

Exact Quantum Yang Mills Propagators

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Abstract:

We show how to obtain an exact propagator in quantum Yang Mills theory, without compromising the gauge symmetry by splitting the Lagrangian into harmonic and perturbative parts, and without foregoing the Lorentz symmetry via a lattice gauge theory. In the course of reviewing the gauge symmetry of the exact propagator, we find that a perturbation tensor in the exact propagator transforms in exactly the same way as the gravitational field is known to transform.

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1. Introduction

It is well known, and can be found in virtually any elementary textbook on particle physics or quantum field theory e.g., [1], equation (14.40), that the field strength tensor for a Yang-Mills (non-Abelian) gauge theory is:

$$F^{i\mu\nu} = \partial^\mu G^{i\nu} - \partial^\nu G^{i\mu} - gf^{ijk} G_j^\mu G_k^\nu \quad (1.1)$$

where the G_i^μ are the gauge bosons (classical potentials) of whatever Yang-Mills group one is using (for instance, weak SU(2) or SU(3) QCD), f^{ijk} are the group structure constants, g is the group charge strength, and the Latin internal symmetry index $i = 1, 2, 3, \dots, N^2 - 1$ for SU(N) is raised and lowered with the unit matrix δ_{ij} . Multiplying (1.1) through by the group generators T^i which are $N \times N$ matrices for SU(N), and employing the group structure $f^{ijk} T_i = -i[T^j, T^k]$, one can readily rewrite (1.1) as:

$$F^{\mu\nu} = \partial^\mu G^\nu - \partial^\nu G^\mu + ig[G^\mu, G^\nu], \quad (1.2)$$

where $F^{\mu\nu} \equiv T^i F_i^{\mu\nu}$ and $G^\mu \equiv T^i G_i^\mu$ are $N \times N$ matrices for SU(N). Multiplying through by $dx_\mu dx_\nu$, and using the forms $G = G^\mu dx_\mu$, $F = \frac{1}{2!} F^{\mu\nu} dx_\mu \wedge dx_\nu = F^{\mu\nu} dx_\mu dx_\nu$, $G^2 = [G, G] = \frac{1}{2!} [G^\mu, G^\nu] dx_\mu \wedge dx_\nu = [G^\mu, G^\nu] dx_\mu dx_\nu$, $dG = \partial^\mu G^\nu dx_\mu \wedge dx_\nu = (\partial^\mu G^\nu - \partial^\nu G^\mu) dx_\mu dx_\nu$ in well-known fashion, this further compacts in the language of differential forms to (see [2], Chapter (4.5)):

$$F = dG + igG^2. \quad (1.3)$$

In QED, the potential one-form is designated as A , the field density two-form is simply $F = dA$, and using duality notation which will be further explored momentarily, the electric charge (source) current density three-form equation is $*J = d*F = d*dA$. It is well-known how to exactly invert this to obtain an exact propagator for both massive bosons, and via gauge fixing, for massless field quanta such as the photon.

But it is not known how to invert and analytically deduce exact propagators in Yang-Mills gauge theories from the corresponding expression $*J = d*F = d*dG + igd*G^2$, due to the extra G^2 which is the hallmark of Yang Mills gauge theories. This paper shows how to do this, without the usual compromises inherent in either perturbation theory or lattice gauge theory, see [2] section VII.1.

2. Integration by Parts of the ‘‘Holistic’’ Yang-Mills Lagrangian

In QED, as noted just above, in the language of compact differential forms, Maxwell’s equation for the electric charge (source) current density is given by the three-form equation:

$$*J = d*F = d*dA \quad (2.1)$$

with $F = dA$, where “*” denotes “duality,” defined in tensor notation as $*F^{\mu\nu} = \frac{1}{2!}\epsilon^{\mu\nu\sigma\tau}F_{\sigma\tau}$ for the field strength (and for a second rank antisymmetric tensor generally) and $*J^{\sigma\mu\nu} = \epsilon^{\sigma\mu\nu\tau}J_{\tau}$ for the current density (and for first and third rank antisymmetric tensors generally), and where $\epsilon^{\mu\nu\sigma\tau}$ is the totally-antisymmetric Levi-Civita tensor. The duality formalism was first developed by Reinich [3], and later elaborated by Wheeler, see [4], and [5], sections 3 and 4. This expands in tensor notation to the equivalent relationship:

$$J^{\nu} = \partial_{\mu}F^{\mu\nu} = \partial_{\mu}\partial^{[\mu}A^{\nu]} = \partial_{\mu}(\partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}) = \partial_{\mu}\partial^{\mu}A^{\nu} - \partial_{\mu}\partial^{\nu}A^{\mu} = (g^{\mu\nu}\partial^{\sigma}\partial_{\sigma} - \partial^{\mu}\partial^{\nu})A_{\mu}, \quad (2.2)$$

which includes the familiar $F^{\mu\nu} = \partial^{[\mu}A^{\nu]} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}$. To go from (2.1) to (2.2) one uses $*J = \frac{1}{3!}*J^{\sigma\mu\nu}dx_{\sigma} \wedge dx_{\mu} \wedge dx_{\nu} = *J^{\sigma\mu\nu}dx_{\sigma}dx_{\mu}dx_{\nu}$ to expand the current term, the duality relations $*J^{\sigma\mu\nu} = \epsilon^{\sigma\mu\nu\tau}J_{\tau}$ and $*F^{\mu\nu} = \frac{1}{2!}\epsilon^{\mu\nu\alpha\tau}F_{\alpha\tau}$, and of course, the expansion $F = \frac{1}{2!}F^{\mu\nu}dx_{\mu} \wedge dx_{\nu} = F^{\mu\nu}dx_{\mu}dx_{\nu}$ together with the definition of the exterior derivative d . For generality, to consider the situation when the vector boson A^{μ} has a non-zero mass m (knowing that the photon itself is massless), a mass term is often added to (2.2), thus:

$$J^{\nu} = (g^{\mu\nu}(\partial^{\sigma}\partial_{\sigma} + m^2) - \partial^{\mu}\partial^{\nu})A_{\mu}. \quad (2.3)$$

The inclusion of a mass (which formally speaking enters via spontaneous symmetry breaking using Goldstone bosons via the Higgs mechanism) also has the benefit of supplying the transverse degree of freedom for the vector boson, because $g^{\mu\nu}(\partial^{\sigma}\partial_{\sigma} + m^2) - \partial^{\mu}\partial^{\nu}$ has an inverse, while $g^{\mu\nu}\partial^{\sigma}\partial_{\sigma} - \partial^{\mu}\partial^{\nu}$ does not and so requires gauge fixing to obtain a propagator.

The above expresses J^{ν} as a function of A_{μ} , that is $J^{\nu}(A_{\mu})$. But it is desirable to also know the inverse expression $A_{\mu}(J^{\nu})$. The most direct way to do this, is to develop the propagator $D_{\nu\lambda}(p^{\sigma})$ in Fourier momentum space, which is specified as the inverse of the configuration space differential operator $g^{\mu\nu}(\partial^{\sigma}\partial_{\sigma} + m^2) - \partial^{\mu}\partial^{\nu}$, thus:

$$D_{\nu\lambda}(p^{\sigma})(g^{\mu\nu}(\partial^{\sigma}\partial_{\sigma} + m^2) - \partial^{\mu}\partial^{\nu})e^{ip^{\sigma}x_{\sigma}} = \delta^{\mu}_{\lambda}e^{ip^{\sigma}x_{\sigma}}, \quad (2.4)$$

where we have also multiplied each side by the Fourier kernel $e^{ip^{\sigma}x_{\sigma}}$.^{*} Taking the derivatives $\partial^{\mu}e^{ip^{\sigma}x_{\sigma}} = ip^{\mu}e^{ip^{\sigma}x_{\sigma}}$ and then removing $e^{ip^{\sigma}x_{\sigma}}$, we arrive at:

$$D_{\nu\lambda}(p^{\sigma})(-g^{\mu\nu}(p^{\sigma}p_{\sigma} - m^2) + p^{\mu}p^{\nu}) = \delta^{\mu}_{\lambda}, \quad (2.5)$$

As is well-known, this then inverts in the usual way to:

^{*} Often, one just makes the substitution $\partial^{\mu} \rightarrow ip^{\mu}$, but here we use the Fourier kernel to illustrate the underlying Fourier transform that is entailed in doing this.

$$D_{\mu\nu}(p^\sigma) = \frac{-g_{\mu\nu} + p_\mu p_\nu / m^2}{p^\sigma p_\sigma - m^2 + i\mathcal{E}}, \quad (2.6)$$

where we have also applied the “ $+i\mathcal{E}$ prescription.” Thus, with some renaming and reshuffling of indexes, we may write the inverse momentum space relation for $A_\mu(J^\nu)$ as:

$$A^\mu(p^\sigma) = D^{\mu\nu}(p^\sigma) J_\nu = \frac{-g^{\mu\nu} + p^\mu p^\nu / m^2}{p^\sigma p_\sigma - m^2 + i\mathcal{E}} J_\nu(p^\sigma). \quad (2.7)$$

For a zero-mass boson (e.g., the photon), via well-researched gauge fixing techniques, (see, for example, [2] section III.4) the corresponding expression is:

$$A^\mu = \frac{-g^{\mu\nu} + (1-\xi)p^\mu p^\nu / p^\sigma p_\sigma}{p^\sigma p_\sigma + i\mathcal{E}} J_\nu. \quad (2.8)$$

Finally, in the Feynman gauge, $\xi = 1$, and with $\mathcal{E} = 0$, this simplifies to:

$$A^\mu(p^\sigma) = -\frac{1}{p^\sigma p_\sigma} J^\mu(p^\sigma). \quad (2.9)$$

We now seek to develop similar inverse relationships for Yang-Mills theory.

