2.8 - Prelude to Preons: The Spinor Decomposition of Four Real Spacetime Dimensions, ct,x,y,z, into Two Complex Spinor Dimensions, $\uparrow$, $\downarrow$, Using the Covariant Polarization States of Vector Bosons

At this juncture, we begin to examine the spin composition of the various polarization states, both real and virtual, for a given vector boson. It is convenient here to begin by reintroducing the Pauli spin matrices (2.9), along with the chiral matrix $\sigma^5 = \sigma^0$, see (2.12)(b):

$$\sigma^U = \left(\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}, \begin{pmatrix}0 & -i\\i & 0\end{pmatrix}, \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}, \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\right). \tag{2.217}$$

Unless explicitly specified, we shall not be concerned here with the chiral matrix σ^5. Recalling the discussion of Section 2.6, it is convenient to seperate out the "spin up" and "spin down" states $\uparrow$ and $\downarrow$, and to form these states into the spin doublets:

$$\bar\phi \equiv (\bar\uparrow\ \bar\downarrow) \quad ; \quad \phi \equiv \begin{pmatrix}\uparrow\\ \downarrow\end{pmatrix} . \tag{2.218}$$

Out of (2.217) and (2.218) above, it is straightforward to form the four-vector:

$$\begin{aligned}\tfrac{1}{2}\bar\phi\sigma^u\phi = \tfrac{1}{2}\Bigg(& (\bar\uparrow\ \bar\downarrow)\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}\uparrow\\ \downarrow\end{pmatrix},\ (\bar\uparrow\ \bar\downarrow)\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}\uparrow\\ \downarrow\end{pmatrix},\\ & (\bar\uparrow\ \bar\downarrow)\begin{pmatrix}0 & -i\\i & 0\end{pmatrix}\begin{pmatrix}\uparrow\\ \downarrow\end{pmatrix},\ (\bar\uparrow\ \bar\downarrow)\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}\uparrow\\ \downarrow\end{pmatrix}\Bigg)\\ = \tfrac{1}{2}(& \bar\uparrow\uparrow + \bar\downarrow\downarrow\ ,\ \bar\downarrow\uparrow + \bar\uparrow\downarrow\ ,\ -i(\bar\downarrow\uparrow - \bar\uparrow\downarrow),\ \bar\uparrow\uparrow - \bar\downarrow\downarrow\ ).\end{aligned} \tag{2.219}$$

Similarly, it is helpful to form the spin matrix square product: $(\sigma_u = g_{uv}\sigma^v)$

$$\begin{aligned}\tfrac{1}{2}\sigma^u(\bar\phi\sigma_u\phi) &= \tfrac{1}{2}\begin{pmatrix}\bar\phi(\sigma_0 + \sigma_3)\phi & \bar\phi(\sigma_1 - i\sigma_2)\phi\\ \bar\phi(\sigma_1 + i\sigma_2)\phi & \bar\phi(\sigma_0 - \sigma_3)\phi\end{pmatrix}\\ &= \begin{pmatrix}(\bar\uparrow\bar\downarrow)\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}\uparrow\\ \downarrow\end{pmatrix} & (\bar\uparrow\bar\downarrow)\begin{pmatrix}0 & -1\\0 & 0\end{pmatrix}\begin{pmatrix}\uparrow\\ \downarrow\end{pmatrix}\\ (\bar\uparrow\bar\downarrow)\begin{pmatrix}0 & 0\\-1 & 0\end{pmatrix}\begin{pmatrix}\uparrow\\ \downarrow\end{pmatrix} & (\bar\uparrow\bar\downarrow)\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}\uparrow\\ \downarrow\end{pmatrix}\end{pmatrix}\\ &= \begin{pmatrix}\bar\downarrow\downarrow & -\bar\uparrow\downarrow\\ -\bar\downarrow\uparrow & \bar\uparrow\uparrow\end{pmatrix}\end{aligned} \tag{2.220}$$

Additionally, following an analogous procedure to that used to arrive at (2.67), one may form the Casimir operator:

$$S(S+1) = \tfrac{1}{4}\,(\sigma^1\sigma^1 + \sigma^2\sigma^2 + \sigma^3\sigma^3) = \begin{pmatrix} 3/4 & 0 \\ 0 & 3/4 \end{pmatrix}, \qquad (2.221)$$

or, more simply,

$$S = \begin{pmatrix} \tfrac{1}{2} & 0 \\ 0 & \tfrac{1}{2} \end{pmatrix} . \qquad (2.222)$$

