concerns us here, it is redundent to show an extra set of diagrams for positrons. (Besides, since electrons and positrons are conjugally related, $v = u_c$ and $u = v_c$, one arrives at all of the conjugally independent transitions simply by examining the positron diagrams (2.173).) To calculate more precisely the amplitudes and cross-sections for these various transitions at various energies, it is necessary to examine scattering phenomena in somewhat more depth. The basis for such calculations will be developed as part of the section to follow.

Finally, as it is often very useful to examine scattering between a pair of fermion worldlines, mediated by an intermediate vector boson, it is useful to generalize the electron diagrams from (2.173), involving $u^{(1,2)}$, $u^{(1,2)}$ so as to give:




(2.175)

This form of interaction diagram will be of particular interest in the discussion to follow.