In Yang-Mills theory, the relationship (2.1) now employs the non-Abelian field $F = dG + igG^2$ of (1.3), and so reads:

$$*J = d * F = d * (dG + igG^2) = d * dG + igd * G^2. \quad (2.10)$$

When expanded out similarly to (2.2), this is equivalent to:

$$J^\nu = \partial_\mu F^{\mu\nu} = \partial_\mu (\partial^\mu G^\nu - \partial^\nu G^\mu + ig[G^\mu, G^\nu]). \quad (2.11)$$

This now employs the non-Abelian field strength $F^{\mu\nu} = \partial^\mu G^\nu - \partial^\nu G^\mu + ig[G^\mu, G^\nu]$ of (1.2). At first sight, developing the inverse $G_\mu(J^\nu)$ appears daunting because of the added term $igd * G^2$ which gives Yang-Mills theory its distinguishing non-linear features. In fact, the difficulty of quantizing Yang-Mills gauge theory starting with propagator development is what has led to perturbative approaches or lattice gauge theory which are not exact and which “brutalize Yang-Mills theory,” either causing the “mangling of gauge invariance” by splitting the Lagrangian term $\text{Tr}(F^{\mu\nu} F_{\mu\nu})$ into a “harmonic oscillator” (the Abelian terms) and a “perturbation” (with all the non-Abelian terms), or by maintaining gauge symmetry but spoiling the Lorentz invariance ([2] at 356). In fact, one can see just how daunting is to try to invert (2.11), by writing out the fully expanded, and rather ugly expression:

$$J^{i\nu} = \partial_\mu F^{i\mu\nu} = \partial_\mu \partial^\mu G^{i\nu} - \partial_\mu \partial^\nu G^{i\mu} + gf^{ijk} \partial_\mu G_j^\mu G_k^\nu + gf^{ijk} G_j^\mu \partial_\mu G_k^\nu. \quad (2.12)$$

And if this is not ugly enough, the pure field term which appears in the Yang-Mills Lagrangian is even worse:

$$\begin{aligned} -\frac{1}{4} F_i^{\mu\nu} F^{i\mu\nu} &= -\frac{1}{4} \left(\partial^\mu G_i^\nu - \partial^\nu G_i^\mu + gf_{ijk} G^j{}^\mu G^k{}^\nu \right) \left(\partial_\mu G^i{}_\nu - \partial_\nu G^i{}_\mu + gf^{imn} G_{m\mu} G_{n\nu} \right) \\ &= -\frac{1}{4} \left(\partial^\mu G_i^\nu - \partial^\nu G_i^\mu \right) \left(\partial_\mu G^i{}_\nu - \partial_\nu G^i{}_\mu \right) \\ &\quad - \frac{1}{2} g \left(\partial^\mu G_i^\nu - \partial^\nu G_i^\mu \right) f^{imn} G_{m\mu} G_{n\nu} - \frac{1}{4} g^2 f_{ijk} f^{imn} G^j{}^\mu G^k{}^\nu G_l{}_\mu G_m{}_\nu \end{aligned} \quad (2.13)$$

The harmonic terms are in $\left(\partial^\mu G_i^\nu - \partial^\nu G_i^\mu \right) \left(\partial_\mu G^i{}_\nu - \partial_\nu G^i{}_\mu \right)$, and the remaining terms of third and fourth order in $G_i{}^\mu$ represent perturbation. Being able to obtain exact solutions quantizing Yang-Mills theories which do not require separate treatment of these two sets of terms or other workarounds remains an unsolved problem, to date.

But this problem can be solved starting with some clever rearrangement of (2.11). We can first write the non-Abelian $F^{\mu\nu}$ of (2.11) in very compact form, as:

$$F^{\mu\nu} = \left(\partial^\mu + igG^\mu \right) G^\nu - \left(\partial^\nu + igG^\nu \right) G^\mu = D^\mu G^\nu - D^\nu G^\mu = D^{[\mu} G^{\nu]}. \quad (2.14)$$

where we have defined the gauge-covariant derivative:

$$D^\mu \equiv \partial^\mu + igG^\mu. \quad (2.15)$$

This enables the Yang-Mills current density (2.11) to be written in the highly simplified form:

$$J^\nu = \partial_\mu F^{\mu\nu} = \partial_\mu D^{[\mu} G^{\nu]}, \quad (2.16)$$

Compare the non-Abelian $J^\nu = \partial_\mu F^{\mu\nu} = \partial_\mu \partial^{[\mu} A^{\nu]}$ of (2.2) which is Maxwell's charge equation. From this view, Yang-Mills theory differs in form from Abelian gauge theory simply insofar as the $F^{\mu\nu} = \partial^{[\mu} G^{\nu]}$ of Abelian gauge theories is replaced by $F^{\mu\nu} = D^{[\mu} G^{\nu]}$, with $\partial^\mu \rightarrow D^\mu \equiv \partial^\mu + igG^\mu$ employed to go over to Yang-Mills theory. We are, in some sense, "applying gauge theory to vector gauge fields," by using the gauge-covariant derivative D^μ . We now wish to exploit this simplified form (2.16) to obtain an exact Yang Mills propagator analogous to (2.6).

Taking the inverse directly from (2.16) as we did starting with (2.3) and (2.4) is still difficult, in particular because (2.16) contains both an ordinary derivative ∂_μ and the gauge-covariant derivative D^μ . A fruitful path, which has the collateral benefit of avoiding the separation of Yang-Mills fields into "harmonic" and "perturbation" terms and so avoids "the perturbative approach with its mangling of gauge invariance" ([2] at 356) and which preserves the Lorentz invariance as well, employs careful integration-by-parts of the Lagrangian for the Yang-Mills field $F^{\mu\nu}$ as specified in (2.14), as the first step to developing the exact Yang-Mills propagator.

In this approach, we start with the Lagrangian for the complete, "holistic" ([2] at 356)

unseparated Yang-Mills field density, using our compact notation:

$$\begin{aligned}\mathcal{L} &= -\frac{1}{2}\text{Tr}(F^{\mu\nu}F_{\mu\nu}) = -\frac{1}{2}\text{Tr}(D^{[\mu}G^{\nu]}D_{[\mu}G_{\nu]}) = -\frac{1}{2}\text{Tr}((D^\mu G^\nu - D^\nu G^\mu)(D_\mu G_\nu - D_\nu G_\mu)) \\ &= -\text{Tr}(D^\mu G^\nu D_{[\mu}G_{\nu]})\end{aligned}\quad (2.17)$$

Following a tedious, but totally-algebraic calculation based on the term $D^\mu G^\nu D_{[\mu}G_{\nu]}$, one finds that (2.17) above can be rewritten as:

$$\mathcal{L} = -\frac{1}{2}\text{Tr}(F^{\mu\nu}F_{\mu\nu}) = -\text{Tr}(D^\mu G^\nu D_{[\mu}G_{\nu]}) = -\text{Tr}(\partial^\mu(G^\nu D_{[\mu}G_{\nu]}) - G^\nu D^\mu D_{[\mu}G_{\nu]}). \quad (2.18)$$

This still contains a mix of ordinary (∂^μ) and gauge-covariant (D^μ) derivatives, but we are not done yet. Now, we integrate by parts. First, we integrate throughout to write:

$$-\frac{1}{2}\text{Tr}\int(F^{\mu\nu}F_{\mu\nu})d^4x = -\text{Tr}\int(\partial^\mu(G^\nu D_{[\mu}G_{\nu]}))d^4x + \text{Tr}\int(G^\nu D^\mu D_{[\mu}G_{\nu]})d^4x, \quad (2.19)$$

Upon definite integration from $-\infty$ to $+\infty$ in configuration space, the first term can be set to zero by imposing the boundary conditions $G^\nu(x^\mu = +\infty) = G^\nu(x^\mu = -\infty) = 0$, as such:*

$$-\text{Tr}\int(\partial^\mu(G^\nu D_{[\mu}G_{\nu]}))d^4x = -\text{Tr}\int\left(\frac{\partial}{\partial x_\mu}(G^\nu D_{[\mu}G_{\nu]})\right)d^4x = -\text{Tr}\int d^3x(G^\nu D_{[\mu}G_{\nu]})\Big|_{G^\nu(-\infty)}^{G^\nu(+\infty)} = 0. \quad (2.20)$$