Now, as noted on a number of occasions in the earlier discussion, spin provides one very important example of what we have referred to as a quantized degree of freedom, see for example eqs. (2.68) and the accompanying discussion. As noted at that time, one obtains these degrees of freedom by examining eigenvalues and eigenvectors associated with those matrices which can be simultaneously diagonalized. If $\frac{1}{2}\sigma^3$ is used to designate S^3, the intrinsic spin directed along the z-axis of particle propagation, while S above designates the Casimir spin, then the eigenvalues and eigenvector solutions for these Pauli matrices (2.217) can be used, as in (2.69), to classify the quantum degrees of freedom associated with various spin states and combinations thereof. Specifically, using σ^0 rather than $\gamma^5\gamma^5$ as the unit matrix, one may write:

$$\begin{aligned} &\det\,(\sigma^3 - S^3\sigma^0) = 0 \\ &(\sigma^3 - S^3\sigma^0)\,\phi = 0 \end{aligned} \qquad (2.223)(a)$$

$$\begin{aligned} &\det\,((1/6)(\sigma^1\sigma^1 + \sigma^2\sigma^2 + \sigma^3\sigma^3) - S\sigma^0) = 0 \\ &((1/6)(\sigma^1\sigma^1 + \sigma^2\sigma^2 + \sigma^3\sigma^3) - S\sigma^0)\,\phi = 0 \, , \end{aligned} \qquad (2.223)(b)$$

noting again that S+1 = 3/2, hence (2/3)x(1/4) = 1/6 for the factor in (2.223)(b) above, using (2.221) and (2.222). The eigenvector solutions to the above, along with associated eigenvalues, along the lines of (2.70), are the following:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{matrix} S^3 = \tfrac{1}{2} \\ S = \tfrac{1}{2} \end{matrix} \ (a), \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{matrix} S^3 = -\tfrac{1}{2} \\ S = \tfrac{1}{2} \end{matrix} \ (b). \qquad (2.224)$$

The next step is to determine the appropriate spin eigenvalues to be associated with each of the eigenstates in the spin doublets (2.218). Clearly, one will wish for ϕ to associate the solution (2.224)(a) with the spin up state $\uparrow$, and (2.224)(b) with the spin down state $\downarrow$. For the adjoint doublet $\bar{\phi}$ however, we must be a little more careful. In particular, the association of the above solutions to the adjoint doublets should be carried out in such a way as to ensure that the antiparticle column doublet transforms identically to the particle column doublet under the SU(2)xU(1) Lie group generated by the (four-dimensional) Pauli matrices σ^u from (2.217). Ironically, or perhaps not so, this requirement simultaneously serves to ensure that it is the time component $\frac{1}{2}\bar{\phi}\sigma^0\phi$, and not the z-axis space component $\frac{1}{2}\bar{\phi}\sigma^3\phi$ of (2.219), which is used to represent the virtual (scalar) polarization of a massive vector boson. In short, it is the time component of (2.219) which we choose to represent the spin composition of the scalar polarization in the spin sum (2.211), which is carried in the term E_uE_v/m^2c^4; and which term in turn is related directly to the energy momentum tensor for a single particle of mass m, $t^{uv} = mu^uu^v$, see eqs. (2.1)-(2.9), with $\hbar$=c=1. For this reason, and also by virtue of the conjugacy relationships developed in section 5, specifically eq. (2.126), it is necessary to reorder the adjoint doublet in (2.218), and to ensure that the upper member of this doublet contains a minus sign, when assigning the eigenvalues in (2.224).[-2.5]

Following the above outlined procedure, and following the conventional rules for the addition of angular and spin angular momentum, for example, eqs. (2.54)-(2.61), one may utilize the eigenvalues of $\frac{1}{2}\sigma^3$ in (2.217) and of S in (2.222), in order to classify by quantum number the various states, or linear combinations of states, as shown in (2.218), (2.219) and (2.220). This classification, which is similar to that of Table 2.1, may be depicted as follows: (Note: The sign of S does not change for $-\bar{\downarrow}, \bar{\uparrow}$ because S is a non-linear Casimir generator, see (2.221), (2.222)).

