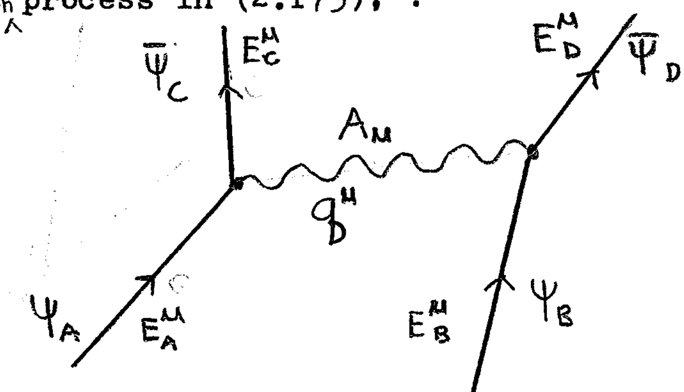
2.7 - Simple Unpolarized s,t,u Scattering Channels with a Covariant Propagator, and the Covariant (Real and Virtual) Polarization States of Massive and Massless Vector Bosons

We begin discussion of scattering phenomena by considering the generalization of the process in (2.175), :



(2.176)

in which a photon A_μ (or vector boson B_μ) is exchanged between two fermionic worldlines, involving four fermionic wavefunctions A,B,C,D. By virtue of energy conservation, one may write:

$$e^{i(E_C^u + E_D^u - E_A^u - E_B^u)} = e^0 = 1 \qquad (2.177)(a)$$

and

$$e^{iq^u} \equiv e^{i(E_D^u - E_B^u)} = e^{i(E_A^u - E_C^u)} \;. \qquad (2.177)(b)$$

Note, we are not concerned here with the chiral energy E^5, though a similar sort of analysis should ultimately be examined including E^5. The Mandelstam variables s,t,u are defined by:

$$s \equiv (E_A^u + E_B^u)(E_{Au} + E_{Bu}) \qquad (2.178)(a)$$

$$t \equiv (E_A^u - E_C^u)(E_{Au} - E_{Cu}) = q^u q_u \qquad (2.178)(b)$$

$$u \equiv (E_A^u - E_D^u)(E_{Au} - E_{Du}) \;, \qquad (2.178)(c)$$

and in the center of mass frame, with K^u defining the momenta of incident and scattered fermions, and θ the scattering angle, these are equal to:[-2.4]

$$s = 4(k^u k_u + m^2) \quad (2.179)(a)$$

$$t = -2k^u k_u(1-\cos\Theta) \quad (2.179)(b)$$

$$u = -2k^u k_u(1+\cos\Theta) \quad . \quad (2.179)(c)$$

With these preliminary definitions, it is helpful now to consider Q.E.D. Lagrangian terms of the form $J^u_{(em)}A_u$ or, for bosons and currents in general, $J^u B_u$. First we recall, from (2.160), that:

$$\Psi = N\ u\ e^{-iE^u x_u} \quad (2.180)(a)$$

$$\overline{\Psi} = N^{\dagger}\,\overline{u}\ e^{\ iE^u x_u} \quad . \quad (2.180)(b)$$

Consequently, referring back to (2.176), we note that:

$$\overline{\Psi}_C \gamma^u \Psi_A = (N_C^{\dagger} N_A)\ (\overline{u}_C \gamma^u u_A)\ e^{i(E_C^u - E_A^u)x_u} \quad (2.181)(a)$$

$$\overline{\Psi}_D \gamma^u \Psi_B = (N_D^{\dagger} N_B)\ (\overline{u}_D \gamma^u u_B)\ e^{i(E_D^u - E_B^u)x_u} \quad (2.181)(b)$$

Next, it is necessary to define the polarization vector ϵ^μ for B^u, according to:

$$B^u = \epsilon^u\ e^{-iq^u x_u} \quad (2.182)(a)$$

$$B^u_* = \epsilon^u_*\ e^{\ iq^u x_u} \quad . \quad (2.182)(b)$$

consequently, it is easily shown that

$$B^u_* B^v = \epsilon^u_* \epsilon^v \quad . \quad (2.183)$$

If we consider both real and virtual particles, the various polarization states of the boson B^u must follow the completeness relation

$$-g^{uv} = \sum_{N=4} B^u_*\ B^v = \sum_{N=4} \epsilon^u_*\ \epsilon^v \quad (2.184)$$

over all four polarization states (2 transverse, 1 longitudinal, 1 scalar). Consideration of chirality brings about non-trivial changes in this analysis.