Consequently, (2.19) simplifies to:

$$-\frac{1}{2}\text{Tr}\int(F^{\mu\nu}F_{\mu\nu})d^4x = +\text{Tr}\int(G^\nu D^\mu D_{[\mu}G_{\nu]})d^4x. \quad (2.21)$$

This finally eliminates the term with the ordinary derivative ∂^μ (which was pesky on the one hand, but essential on the other hand to carry out the definite integration in (2.20)). Now, with expansion of $D_{[\mu}G_{\nu]}$ and a bit of algebraic rearranging, this becomes:

$$\boxed{-\frac{1}{2}\text{Tr}\int(F^{\mu\nu}F_{\mu\nu})d^4x = +\text{Tr}\int G_\mu(g^{\mu\nu}D^\sigma D_\sigma - D^\nu D^\mu)G_\nu d^4x.} \quad (2.22)$$

It is a very significant result that this is identical in form to the analogous QED relations:

$$-\frac{1}{4}\int(F^{\mu\nu}F_{\mu\nu})d^4x = \frac{1}{2}\int A_\mu(g^{\mu\nu}\partial^\sigma\partial_\sigma - \partial^\nu\partial^\mu)A_\nu d^4x. \quad (2.23)$$

And (2.22) differs from (2.23) only insofar as the term $g^{\mu\nu}\partial^\sigma\partial_\sigma - \partial^\nu\partial^\mu$ is replaced by the term $g^{\mu\nu}D^\sigma D_\sigma - D^\nu D^\mu$ which employs the gauge-covariant derivative D^μ rather than the ordinary derivative ∂^μ . All of the unwieldy non-linearity of non-Abelian Yang-Mills gauge theory, originating in the term igG^2 in (1.1), is cleanly carried through the integration by parts by the

* One may actually impose the less-stringent ‘‘isotropic’’ condition $G^\nu(x^\mu = +\infty) = G^\nu(x^\mu = -\infty)$ to obtain the same result, though the benefit of so doing is unclear and the end result is the same.

covariant derivative $D^\mu = \partial^\mu + igG^\mu$. The result (2.22) provides the great benefit of not mangling the gauge invariance or the Lorentz invariance of Yang Mills theory. And, at the end of the day, the differential operator $g^{\mu\nu}D^\sigma D_\sigma - D^\nu D^\mu$ may now be used to develop exact Yang-Mills propagators.

3. Development of Yang Mills Propagators

Before proceeding further, there is one other detail which need to be addressed, dealing with the commutation of $\partial^\nu \partial^\mu$ and $D^\nu D^\mu$. In flat spacetime partial derivatives commute, but in curved spacetime they do not. Indeed, $R^\alpha{}_{\beta\mu\nu}A_\alpha \equiv \partial_{;\nu}\partial_{;\mu}A_\beta - \partial_{;\mu}\partial_{;\nu}A_\beta = [\partial_{;\nu}, \partial_{;\mu}]A_\beta$, the Riemann curvature tensor, is *defined* via parallel transport as a measure of the degree to which gravitational covariant derivatives $\partial_{;\mu}$ commute. If $R^\alpha{}_{\beta\mu\nu} = 0$, then $[\partial_{;\nu}, \partial_{;\mu}] = [\partial_\nu, \partial_\mu] = 0$, and the covariant derivatives become ordinary derivatives and do commute. Otherwise, they do not commute.

But when it comes to the $D^\nu D^\mu$ term in $g^{\mu\nu}D^\sigma D_\sigma - D^\nu D^\mu$, we do not have the luxury of commuting D^ν and D^μ , *even in flat spacetime*. This is because

$$D^\nu D^\mu = (\partial^\nu + igG^\nu)(\partial^\mu + igG^\mu), \quad (3.1)$$

and it is very clear that the Yang-Mills gauge theory $[G^\mu, G^\nu] \neq 0$. Indeed, $[G^\mu, G^\nu] \neq 0$ is the whole point of Yang-Mills, see (1.2).

Now, in the term $g^{\mu\nu}D^\sigma D_\sigma - D^\nu D^\mu$, we require a tensor which is symmetric in μ, ν , because if not, then the propagators will also be non-symmetric in μ, ν . Thus, we take the active step of fully symmetrizing $g^{\mu\nu}D^\sigma D_\sigma - D^\nu D^\mu$, by using an anticommutator to set $D^\nu D^\mu \rightarrow \frac{1}{2}\{D^\nu, D^\mu\}$. Nothing needs to be done with $g^{\mu\nu}$, because that is already symmetric. Thus, we change (2.22) to:

$$-\frac{1}{2}\text{Tr} \int (F^{\mu\nu} F_{\mu\nu}) d^4x = +\text{Tr} \int G_\mu (g^{\mu\nu} D^\sigma D_\sigma - \frac{1}{2}\{D^\mu, D^\nu\}) G_\nu d^4x. \quad (3.2)$$

Thus, in place of (2.4), we use the differential configuration space operator in (3.2) to specify the Yang-Mills propagator as:

$$D_{\nu\lambda}(p^\sigma) (g^{\mu\nu} (D^\sigma D_\sigma + m^2) - \frac{1}{2}\{D^\mu, D^\nu\}) e^{ip^\sigma x_\sigma} = \delta^\mu{}_\lambda e^{ip^\sigma x_\sigma}, \quad (3.3)$$

where we have added a mass term in the same fashion as is done in Abelian QED. Now, we expand this a bit, to write:

$$\begin{aligned} & D_{\nu\lambda}(p^\sigma) (g^{\mu\nu} ((\partial^\sigma + igG^\sigma)(\partial_\sigma + igG_\sigma) + m^2) - \frac{1}{2}\{(\partial^\nu + igG^\nu), (\partial^\mu + igG^\mu)\}) e^{ip^\sigma x_\sigma} \\ &= D_{\nu\lambda}(p^\sigma) (g^{\mu\nu} (-(p^\sigma + gG^\sigma)(p_\sigma + gG_\sigma) + m^2) + \frac{1}{2}\{(p^\nu + gG^\nu), (p^\mu + gG^\mu)\}) e^{ip^\sigma x_\sigma}, \quad (3.4) \\ &= D_{\nu\lambda}(p^\sigma) (-g^{\mu\nu} (\pi^\sigma \pi_\sigma - m^2) + \frac{1}{2}\{\pi^\nu, \pi^\mu\}) e^{ip^\sigma x_\sigma} = \delta^\mu{}_\lambda e^{ip^\sigma x_\sigma} \end{aligned}$$

where for notational compactness, we have defined a Yang-Mills momentum term:

$$\pi^\sigma \equiv p^\sigma + gG^\sigma. \quad (3.5)$$

Stripping the Fourier kernel $e^{ip^\sigma x_\sigma}$ from (3.4) then yields:

$$D_{\nu\lambda}(p^\sigma) \left(-g^{\mu\nu} (\pi^\sigma \pi_\sigma - m^2) + \frac{1}{2} \{ \pi^\nu, \pi^\mu \} \right) = \delta^\mu_\lambda, \quad (3.6)$$

The above is identical in form to (2.5), with the sole replacement $p^\sigma \rightarrow \pi^\sigma$, and the use of the anticommutator $\frac{1}{2} \{ \pi^\nu, \pi^\mu \}$ to ensure μ, ν symmetry. For here, inversion is strictly and simply an algebraic operation, and (2.6) tells us the basic form of the result. So with the “ $+i\epsilon$ prescription,” we find that the exact Yang-Mills propagator is:

$$\boxed{D_{\mu\nu}(p^\sigma) = \left(-g_{\mu\nu} + \frac{\frac{1}{2} \{ \pi_\mu, \pi_\nu \}}{m^2} \right) \times (\pi^\sigma \pi_\sigma - m^2 + i\epsilon)^{-1} \\ = \left(-g_{\mu\nu} + \frac{\frac{1}{2} \{ (p_\mu + gG_\mu), (p_\nu + gG_\nu) \}}{m^2} \right) \times ((p^\sigma + gG^\sigma)(p_\sigma + gG_\sigma) - m^2 + i\epsilon)^{-1}}, \quad (3.7)}$$

Aside from the migration of $p^\sigma \rightarrow \pi^\sigma \equiv p^\sigma + gG^\sigma$ (which directly originates from the “trick” of the fact that $\partial^\mu \rightarrow D^\mu \equiv \partial^\mu + igG^\mu$ in the field density $F^{\mu\nu} = \partial^{[\mu} G^{\nu]}$ $\rightarrow D^{[\mu} G^{\nu]}$ when going from Abelian to Yang-Mills gauge theory), and the symmetrizing in $\frac{1}{2} \{ \pi_\mu, \pi_\nu \}$, the only difference between this and (2.6), is that we do not place the term $\pi^\sigma \pi_\sigma - m + i\epsilon$ in a denominator, because the G^σ are $N \times N$ matrices for the Yang-Mills group $SU(N)$, and so this term need to be properly inverted using matrix inversion mathematics. While it should be clear that this matrix inversion to obtain $((p^\sigma + gG^\sigma)(p_\sigma + gG_\sigma) - m + i\epsilon)^{-1}$ is a very non-trivial piece of arithmetic work for any Yang-Mills group larger than $SU(2)$ (and even takes some work for $SU(2)$), nonetheless, (3.7) is an *exact* expression which does not perturbatively split Yang Mills theory into harmonic and perturbative parts and which fully maintains Lorentz invariance. This is a central piece of the puzzle for how to obtain the *exact* quantization of Yang Mills theories, and forms the basis for being able to carry out other exact calculations using quantum Yang-Mills gauge theory.