	s^3	s	
$\uparrow$	$\frac{1}{2}$	$\frac{1}{2}$	
$\downarrow$	$-\frac{1}{2}$	$\frac{1}{2}$	
$-\bar{\downarrow}$	$\frac{1}{2}$	$\frac{1}{2}$	
$\bar{\uparrow}$	$-\frac{1}{2}$	$\frac{1}{2}$	
$-\bar{\downarrow}\uparrow$	1	1	$\vert s=1, s^3=1\rangle$
$\bar{\uparrow}\downarrow$	-1	1	$\vert s=1, s^3=-1\rangle$
$\bar{\uparrow}\uparrow$	0	1	
$-\bar{\downarrow}\downarrow$	0	1	
$\frac{1}{2}[\bar{\uparrow}\uparrow - \bar{\downarrow}\downarrow]$	0	1	$\vert s=1, s^3=0\rangle$
$\frac{1}{2}[\bar{\uparrow}\uparrow + \bar{\downarrow}\downarrow]$	0	0	$\vert s=0, s^3=0\rangle$.

Table 2.2 - Spin Degrees of Freedom for Fermion and Boson Polarizations

where,

$$s^3 = \tfrac{1}{2}\sigma^3 \qquad (2.225)(a)$$

$$s = (2/3)\left[(\tfrac{1}{2}\sigma^1)^2 + (\tfrac{1}{2}\sigma^2)^2 + (\tfrac{1}{2}\sigma^3)^2\right] , \qquad (2.225)(b)$$

where the various fermion polarizations are given by: (f=Fermion)

$$\uparrow \;= \left| s=\tfrac{1}{2} \;,\; s^3= \tfrac{1}{2}\right\rangle \qquad (2.226)(a)$$

$$\downarrow \;= \left| s=\tfrac{1}{2} \;,\; s^3=-\tfrac{1}{2}\right\rangle \qquad (2.226)(b)$$

$$-\bar{\downarrow} \;= \left| s=\tfrac{1}{2} \;,\; s^3= \tfrac{1}{2}\right\rangle \qquad (2.226)(c)$$

$$\bar{\uparrow} \;= \left| s=\tfrac{1}{2} \;,\; s^3=-\tfrac{1}{2}\right\rangle , \qquad (2.226)(d)$$

while the various real and virtual vector boson polarizations are given by: (B=Boson)

$$B^u\left[\tfrac{1}{2}\bar{\phi}\sigma^0\phi\right] = B^u\left[\tfrac{1}{2}(\bar{\uparrow}\uparrow+\bar{\downarrow}\downarrow)\right] = \left|S=0 \;,\; s^3=0\right\rangle \qquad (2.227)(a)$$

$$\begin{cases} B^u\left[-\tfrac{1}{2}\bar{\phi}(\sigma^1 + i\sigma^2)\phi\right] = B^u\left[-\bar{\downarrow}\uparrow\right] = \left|S=1 \;,\; s^3=1\right\rangle & (2.227)(b) \\ B^u\left[\tfrac{1}{2}\bar{\phi}\sigma^3\phi\right] = B^u\left[\tfrac{1}{2}(\bar{\uparrow}\uparrow - \bar{\downarrow}\downarrow)\right] = \left|S=1 \;,\; s^3=0\right\rangle & (2.227)(c) \\ B^u\left[\tfrac{1}{2}\bar{\phi}(\sigma^1 - i\sigma^2)\phi\right] = B^u\left[\bar{\uparrow}\downarrow\right] = \left|S=1 \;,\; s^3=-1\right\rangle . & (2.227)(d) \end{cases}$$

For a massive boson at rest, using (2.198) and defining $\epsilon_u{}^{(3)}=(0,0,0,1)$, the above may be used to label the polarization vectors (2.212) and single particle energy tensor (2.210) of the spin sum (2.211), in a little more detail. Specifically, if we label each polarization state with the ordered pair (S,S^3) from (2.227), it is now possible to write:

$$\epsilon_u^{(1,1)} = (\ 0,\ -1,\ -i,\ 0)/\sqrt{2} = -(\epsilon_u^{(1)} + i\epsilon_u^{(2)}) \qquad (2.228)(a)$$

$$\epsilon_u^{(1,0)} = (\ 0,\ 0,\ 0,\ 1) = \epsilon_u^{(3)} \qquad (2.228)(b)$$

$$\epsilon_u^{(1,-1)} = (\ 0,\ 1,\ -i,\ 0)/\sqrt{2} = (\epsilon_u^{(1)} - i\epsilon_u^{(2)}) \qquad (2.228)(c)$$

for the polarization vectors of a massive Boson, and

$$\frac{E_u E_v}{m^2c^4} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = -\epsilon_u^{(0,0)*}\epsilon_v^{(0,0)} \qquad (2.229)$$