Now we turn to the Maxwell current equation for coupling to a massive vector particle

$$J^v_{(em)} = eQJ^v = F^{uv}{}_{;u} + m^2 A^v \quad (2.185)$$

with

$$F^{uv} = A^{v;u} - A^{u;v} = A^{v,u} - A^{u,v} . \quad (2.186)$$

For the massless photon, m=0 in (2.185), though it is helpful to leave m in this equation for generality. Note, in non-abelian field theory, which will be the primary focus of subsequent chapters, (2.186) acquires an additional non-symmetric term. Additionally, the masslessness of the photon is imposed as a gauge condition, as discussed in the introduction.

If we utilize the definition (2.182)(a), and we also impose the continuity equation in spacetime, $J^{u}{}_{;u} = 0$, which is part of (2.119) and (2.120) (see also (2.181)), one may deduce from (2.185)(2.186) that: ($\hbar = c = 1$)

$$J^{v}{}_{;v} = m^2 A^{v}{}_{;v} = 0 \ . \tag{2.187}$$

For $m \neq 0$, one therefore has no choice but to choose the Lorentz gauge:

$$A^{v}{}_{;v} = 0 \ , \tag{2.188}$$

while for m=0, the above is nevertheless imposed as a gauge condition, along with the non-covariant Coulomb gauge:

$$A^{0}{}_{;0} = A^{k}{}_{;k} = 0 \ . \tag{2.189}$$

Consequently, using (2.187) specifically, (2.185)(2.186) may be combined and reduced to: (use also (2.182))

$$\begin{aligned} J^{u}_{(em)} &= eQJ^{u} = A^{u;\sigma}{}_{;\sigma} + m^2 A^{u} \\ &= \left[-q^{\sigma} q_{\sigma} + m^2\right] A^{u} \ . \end{aligned} \tag{2.190}$$

This is then readily rewritten by:

$$A^{u} = \frac{-1}{q^{\sigma} q_{\sigma} - m^2} \cdot eQJ^{u} \ . \tag{2.191}$$

Finally, calling again on the continuity equations (2.119)(2.120), we recall that:

$$J^{u} = \overline{\psi} \gamma^{u} \psi \ . \tag{2.192}$$

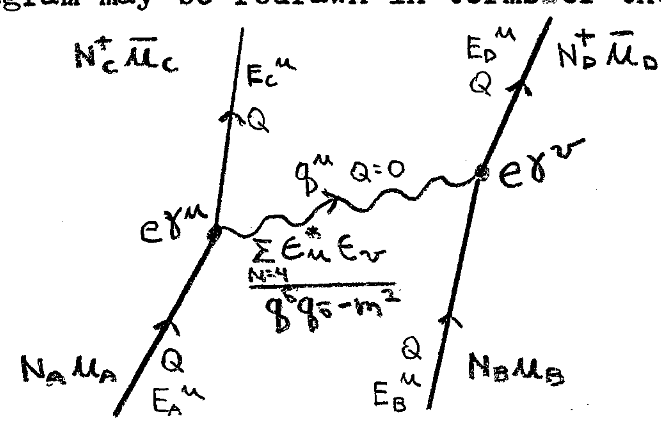
Therefore, utilizing all of (2.177),(2.181),(2.184),(2.191) and (2.192), it is possible to expand the term $J^{u}A_{u}$ as follows:

$$
\begin{aligned}
J^u_{(em)}A_u &= eQJ^uA_u \\
&= eQJ^u\, g_{uv}\, A^v \\
&= eQJ^u \frac{-g_{uv}}{q^\sigma q_\sigma - m^2}\quad eQJ^v \qquad (2.193) \\
&= eQ\ (\bar{\Psi}_C \gamma^u \Psi_A)\frac{\sum_{N=4} A_u A_v}{q^\sigma q_\sigma - m^2}\quad eQ\ (\bar{\Psi}_D \gamma^u \Psi_B) \\
&= (N_C^\dagger N_A)(N_D^\dagger N_B)\ eQ(\bar{\mu}_C \gamma^u \mu_A)\frac{\sum_{N=4} \epsilon_u^* \epsilon_v}{q^\sigma q_\sigma - m^2}\ eQ(\bar{\mu}_D \gamma^u \mu_B)\ .
\end{aligned}
$$

While we are not particularly concerned here with the normalization constant N, we note in the covariant normalization, that

$$
N = \sqrt{E^0 + mc^2}\ , \qquad (2.194)
$$

compare with eqs. (2.75). All of the terms on the last line of (2.193) correspond to elements of the Feynman diagram (2.176). Specifically this (unpolarized, covariant, virtual) diagram may be redrawn in terms of the various factors in (2.193), as:



(2.195)

The above forms the basis for detailed scattering computations in any given situation.