Now, from (3.7), let us focus for a moment on the terms:

$$\frac{1}{2} \{ (p_\mu + gG_\mu), (p_\nu + gG_\nu) \} = \frac{1}{2} \{ p_\mu, p_\nu \} + \frac{1}{2} g \{ \{ p_\mu, G_\nu \} + \{ G_\mu, p_\nu \} \} + \frac{1}{2} g^2 \{ G_\mu, G_\nu \} \quad (3.8)$$

and

$$(p^\sigma + gG^\sigma)(p_\sigma + gG_\sigma) = p^\sigma p_\sigma + g(p^\sigma G_\sigma + G^\sigma p_\sigma) + g^2 G^\sigma G_\sigma. \quad (3.9)$$

In QED, the Klein-Gordon equation may be written as $(\partial_\mu \partial^\mu + m^2)\phi = -V\phi$, where:

$$-V = ie(\partial^\sigma A_\sigma + A_\sigma \partial^\sigma) + e^2 A^\sigma A_\sigma \quad (3.10)$$

with V being the electromagnetic perturbation. (e.g., [1], page 85) If we convert (3.10) into momentum space via $\partial_\mu \rightarrow ip_\mu$ and then change $-e \rightarrow +g$ (the “+” sign here is because unlike in QED, we define the running charge strength to be positive $+g$ rather than negative $-e$) and make the symbolic substitution $A^\mu \rightarrow G^\mu$, then (3.10) migrates over to:

$$-V = g(p^\sigma G_\sigma + G_\sigma p^\sigma) + g^2 G^\sigma G_\sigma. \quad (3.11)$$

We see that this is *identical* to the last few terms in (3.9), which means that these terms in (3.9) are simply the Yang-Mills perturbation, carried through the entire integration by parts and the inversion used to derive the propagator. Therefore, let us define a second rank symmetric perturbation tensor:

$$-V_{\mu\nu} \equiv \frac{1}{2} g(\{p_\mu, G_\nu\} + \{G_\mu, p_\nu\}) + \frac{1}{2} g^2 \{G_\mu, G_\nu\} \quad (3.12)$$

which has a trace (scalar) equation for $V = V^\sigma_\sigma$ given by (3.11). Thus, (3.8) and (3.9) may be written as*:

$$\frac{1}{2} \{(p_\mu + gG_\mu), (p_\nu + gG_\nu)\} = \frac{1}{2} \{p_\mu, p_\nu\} - V_{\mu\nu} \quad (3.13)$$

and

$$(p^\sigma + gG^\sigma)(p_\sigma + gG_\sigma) = p^\sigma p_\sigma - V. \quad (3.14)$$

Keep in mind, however, that V is now a non-trivial $N \times N$ matrix for $SU(N)$, so we still cannot put it in a denominator but must properly use matrix inversion.

Using (3.13) and (3.14), we write the Yang-Mills propagator in a simpler form as:

$$\boxed{D_{\mu\nu}(p^\sigma) = \left(-g_{\mu\nu} + \frac{\frac{1}{2} \{p_\mu, p_\nu\} - V_{\mu\nu}}{m^2} \right) \times (p^\sigma p_\sigma - m^2 - V + i\epsilon)^{-1}}. \quad (3.15)$$

While this does show the contribution of the Yang-Mills “perturbation” $V_{\mu\nu}$ separately from that of the harmonic terms, this is an exact expression because it was derived by never separating and treating separately, the third and second lines of (2.13), respectively. We arrived at (3.15) by maintaining $F_i^{\mu\nu} F^i_{\mu\nu}$ as a “holistic” entity. It is worth noting that because V is an $N \times N$ matrix for $SU(N)$, the $+i\epsilon$ prescription is no longer necessary to avoid propagator poles for an on-shell

* In flat spacetime, $R^\alpha_{\beta\mu\nu} = 0$, we may still commute $[p_\mu, p_\nu] = 0$ so that $\frac{1}{2} \{p_\mu, p_\nu\} = p_\mu p_\nu = p_\nu p_\mu$, but otherwise we cannot assume that anything commutes. It is always safer to use the anticommutator, because even when the variables in the anticommutator *do* commute, the anticommutator will simply revert to a multiplication – in either order – of the two variables.

$p^\sigma p_\sigma - m^2 = 0$ particle, because in general, the on-shell expression $p^\sigma p_\sigma - m^2 - V \rightarrow -V$ is invertible, because in the general case, V is invertible. It will also be stated, without elaboration at this time, that the inverted matrix V^{-1} may provide a basis for calculating observed meson masses in a precise way, thus filling the so-called “mass gap.”

For a *massless* vector boson, following the usual gauge fixing, we get a propagator analogous to that used in (2.8), but with $p_\mu p_\nu \rightarrow p_\mu p_\nu - V_{\mu\nu}$, so that:

$$\boxed{D_{\mu\nu}(p^\sigma) = \left(-g_{\mu\nu} + (1 - \xi) \left(\frac{1}{2} \{p_\mu, p_\nu\} - V_{\mu\nu} \right) \times (p^\sigma p_\sigma - V)^{-1} \right) \times (p^\sigma p_\sigma - V + i\varepsilon)^{-1}}. \quad (3.16)$$

Here again, we must be careful not to put inverses of the $N \times N$ Yang-Mills matrices into any denominators.

In both of the propagators (3.15) and (3.16), we see the importance of the earlier symmetrizing $D^\nu D^\mu \rightarrow \frac{1}{2} \{D^\nu, D^\mu\}$, because this is what ensures that these propagators $D_{\mu\nu}(p^\sigma)$ are also symmetric in μ, ν . In Feynman gauge, $\xi = 1$, (3.16) simplifies to:

$$D_{\mu\nu}(p^\sigma) = -g_{\mu\nu} \times (p^\sigma p_\sigma - V + i\varepsilon)^{-1}. \quad (3.17)$$

Finally, for small perturbation, that is, for $V \rightarrow 0$, we can treat the matrix inverse $(p^\sigma p_\sigma - V + i\varepsilon)^{-1} \rightarrow 1/(p^\sigma p_\sigma + i\varepsilon)$ as an ordinary denominator. And if we further set $\varepsilon = 0$, then for a zero-mass vector boson, with $V \rightarrow 0$, (3.17) becomes:

$$D_{\mu\nu}(p^\sigma) = -\frac{1}{p^\sigma p_\sigma} g_{\mu\nu}, \quad (3.18)$$

just as it is in QED.

The propagator (3.15), which is *exact*, will be very useful for finally being able to obtain exact solutions to Yang-Mills gauge theories in a wide range of situations, and will finally bring Yang-Mills theories beyond the point where non-exact methods such as lattice gauge theories must be employed in conjunction with numerical / computational methods, because exact analytic solutions are not known.

Finally, the inverse relationship $G_\mu(J^\nu)$ we sought following (2.11), for a Yang-Mills theory, via (3.15), is given generally by:

$$G^\mu = D^{\mu\nu} J_\nu = \left(-g^{\mu\nu} + \frac{\frac{1}{2} \{p^\mu, p^\nu\} - V^{\mu\nu}}{m^2} \right) (p^\sigma p_\sigma - V - m^2 + i\varepsilon)^{-1} J_\nu. \quad (3.19)$$

One should observe, however, that because $-V_{\mu\nu} \equiv \frac{1}{2} g \left(\{p_\mu, G_\nu\} + \{G_\mu, p_\nu\} \right) + \frac{1}{2} g^2 \{G_\mu, G_\nu\}$, G^μ is now specified not only as a function of J^ν , but also as a function of itself, $G_\mu(J^\nu, G^\sigma)$, in a “recursive” manner. This reflects the inherent non-linearity of Yang-Mill gauge theory. For a massless vector boson in Feynman gauge, this reduces to:

$$G^\mu = D^{\mu\nu} J_\nu = -(p^\sigma p_\sigma - V + i\varepsilon)^{-1} J^\mu(p^\sigma). \quad (3.20)$$

Finally, for small perturbation $V \rightarrow 0$ and $\varepsilon = 0$, this becomes:

$$G^\mu = -\frac{1}{p^\sigma p_\sigma} J^\mu. \quad (3.21)$$

We shall show how a path to quantum gravitation may be contained in the results (3.15) and (3.16).