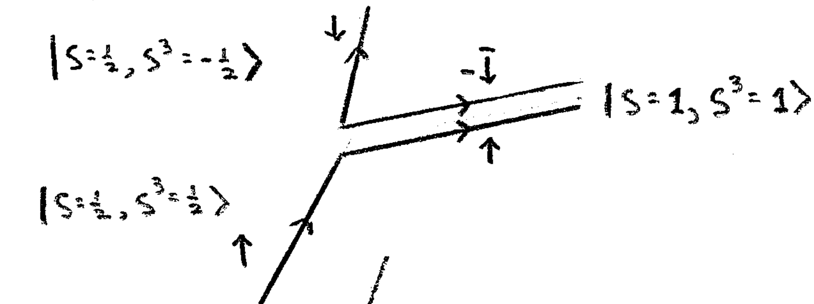
for the virtual polarization state. (2.229) above in particular, shows with particular clarity the manner in which the energy tensor for a single particle (at rest) enters into the overall spin sum. If we further form the square matrix product of the Dirac matrices γ^k, somewhat analogously to eq. (2.220) for the Pauli matrices, one may show explicitly the polarization vectors (2.228), ie., (Note, $-\epsilon^{(1,-1)u} = \epsilon_*^{(1,1)u}$, $-\epsilon^{(1,1)u} = \epsilon_*^{(1,-1)u}$)

$$\begin{pmatrix} 0 & 0 & \gamma_3 & \gamma_1 - i\gamma_2 \\ 0 & 0 & \gamma_1 + i\gamma_2 & -\gamma_3 \\ -\gamma_3 & -\gamma_1 + i\gamma_2 & 0 & 0 \\ \gamma_1 - i\gamma_2 & \gamma_3 & 0 & 0 \end{pmatrix}$$

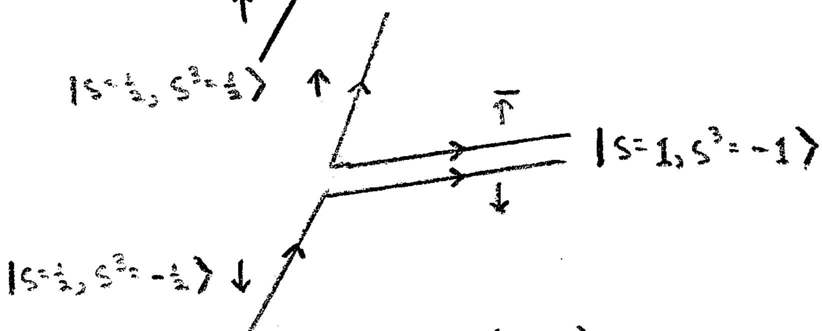
$$= \begin{pmatrix} 0 & 0 & \epsilon^{(1,0)u}\gamma_u & \sqrt{2}\epsilon^{(1,-1)u}\gamma_u \\ 0 & 0 & -\sqrt{2}\epsilon^{(1,1)u}\gamma_u & -\epsilon^{(1,0)u}\gamma_u \\ -\epsilon^{(1,0)u}\gamma_u & -\sqrt{2}\epsilon^{(1,-1)u}\gamma_u & 0 & 0 \\ \sqrt{2}\epsilon^{(1,1)u}\gamma_u & \epsilon^{(1,0)u}\gamma_u & 0 & 0 \end{pmatrix}. \qquad (2.230)$$

In fact, these sort of square matrix products are particularly useful mathematical tools in a variety of situations, and it pays to become familiar with the various ways in which these types of product can be utilized. (2.220) is another example of this.

With this, one may return to the spin state diagrams of Section 6, to develop these diagrams a little more precisely. In particular, one may start with the particular polarization compositions given in (2.227) and work backwards toward the relevent Feynman diagrams for the various real and virtual (covariant) spin vertices. From the (1,1) and (1,-1) states, one may construct the diagrams:



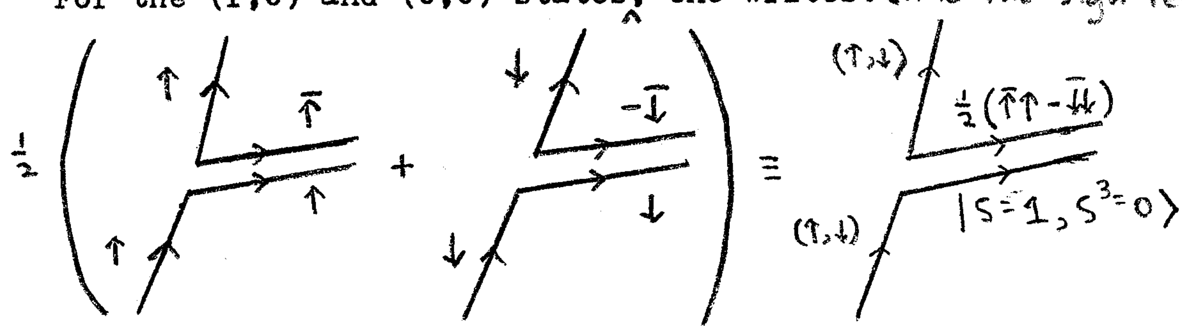
E. (2.228)(a)



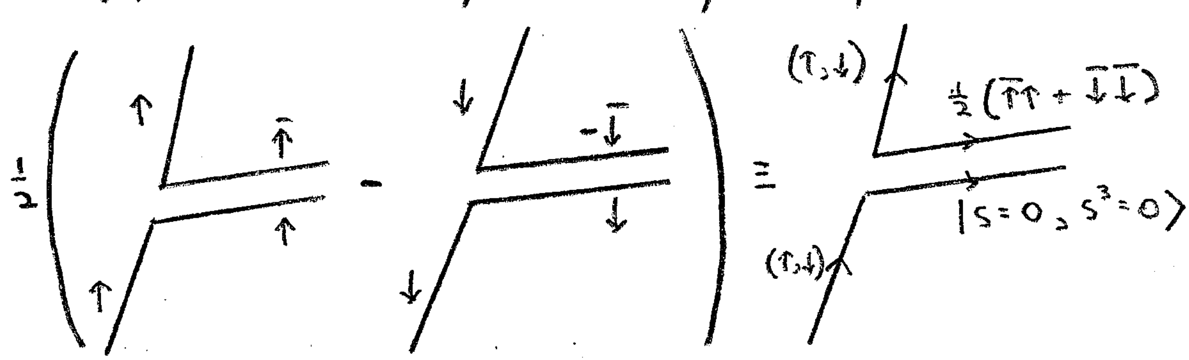
E. (2.228)(b)

.

For the (1,0) and (0,0) states, in (2.227) one writes: (note the sign reversal)



E.(2.228)(c)



E.(2.228)(d)

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Note that these are constructed from various linear combinations of the diagrams from Section 6.

It is also useful to graphically represent these various polarizations in terms of their spin space decomposition. In particular, one may represent the $2 \otimes \bar{2} = 3 \oplus 1$ composition of the boson spin states (2.227) out of the fermion states (2.226) according to Table 2.2 in the following manner:

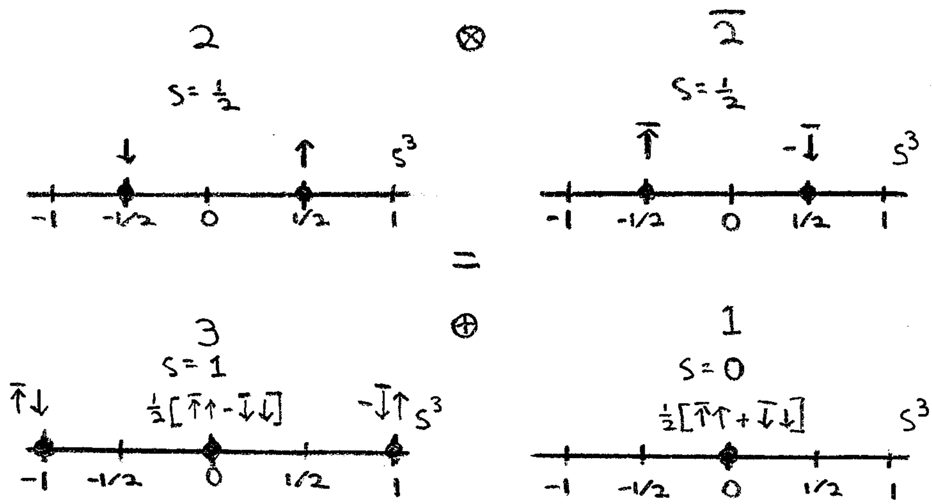

Figure 2.1 - Spin Space Decomposition of Boson Polarizations Into Fermion Polarizations

This form of graphical representation is very important, and will be utilized extensively in the later discussions of flavor symmetry and grand unification.