Of particular interest here are the polarization vectors ϵ^μ, since these provide another example of the spin "degree of freedom," for a spin 1 particle. This turns out to be similar to the isospin degree of freedom that is particularly important in examination of electroweak beta-decay, and it provides a basis to begin the more detailed consideration of flavor freedom in general.

(with $A^{\nu} \to B^{\nu}$)

For a massive vector particle, the Lorentz gauge (2.188) in combination with (2.182)(a) implies that:

$$q_u \epsilon^u = 0 . \tag{2.196}$$

This reduces the number of independent components of ϵ^{μ} from four to three in a covariant manner. For a massless vector particle, the above, along with the Coulomb gauge (2.189), implies that:

$$q_0 \epsilon^0 = q_k \epsilon^k = 0 . \tag{2.197}$$

This further reduces the number of independent polarizations from three down to two. As (2.196) and (2.197) above can be used to set $\epsilon^0 = \epsilon^3 = 0$, we may take as the independent polarizations for a massless particle:

$$\epsilon_u^{(1)} \equiv (0, 1, 0, 0) \tag{2.198(a)}$$

$$\epsilon_u^{(2)} \equiv (0, 0, 1, 0) \quad , \tag{2.198(b)}$$

or linear combinations thereof. Note that (1) and (2) in the above are labels, not indices. By taking the specific linear combination:

$$\epsilon_u^{(+1)} = -(1/\sqrt{2})\,(\epsilon^{(1)u} + i\,\epsilon^{(2)u}) = (0,-1,-i,0)/\sqrt{2} \tag{2.199(a)}$$

$$\epsilon_u^{(-1)} = (1/\sqrt{2})\,(\epsilon_u^{(1)} \pm i\,\epsilon_u^{(2)}) = (0, 1,-i,0)/\sqrt{2} \tag{2.199(b)}$$

it is possible to show, for a photon rotated through an angle $\theta = a_3$ about the z-azis of propagation, that

$$\epsilon^u \to \epsilon^{u'} = e^{-i\sigma^3 a_3}\epsilon^{\mu} , \tag{2.200}$$

for each of (2.199), where

$$\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.201}$$

is the Pauli spin matrix σ^3, see (2.9). Thus, we see that (2.199) describe photons with circular polarizations of +1 and -1 respectively (right and left handed), transverse to the z-axis of propagation. This is the origin of the labels introduced in (2.199). Because:

$$-g_{jk}^{(+1)} = \epsilon_j^{(+1)*}\epsilon_k^{(+1)} = \tfrac{1}{2}\begin{pmatrix} 1 & -i & 0 \\ i & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (2.202)(a)$$

$$-g_{jk}^{(-1)} = \epsilon_j^{(-1)*}\epsilon_k^{(-1)} = \tfrac{1}{2}\begin{pmatrix} 1 & i & 0 \\ -i & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (2.202)(b)$$

and, for motion along the z-axis,

$$\hat{q}_j\hat{q}_k = \frac{q_j\ q_k}{|q|\ |q|} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad (2.202)(c)$$

the above may be used to write the completeness relation for a massless vector particle:

$$-g_{jk} = \sum_{N=1} \epsilon_j{}^*\epsilon_k + \hat{q}_j\hat{q}_k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad (2.203)$$

where the sum is taken over the two real polarizations (2.202)(a) and (b). By virtue of (2.198)-(2.201), one may summarize the spin degree of freedom for these two real polarizations by:

$$\epsilon^{(+1)u} = (0,\ 1,\ i,\ 0)/\sqrt{2} = |S=1,\ S^3=+1\rangle \qquad (2.204)(a)$$

$$\epsilon^{(-1)u} = (0,-1,\ i,\ 0)/\sqrt{2} = |S=1,\ S^3=-1\rangle\ . \qquad (2.204)(b)$$

Further, whereas (2.201) is simply the Pauli generator matrix for rotation about the z-axis, one may similarly introduce the hermitian generators of x and y rotations. By (2.21), one may write:

$$\sigma^k = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) \qquad (2.205)$$

and where (see (2.200))

$$\epsilon^u \rightarrow \epsilon^{u'} = e^{-i\sigma^k \alpha_k}\,\epsilon^u \qquad (2.206)$$

with k=1,2,3 may be used to describe spatial rotations of the polarization vectors in general. We have purposefully avoided introduction of the unit matrix σ^0 in (2.205) above, because it is associated, not with real, but with virtual photons, as we shall see shortly.