4. Similar Gauge Transformations of the Propagator Perturbation and the Gravitational Field

In non-Abelian gauge theory, $SU(N)$ for example, we define a series of $N^2 - 1$ local phase angles $\theta_\alpha(x^\mu)$ which we then form into the $N \times N$ matrix $\theta = T^\alpha \theta_\alpha(x^\mu)$. The non-Abelian gauge transformation, written as an $N \times N$ matrix equation, then takes the form:

$$G_\nu \rightarrow G_\nu + \partial_\nu \theta + i[\theta, G_\nu]. \quad (4.1)$$

We see embedded in this, the Abelian gauge transformation $G_\nu \rightarrow G_\nu + \partial_\nu \theta$. But it is this new term $i[\theta, G_\nu]$ which requires us to establish $F^{\mu\nu} = \partial^\mu G^\nu - \partial^\nu G^\mu + ig[G^\mu, G^\nu]$ for the Yang-Mills field density rather than the Abelian $F^{\mu\nu} = \partial^\mu G^\nu - \partial^\nu G^\mu$, and which generally drives the non-linear features of Yang-Mills gauge theory which make this distinctive over Abelian gauge theories such as QED.

Now, if we take a careful look at the propagators (3.15) and (3.16), we see that with the introduction of the perturbation $V^{\mu\nu}$, these propagators appear to be not invariant under a non-Abelian gauge transformation (4.1). Take $-V_{\mu\nu} \equiv \frac{1}{2} g \{ \{ p_\mu, G_\nu \} + \{ G_\mu, p_\nu \} \} + \frac{1}{2} g^2 \{ G_\mu, G_\nu \}$ from (3.12). Under a Yang-Mills gauge transformation (4.1), this transforms as (there is some cumbersome, but wholly algebraic manipulation used to get to the final equality):

$$\begin{aligned} -V_{\mu\nu} &= \frac{1}{2} g \{ \{ p_\mu, G_\nu \} + \{ G_\mu, p_\nu \} \} + \frac{1}{2} g^2 \{ G_\mu, G_\nu \} \\ \rightarrow -V'_{\mu\nu} &= \frac{1}{2} g \{ \{ p_\mu, (G_\nu + \partial_\nu \theta + i[\theta, G_\nu]) \} + \{ (G_\mu + \partial_\mu \theta + i[\theta, G_\mu]), p_\nu \} \} \\ &\quad + \frac{1}{2} g^2 \{ \{ (G_\mu + \partial_\mu \theta + i[\theta, G_\mu]), (G_\nu + \partial_\nu \theta + i[\theta, G_\nu]) \} \} \\ &= \frac{1}{2} g \{ \{ p_\mu, G_\nu \} + \{ G_\mu, p_\nu \} \} + \frac{1}{2} g^2 \{ G_\mu, G_\nu \} \\ &\quad + \frac{1}{2} \{ \{ p_\mu + gG_\mu, (g\partial_\nu \theta + ig[\theta, G_\nu]) \} + \{ (g\partial_\mu \theta + ig[\theta, G_\mu]), (p_\nu + gG_\nu) \} \} \\ &\quad + \frac{1}{2} \{ \{ g\partial_\mu \theta + ig[\theta, G_\mu], (g\partial_\nu \theta + ig[\theta, G_\nu]) \} \} \end{aligned} \quad (4.2)$$

If we now define the vector $i\Lambda_\mu$ as (the reason for including i in this definition will become apparent momentarily):

$$i\Lambda_\mu \equiv g\partial_\mu\theta + ig[\theta, G_\mu], \quad (4.3)$$

and also employ $\pi^\sigma \equiv p^\sigma + gG^\sigma$ from (3.5), then the essence of gauge transformation (4.2) of $-V_{\mu\nu}$, is:

$$\begin{aligned} -V_{\mu\nu} \rightarrow -V'_{\mu\nu} &= -V_{\mu\nu} + \frac{1}{2}(\{\pi_\mu, i\Lambda_\nu\} + \{i\Lambda_\mu, \pi_\nu\}) - \frac{1}{2}\{\Lambda_\mu, \Lambda_\nu\} \\ &= -V_{\mu\nu} + \frac{1}{2}(\pi_\mu i\Lambda_\nu + \pi_\nu i\Lambda_\mu + i\Lambda_\mu \pi_\nu + i\Lambda_\nu \pi_\mu) - \frac{1}{2}(\Lambda_\mu \Lambda_\nu + \Lambda_\nu \Lambda_\mu). \end{aligned} \quad (4.4)$$

Now, in gravitational theory, we know that under a general coordinate transformation $x^\mu \rightarrow x^\mu + \Lambda^\mu$, the metric tensor transforms as $g'^{\mu\nu} = (\partial x'^\mu / \partial x'^\alpha)(\partial x'^\nu / \partial x'^\beta)g^{\alpha\beta}$, while the gravitational field $h_{\mu\nu}$ undergoes a gravitational gauge-like transformation of the form:

$$h^{\mu\nu} \rightarrow h'^{\mu\nu} = h^{\mu\nu} + \partial^{[\mu}\Lambda^{\nu]} = h^{\mu\nu} + \partial^\mu\Lambda^\nu + \partial^\nu\Lambda^\mu. \quad (4.5)$$

The metric tensor is of course related to $h_{\mu\nu}$ according to $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} + O(h^2)$... with $\kappa = \sqrt{16\pi G/c^4}$.

Let us now Fourier transform (4.5) from configuration space to momentum space via $\partial^\mu \rightarrow ip^\mu$, thus:

$$h^{\mu\nu} \rightarrow h'^{\mu\nu} = h^{\mu\nu} + ip^\mu\Lambda^\nu + ip^\nu\Lambda^\mu. \quad (4.6)$$

Closely inspecting (4.4) for the Yang-Mills gauge transformation of the perturbation $-V_{\mu\nu}$, we see that this very much resembles the gravitational gauge transformation (4.6). In fact, this expression (4.6) assumes that $[p^\mu, \Lambda^\nu] = 0$, but if this was *not* the case, then it would be necessary to symmetrize (4.6) as we first did earlier in (3.2), in order to maintain a symmetric gravitational field $h^{\mu\nu} = h^{\nu\mu}$. In this event, we would need to write (4.6) as:

$$\begin{aligned} h^{\mu\nu} \rightarrow h'^{\mu\nu} &= h^{\mu\nu} + \frac{1}{2}(\{p^\mu, i\Lambda^\nu\} + \{p^\nu, i\Lambda^\mu\}) \\ &= h^{\mu\nu} + \frac{1}{2}(p^\mu i\Lambda^\nu + p^\nu i\Lambda^\mu + i\Lambda^\mu p^\nu + i\Lambda^\nu p^\mu). \end{aligned} \quad (4.7)$$

As noted in an earlier footnote just before (3.13), in any situation where $[p^\mu, \Lambda^\nu] = 0$, (4.7) will revert to (4.6), so (4.7) is the more general expression to ensure a symmetric $h^{\mu\nu}$ in any circumstance. Except for the final term $\frac{1}{2}\{\Lambda_\mu, \Lambda_\nu\}$, (4.4) and (4.7) are remarkably-similar transformations. Thus we ask: given these very similar gauge transformations, might the perturbation $V_{\mu\nu}$ be related in some way to the gravitational field $h_{\mu\nu}$?

The $V_{\mu\nu}$ in (4.4) is, for $SU(N)$, is an $N \times N$ matrix of 4×4 tensors, while the $h^{\mu\nu}$ in (4.5) is just a single 4×4 tensor. So if $V_{\mu\nu}$ is related to the gravitational field, then it must be related to a *non-Abelian, Yang-Mills gravitational field*.

We saw in the previous section, that in going from Abelian to Yang-Mills gauge theory,

one simply needs to use the gauge-covariant derivative $D^\mu \equiv \partial^\mu + igG^\mu$ to form the field density $F^{\mu\nu} = D^{[\mu}G^{\nu]}$, see (2.14) and (2.15). Let us use this same $D^\mu \equiv \partial^\mu + igG^\mu$ in (4.5) to take the partial derivatives of $h^{\mu\nu}$ in (4.5), thus writing:

$$h^{\mu\nu} \rightarrow h^{\mu\nu'} = h^{\mu\nu} + D^\mu \Lambda^\nu + D^\nu \Lambda^\mu = h^{\mu\nu} + (\partial^\mu + igG^\mu)\Lambda^\nu + (\partial^\nu + igG^\nu)\Lambda^\mu. \quad (4.8)$$

Once again, as we did in (2.16), we are applying gauge theory to gauge fields, in this case, the gravitational field $h^{\mu\nu}$, to go from Abelian to non-Abelian gauge theory. In a pejorative sense, Yang-Mills theory can be thought of as gauge theory on steroids. Then, we can Fourier transform (4.8) over from configuration space to momentum space via $\partial^\mu \rightarrow ip^\mu$, and also use $\pi^\sigma \equiv p^\sigma + gG^\sigma$ from (3.5) to write (4.8) as:

$$h^{\mu\nu} \rightarrow h^{\mu\nu'} = h^{\mu\nu} + \pi^\mu i\Lambda^\nu + \pi^\nu i\Lambda^\mu. \quad (4.9)$$

Then, because we cannot assume in general that $[\pi^\mu, \Lambda^\nu] = 0$, we have to symmetrize (4.9) to maintain a symmetric gravitational field. So (4.9) now becomes:

$$h^{\mu\nu} \rightarrow h^{\mu\nu'} = h^{\mu\nu} + \frac{1}{2}(\pi^\mu i\Lambda^\nu + \pi^\nu i\Lambda^\mu + i\Lambda^\mu \pi^\nu + i\Lambda^\nu \pi^\mu). \quad (4.10)$$

Except for the term $\frac{1}{2}\{\Lambda_\mu, \Lambda_\nu\}$ which appears in (4.4), $h_{\mu\nu}$ and $-V_{\mu\nu}$ transform in exactly the same way! But in a non-Abelian Yang-Mills gauge theory, the essential feature that is new, is the term $ig[\theta, G_\mu]$ in (4.3). This is why $F^{\mu\nu} = \partial^\mu G^\nu - \partial^\nu G^\mu + ig[G^\mu, G^\nu]$ must contain the extra term $ig[G^\mu, G^\nu]$ which makes Yang-Mills distinctive. And, this term $\frac{1}{2}\{\Lambda_\mu, \Lambda_\nu\}$ in (4.4) traces directly to the term the term $ig[\theta, G_\mu]$ in (4.3), combined with the need to maintain a symmetric propagator. Thus, if we want to talk about non-Abelian, Yang-Mills gravitational field theory, we should expect to have to append an additional term $\frac{1}{2}\{\Lambda_\mu, \Lambda_\nu\}$ to the gauge transformation (4.9). Thus, we augment (4.10) to read:

$$h^{\mu\nu} \rightarrow h^{\mu\nu'} = h^{\mu\nu} + \frac{1}{2}(\pi^\mu i\Lambda^\nu + \pi^\nu i\Lambda^\mu + i\Lambda^\mu \pi^\nu + i\Lambda^\nu \pi^\mu) - \frac{1}{2}\{\Lambda_\mu, \Lambda_\nu\}. \quad (4.11)$$

With this change to the transformation for $h_{\mu\nu}$, the gravitational field $h_{\mu\nu}$ and perturbation $-V_{\mu\nu}$ now have the exact same transformation. Consequently, it seems plausible to promote the gravitational field $h^{\mu\nu}$ to an $N \times N$ matrix of 4×4 tensors for the Yang-Mills group $SU(N)$, and then identify $-V_{\mu\nu}$ directly with a new gravitational field *operator* which we designate with the uppercase $H_{\mu\nu}$, multiplied by a suitable constant k which maintains proper mass, length and time dimensions. That is, because $h_{\mu\nu}$ and $-V_{\mu\nu}$ transform in identical ways, we now *postulate* the connection:

$$kH_{\mu\nu} = -V_{\mu\nu}, \quad (4.12)$$

and rescale $i\Lambda^\nu \rightarrow \Lambda^\nu$, in which case this gravitational field operator $H_{\mu\nu}$ now gauge transforms under a general coordinate transformation according to:

$$H^{\mu\nu} \rightarrow H^{\mu\nu'} = H^{\mu\nu} + \frac{1}{2} \left(\{D^\mu, \Lambda^\nu\} + \{D^\nu, \Lambda^\mu\} \right) + \frac{1}{2} \{ \Lambda_\mu, \Lambda_\nu \}. \quad (4.13)$$

The *observed* gravitational field $h_{\mu\nu}$ will simply derive from the Eigenvalue matrices of the $H_{\mu\nu}$ operator, that is, when operating on an $SU(N)$ N -tuple ϕ , we will have:

$$H^{\mu\nu} |\phi\rangle = h^{\mu\nu} |\phi\rangle. \quad (4.14)$$

From (4.12), we can also extract the trace equation:

$$kH = kH^\sigma{}_\sigma = -V = -V^\sigma{}_\sigma, \quad (4.15)$$

Using (4.12) and (4.15) in the propagator (3.15) now allows us to write:

$$\boxed{D_{\mu\nu}(p^\sigma) = \left(-g_{\mu\nu} + \frac{\frac{1}{2} \{p_\mu, p_\nu\} + kH_{\mu\nu}}{m^2} \right) \times (p^\sigma p_\sigma - m^2 + kH + i\epsilon)^{-1}}, \quad (4.16)$$

while for a massless vector boson, (3.16) may now be written

$$\boxed{D_{\mu\nu}(p^\sigma) = \left(-g_{\mu\nu} + (1 - \xi) \left(\frac{1}{2} \{p_\mu, p_\nu\} + kH_{\mu\nu} \right) \right) \times (p^\sigma p_\sigma + kH)^{-1} \times (p^\sigma p_\sigma + kH + i\epsilon)^{-1}}, \quad (4.17)$$

To get back to where we started, the postulate (4.12) and the parallel transformations (4.4) and (4.13) enable us to absorb any gauge transformation into the propagator in invariant fashion. Most importantly, with the postulate (4.12), *the gauge transformation which concerned us starting with (4.2) is now seen in (4.16) and (4.17) to be one and the same as a general coordinate transformation $x^\mu \rightarrow x^\mu + \Lambda^\mu$, with rescaled $\Lambda^\mu \equiv g\partial^\mu\theta + ig[\theta, G^\mu]$ as in (4.3)*. So, there is no need to concern ourselves that these propagators might not be gauge invariant. Rather, the gauge freedom which initially caused concern corresponds to no more, and no less, than the general relativistic freedom to choose and transform coordinates! We can now be perfectly comfortable with the gauge freedom in these propagators. To make this very clear: in non-Abelian general relativity, the general coordinate transformation is:

$$x^\mu \rightarrow x^\mu + g\partial^\mu\theta + ig[\theta, G^\mu]. \quad (4.18)$$

This is of identical form to the gauge field transformation (4.1), and it leads directly to the augmented gravitational field transformation of (4.11).

Finally, let us obtain the constant k in (4.12), $kH_{\mu\nu} = -V_{\mu\nu}$. For this, we use $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} + \dots$ to extract the following terms from the numerator of (4.16), showing c explicitly:

$$-g_{\mu\nu} + \dots + \frac{kH_{\mu\nu}}{m^2 c^4} = -\eta_{\mu\nu} - \kappa h_{\mu\nu} \dots + \frac{kH_{\mu\nu}}{m^2 c^4}, \quad (4.19)$$

We see that this numerator contains both the gravitational field operator $H_{\mu\nu}$ and its Eigenvalues $h^{\mu\nu}$, see (4.14). For these to match up, both dimensionally and in magnitude, we employ the Planck mass $m_p^2 c^4 = \hbar c^5 / G$ in place of $m^2 c^4$ in (4.19) to introduce the correct natural constants so that the dimensions are correct, thus writing:

$$-g_{\mu\nu} + \dots + \frac{kH_{\mu\nu}}{m_p^2 c^4} = -\eta_{\mu\nu} - \kappa h_{\mu\nu} \dots + \frac{kGH_{\mu\nu}}{\hbar c^5}. \quad (4.20)$$

This does not mean that we are looking at a vector boson mass equal to the Planck mass. It is simply a device to arrive at the correct dimensionality using only the natural constants G , c , and \hbar . Then, to put $H_{\mu\nu}$ and $h^{\mu\nu}$ onto the name numeric footing, referring to the Eigenvalue equation (4.14), we use (4.20) to equate:

$$\kappa = \frac{kG}{\hbar c^5}. \quad (4.21)$$

or, via $\kappa = \sqrt{16\pi G / c^4}$:

$$\boxed{k = \hbar c^3 \sqrt{\frac{16\pi}{G}}}. \quad (4.22)$$

As a result, (4.20) becomes:

$$-g_{\mu\nu} + \dots + \frac{kH_{\mu\nu}}{m_p^2 c^4} = -\eta_{\mu\nu} - \kappa h_{\mu\nu} \dots + \kappa H_{\mu\nu}, \quad (4.23)$$

which includes the terms $\kappa(H_{\mu\nu} - h_{\mu\nu})$, i.e., the $H_{\mu\nu}$ operator minus its Eigenvalues $h_{\mu\nu}$, again, see the Eigenvalue equation (4.14) which was used as the guide to arrive at this.