For a massive vector particle, the Coulomb gauge (2.189) no longer applies, ~~hence~~ neither does eq. (2.197). Consequently, we may add to (2.199) a third, longitudinal state of polarization. For this polarization, one adopts:

$$\epsilon_u^{(0)} = (|E|,\ 0,\ 0,\ E^0)/mc^2, \tag{2.207}$$

where $|E| = |\sqrt{E^1E^1 + E^2E^2 + E^3E^3}|$, or, with motion along the z-axis, $E^1=E^2=0$, $|E| = |E^3|$. From the metric equation (2.1), particularly written in the form (2.6), one may write:

$$\frac{E^uE_u}{m^2c^4} = \frac{E^0E^0 - E^1E^1 - E^2E^2 - E^3E^3}{m^2c^4} = 1 \quad . \tag{2.208}$$

The additional longitudinal polarization (2.207) allows one to add to (2.202)(a) and (b) a third relationship: $(E_1=E_2=0)$

$$-g_{uv}^{(0)} = \epsilon_u^{(0)*}\epsilon_v^{(0)} = \begin{pmatrix} E_3E_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & E_0E_0 \end{pmatrix} /m^2c^4 \quad . \tag{2.209}$$

Further, for motion along the z axis,

$$\frac{E_uE_v}{m^2c^4} = \begin{pmatrix} E_0E_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & E_3E_3 \end{pmatrix} /m^2c^4 \quad . \tag{2.210}$$

Thus, we may combine all of (2.202)(a)(b), (2.209) and (2.210), using (2.208), to arrive at the completeness relation for a massive vector particle:

$$-g_{uv} = \sum_{N=3}\epsilon_u{}^*\epsilon_v - \frac{E_uE_v}{m^2c^4} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.211}$$

with metric signature as expected. Including the longitudinal polarization, the spin degree of freedom for a massive vector particle, (see (2.204)) may now be summarized by the three polarization states:

$$\epsilon^{(+1)u} = (\ 0,\ 1,\ i,\ 0)/\sqrt{2} \quad = |S=1,\ S^3=+1\rangle \qquad (2.212)(a)$$

$$\epsilon^{(0)u} = -(\ E\ ,\ 0,\ 0,\ E^0)/mc^2 = |S=1,\ S^3=\ 0\rangle \qquad (2.212)(b)$$

$$\epsilon^{(-1)u} = (\ 0,-1,\ i,\ 0)/\sqrt{2} \quad = |S=1,\ S^3=-1\rangle\ . \qquad (2.212)(c)$$

The fourth polarization state, $|S=0,\ S^3=0\rangle$ is not real, it is virtual; and it is associated with a scalar (spin zero) boson. To maintain covariance, this state must nevertheless also be accounted for, see (2.184). It is this fourth scalar polarization, which leads one to also consider the Pauli unit matrix σ^0, in conjunction with the remaining σ^k of (2.205). This will be examined in detail shortly.

One final note should be added before closing the discussion here. Thus far, we have ignored the chiral dimension associated with γ^5. If we account for γ^5 as well, then eq. (2.120)(c) says that:

$$J^5{}_{;5} = 0\ , \qquad (2.213)$$

as a distinct relationship. Thus, for a massive vector particle, following the same analysis that led earlier to eqs. (2.187),(2.188), one is led to impose what we shall refer to as the "chiral gauge,"

$$A^5{}_{;5} = 0\ . \qquad (2.214)$$

Generalizing (2.182) to five dimensions, this implies that:

$$q_5\epsilon^5 = 0\ , \qquad (2.215)$$

which allows us to set $\epsilon^5 = 0$ for all real polarization vectors. Thus, the chiral dimension has no impact at all on the number of permitted real polarizations for massive or massless vector particles. That is, for a massive vector particle, one starts with five independent components for the polarization vector, which are subsequently reduced to three because of $A^u{}_{;u}=0$ and $A^5{}_{;5} = 0$ (Eqs. (2.120)(b),(c)). For a massless particle, the Coulomb gauge results in a further reduction to two states, namely those in (2.204), but with $\epsilon^5 = 0$ in all real polarizations as well. Hence, the number of real polarization vectors remains exactly the same.

The non-trivial impact of the chiral dimension shows up however in the completness relationships, particularly, because the five dimensional metric $ds^2 = g_{UV} dx^U dx^V$, with signature given in (2.18)(a), requires one to replace (2.208) with the five dimensional:

$$\frac{E^U E_U}{m^2 c^4} = \frac{E^0 E^0 - E^1 E^1 - E^2 E^2 - E^3 E^3 + E^5 E^5}{m^2 c^4} = 1 . \qquad (2.216)$$

This will necessitate a modification in the five-dimensional completeness relationships, see (2.208)-(2.211), and may require that the longitudinal polarization (2.207) be somewhat modified to reflect chiral energy as well. For now, we shall leave this as an open question.