If the connection $kH_{\mu\nu} = -V_{\mu\nu}$ of (4.12) is correct, then the Yang-Mills gauge transformation (4.1) merely corresponds to the general coordinate transformation (4.18) and the transformation (4.13) in the gravitational field operator. And, a Yang-Mills perturbation is synonymous with the gravitational field.

5. Filling the Mass Gap

Let us now consider the so called ‘‘mass gap’’ question. In particular, we wish to understand how a Yang-Mills theory such as QCD can have massless gauge fields (gluons) yet at the same time display a short range (as is displayed, for example, by nuclear interactions

between nucleons) which requires mediation by massive particles. As stated by Jaffe and Witten in [6]:

“By the 1950s, . . . The massless nature of classical Yang-Mills waves was a serious obstacle to applying Yang-Mills theory to the other forces, for the weak and nuclear forces are short range and many of the particles are massive. Hence these phenomena did not appear to be associated with long-range fields describing massless particles.”

“In the 1960s and 1970s, physicists overcame these obstacles to the physical interpretation of non-abelian gauge theory. In the case of the weak force, this was accomplished by the Glashow-Salam-Weinberg electroweak theory . . . with gauge group $H = SU(2) \times U(1)$. By elaborating the theory with an additional ‘Higgs field,’ one avoided the massless nature of classical Yang-Mills waves. . . .”

But, they go on to note:

“The solution to the problem of massless Yang-Mills fields for the strong interactions has a completely different nature . . . classical nonabelian gauge theory is very different from the observed world of strong interactions; for QCD to describe the strong force successfully, it must have at the quantum level the following three properties, each of which is dramatically different from the behavior of the classical theory:”

“(1) It must have a “mass gap;” namely there must be some constant $\Delta > 0$ such that every excitation of the vacuum has energy at least Δ The first point is necessary to explain why the nuclear force is strong but shortranged. . . .”

Simply restated, what Jaffe and Witten are saying is this: Abelian gauge theories such as QED naturally have massless gauge bosons with infinite range and lifetime, such as the photon. The weak interaction required a massive vector boson in order to explain the observed short range of this interaction. Breaking symmetry with a Higgs field provided the ability to take a massless gauge boson with two transverse polarizations and give it a third, longitudinal polarization “swallowed” from a Goldstone scalar, and thus endow the boson with a mass and consequent short range, while at the same time preserving the renormalizability of the gauge theory.

But strong interactions, Jaffe and Witten are saying, are different. The gauge bosons of QCD are massless gluons, and we never spontaneously break the $SU(3)$ color symmetry and so never take the Higgs / Goldstone steps used in electroweak theory which can endow a gauge boson with a mass and thus a short range. But, nonetheless, in nuclear interactions, we see nucleons bonding to nucleons via various vector and scalar mesons which have a mass and thus a short range, but which do not gain their masses via the Higgs / Goldstone mechanism that we learned about in electroweak theory. So, they ask: where does this mass come from? How, in the context of Yang-Mill gauge theory, do we experimentally observe massive, short range vector and scalar mesons mediating nuclear interactions, which mesons do not get their mass through spontaneous Higgs-style symmetry breaking? What other mechanism might exist for

giving mass to these mediators of nuclear interactions? Higgs and Goldstone filled the mass gap for electroweak theory. How do we fill the “mass gap” for nuclear theory?

To get a handle on this question, let’s look closely at two of the propagators developed earlier in this paper, and compare them side by side. First, the propagator (2.6) for a *massive* vector boson is:

$$D_{\mu\nu}(p^\sigma) = \frac{-g_{\mu\nu} + p_\mu p_\nu / m^2}{p^\sigma p_\sigma - m^2 + i\epsilon}, \quad (5.1)$$

This not an SU(N) matrix, but rather, it is a singlet in the context of Yang-Mills gauge theory. In the $p^\sigma p_\sigma \ll m^2$ limit, this propagator approximates to:

$$D_{\mu\nu}(p^\sigma) \cong \frac{g_{\mu\nu}}{m^2 - i\epsilon}, \quad (5.2)$$

This tells us that any time we find a propagator with a non-zero number with mass dimension +2 in its denominator, we will associate this number with the observed squared mass of a massive vector particle. This also tells us, because $i\epsilon$ is an “imaginary” mass contribution which in essence takes an infinite-lifetime mass and gives it a finite lifetime (e.g., see [1] following (6.130)), that any time we find a non-zero *imaginary* number with mass dimension +2 in a propagator denominator, we will be observing a massive vector particle with finite lifetime. In electroweak theory, the mass m in (5.2) originates via spontaneous symmetry breaking. But, if we should find a way for these non-zero masses m and non-zero $i\epsilon$ terms to enter a propagator denominator in a Yang-Mills theory that has *not* had its symmetry broken, then we will have given rise to massive, short-lived vector particles, without symmetry breaking, in a theory that maintains its massless gauge bosons, e.g., gluons, and will perhaps have filled the mass gap.

Second, let us now return to the propagator (3.17) for a *massless* vector boson in Yang-Mills theory, in Feynman gauge $\xi = 1$, namely

$$D_{\mu\nu}(p^\sigma) = -g_{\mu\nu} \times (p^\sigma p_\sigma - V + i\epsilon)^{-1}. \quad (5.3)$$

Unlike (5.1) and (5.2), this is an $N \times N$ matrix for SU(N). In the $p^\sigma p_\sigma \ll V$ limit where the perturbation dominates, this reduces to:

$$D_{\mu\nu}(p^\sigma) \cong g_{\mu\nu} \times (V - i\epsilon)^{-1}. \quad (5.4)$$

Contrasting (5.2) with (5.4), we find that in a Yang-Mills theory, if the perturbation V is sufficient, and even if the gauge bosons are massless (remember, this is the propagator for a *massless* vector field), one will end up with a propagator (5.4) in which V plays the same role as does m^2 in (5.2). That is, stripping out the $i\epsilon$ in (5.2) and (5.4) to make the comparison more directly, V^{-1} in (5.4) for a *massless* vector boson plays the same role as m^{-2} in (5.2) for a *massive* vector boson, i.e.:

$$V^{-1}(\text{massless}) \sim m^{-2}(\text{massive}). \quad (5.5)$$

So, if one is “expecting” a propagator in the massive form of (5.2) and ends up observing vector particles which originate from a propagator in the massless form of (5.4), one will be left wondering how it is that massive vector particles can arise from a theory with massless gauge bosons. This is how it happens: The perturbations give rise to a “pseudo mass” which arises from the observables of mass dimension -2 in the V^{-1} matrix of (5.4) being “mistaken” for observables of mass dimension -2 in the m^{-2} number of (5.2).

Also of interest are the particle lifetimes. It was noted following (3.15) that the matrix V is generally invertible and so may obviate the need to use the $+i\epsilon$ prescription because there is no problem with infinite poles. Let us flesh this out a bit more. Let us now set $+i\epsilon = 0$ in (5.4), and so write this *massless* propagator in its starkest and simplest form:

$$D_{\mu\nu}(p^\sigma) \cong g_{\mu\nu} \times V^{-1}. \quad (5.6)$$

And now, let us compare (5.6) to (5.2) which retains the $+i\epsilon$ term. For any gauge group $SU(N>2)$, it turns out that while some of the elements in V^{-1} will be real-valued, others will be complex, and still others will be imaginary. If one is “expecting” (5.2) but actually “observing” (5.6), and if (5.6) includes some real numbers, some complex numbers, and some imaginary numbers (as it will for $SU(N>2)$), then when viewed through the “expectation” of (5.2), one will be observing massive vector particles which are fully stable (real numbers), massive vector particles with finite lifetimes (complex numbers), and massless vector particles with finite lifetimes (imaginary numbers). And, one will be left wondering how all of these particles can be observed in a theory like QCD that is based on *massless* gauge bosons with an $SU(3)$ symmetry that has never been spontaneously broken. It is the matrix inverse V^{-1} which plays a pivotal and central role in filling the mass gap. And as discovered in section 4, the matrix V may in fact be an alternative guise for a non-Abelian gravitational field matrix H .

This is the theory behind filling the mass gap. To put this into practice and calculate numbers, one needs to actually calculate V^{-1} for $SU(3)$ QCD, as well as for $SU(2)_W \times U(1)_Y$, then use (5.6) in connection with the invariant amplitudes $\sim J^\sigma V^{-1} J_\sigma$, pick out the “numbers” that emerge, and compare those results to the amplitudes $\sim J^\sigma J_\sigma / (m^2 - i\epsilon)$ that one is “expecting” based on (5.2). If done correctly, the result should be the spectrum of observed nuclear vector particles. The key expression for doing this is (3.11), which we use to write the inverse:

$$V^{-1} = \left[-g(p^\sigma G_\sigma + G_\sigma p^\sigma) - g^2 G^\sigma G_\sigma \right]^{-1}. \quad (5.7)$$

This is the matrix that is used to perform the calculations that fill the mass gap and lead to observed boson particle spectra.

6. Calculating with Yang-Mills Part 1: General Approach

Now let us explore the actual mechanics of calculating vector boson masses using Yang-Mills theory, using the simplest gauge group SU(2) as an example. We can also outline what we might expect with larger gauge groups.

Following path integration as first pioneered by Feynman, propagators appear most importantly in the expressions for transition current, which in Yang-Mills gauge theory takes the form of (compare the related Lagrangian density \mathcal{L} of (2.17) et. seq.):

$$W(J) = -\frac{1}{2} \int \frac{d^4 p}{(2\pi)^4} \mathcal{N} = -\int \frac{d^4 p}{(2\pi)^4} \text{Tr} \left[J^\mu(p^\sigma) D_{\mu\nu}(p^\sigma) J^\nu(p^\sigma) \right]. \quad (6.1)$$

Of particular interest here is the invariant amplitude:

$$\frac{1}{2} \mathcal{N} = \text{Tr} \left[J^\mu(p^\sigma) D_{\mu\nu}(p^\sigma) J^\nu(p^\sigma) \right]. \quad (6.2)$$

We note that every object appearing in (6.2) is in momentum space, not configuration space, which is why we write $J^\nu(p^\sigma)$ and $D_{\mu\nu}(p^\sigma)$. From now on we will drop the (p^σ) reminder just to save space, but it is important to be reminded of this at the outset.

Let us first insert the propagator (5.6) for (remember) *massless* Yang-Mills vector bosons into (6.2), to write this as:

$$\frac{1}{2} \mathcal{N} = \text{Tr} \left[J^\mu D_{\mu\nu} J^\nu \right] = \text{Tr} \left[J^\mu \left(g_{\mu\nu} \times V^{-1} \right) J^\nu \right] = \text{Tr} \left(J^\sigma V^{-1} J_\sigma \right). \quad (6.3)$$

Keeping in mind that in Yang-Mills gauge theory, for SU(N), each of these objects is an $N \times N$ matrix, it helps to be more explicit about this by using $J^\mu \equiv T^i J_i^\mu$ to rewrite (6.3) as:

$$\frac{1}{2} \mathcal{N} = \text{Tr} \left(J^\sigma V^{-1} J_\sigma \right) = \text{Tr} J^i{}^\sigma \left(T^i V^{-1} T^j \right) J^j{}_\sigma = \frac{1}{2} J^i{}^\sigma \mathfrak{K}^{ij} J^j{}_\sigma. \quad (6.4)$$

wherein we have defined a “mass kernel” matrix \mathfrak{K}^{ij} as:

$$\frac{1}{2} \mathfrak{K}^{ij} \equiv \text{Tr} \left(T^i V^{-1} T^j \right). \quad (6.5)$$

This is what we really want to get our hands on, because this is the “machine” that we will use to crank out calculable masses. For SU(N), $i, j = 1, 2, 3, \dots, N^2 - 1$. This means that for SU(2), \mathfrak{K}^{ij} will allow us to calculate up to 9=3x3 distinct mass magnitudes. For SU(3) this will generate up to 64=8x8 distinct mass magnitudes. For SU(6) we would have up to 1225 = 35 x 35 magnitudes. For $SU(3)_c \times SU(2)_w$, which of course is eventually a group of keen interest because it encompasses both strong and weak interactions, \mathfrak{K}^{ij} will generate up to 576 = 64 x 9 distinct mass magnitudes. These numbers are *maximum* numbers of magnitudes. Some mass magnitudes will be duplicated because of symmetry considerations, and others will zero out. But this is the basic scheme of what to expect from (6.5).

Finally, we take V^{-1} of (5.7) and place it directly into (6.5), to write:

$$\frac{1}{2}\mathfrak{K}^{ij} \equiv \text{Tr}\left(T^i \left[-g(p^\sigma G_\sigma + G_\sigma p^\sigma) - g^2 G^\sigma G_\sigma \right]^{-1} T^j \right). \quad (6.6)$$

We then calculate each of the components of \mathfrak{K}^{ij} for SU(N), and then compare them to the $m^2 - i\varepsilon$ in the denominator of the massive propagator (5.2), to determine the ‘‘pseudo masses’’ and ‘‘pseudo lifetimes’’ that emerge from (6.6). Looking at the origin of how (6.6) originated from (6.3) and (6.4), we see that the ij component of \mathfrak{K}^{ij} will tell us the mass and statistical half-life of the vector boson that mediates the transition from the current J^j_σ to the current $J^{i\sigma}$. For example, the \mathfrak{K}^{12} entry tells the mass and half-life of the vector boson which mediates from J^2_σ to J^1_σ . In SU(3), the \mathfrak{K}^{83} component would tell us about the vector boson which mediates between the two neutral currents from J^3_σ to J^8_σ . A component \mathfrak{K}^{ij} which is zero (not zero mass, just zero, period) tells us that the two associated currents simply do not interact via any sort of mediating boson. Now, let us turn the crank for SU(2).

7. Calculating with Yang-Mills Part 2: The Example of SU(2)

Now we will explicitly calculate V^{-1} in (6.6) for SU(2). Before we do so, however, we keep in mind the reminder that the propagator $D_{\mu\nu}(p^\sigma) \cong g_{\mu\nu} \times V^{-1}$ of (5.6) is in *momentum space*, and in particular, that the ‘‘mixed’’ terms $p^\sigma G_\sigma + G_\sigma p^\sigma$ in (6.6) is really to be thought of as $p^\sigma G_\sigma(p^\mu) + G_\sigma(p^\mu)p^\sigma$. By Heisenberg, we know that in *configuration space*, $[p^\sigma, G_\sigma(x^\mu)] \neq 0$, because position and momentum are non-commuting. But momentum does commute with other momentum, so that in momentum space, $[p^\sigma, G_\sigma(p^\mu)] = 0$. This enables us to make life a little easier by writing (6.6) as:

$$-\frac{1}{2}\mathfrak{K}^{ij} = \text{Tr}\left(T^i \left[2gp^\sigma G_\sigma + g^2 G^\sigma G_\sigma \right]^{-1} T^j \right) = \text{Tr}\left(T^i \left[2gp \cdot G + g^2 G \cdot G \right]^{-1} T^j \right), \quad (7.1)$$

where we also use $p \cdot G = p^\sigma G_\sigma$ and $G \cdot G = G^\sigma G_\sigma$ to suppress the spacetime indexes so we can more easily focus on the Yang-Mill indexes.

Now, with the group generators normalized to $\text{Tr}(T^{i2}) = \frac{1}{2}$, let us first expand $G = T^i G^i$ and then calculate:

$$\begin{aligned} -V &= 2gp \cdot G + g^2 G \cdot G = 2gp \cdot T^i G^i + g^2 T^i G^i \cdot T^j G^j \\ &= 2gp \cdot \frac{1}{2} \begin{pmatrix} G^3 & G^1 - iG^2 \\ G^1 + iG^2 & -G^3 \end{pmatrix} + g^2 \frac{1}{2} \begin{pmatrix} G^3 & G^1 - iG^2 \\ G^1 + iG^2 & -G^3 \end{pmatrix} \cdot \frac{1}{2} \begin{pmatrix} G^3 & G^1 - iG^2 \\ G^1 + iG^2 & -G^3 \end{pmatrix} \\ &= gp \cdot \begin{pmatrix} G^3 & G^1 - iG^2 \\ G^1 + iG^2 & -G^3 \end{pmatrix} + \frac{1}{4} g^2 \begin{pmatrix} G^i \cdot G^i & 0 \\ 0 & G^i \cdot G^i \end{pmatrix} \quad (7.2) \\ &= \begin{pmatrix} \frac{1}{4} g^2 G^i \cdot G^i + gp \cdot G^3 & gp \cdot G^1 - igp \cdot G^2 \\ gp \cdot G^1 + igp \cdot G^2 & \frac{1}{4} g^2 G^i \cdot G^i - gp \cdot G^3 \end{pmatrix} \end{aligned}$$

Next, we invert this. In general, the inverse of a matrix M is specified by:

$$M^{-1} = \frac{1}{|M|} C, \quad (7.3)$$

where $|M|$ is the matrix determinant and C is a cofactor or *adjugate* matrix. For (7.2) above, this inverse is given by:

$$V^{-1} = \frac{1}{|V|} \begin{pmatrix} -\frac{1}{4} g^2 G^i \cdot G^i + gp \cdot G^3 & gp \cdot G^1 - igp \cdot G^2 \\ gp \cdot G^1 + igp \cdot G^2 & -\frac{1}{4} g^2 G^i \cdot G^i - gp \cdot G^3 \end{pmatrix} \quad (7.4)$$

where the determinant:

$$|V| = \left(-\frac{1}{4} g^2 G^i \cdot G^i - gp \cdot G^3 \right) \left(-\frac{1}{4} g^2 G^j \cdot G^j + gp \cdot G^3 \right) - \left(-gp \cdot G^1 + igp \cdot G^2 \right) \left(-gp \cdot G^1 - igp \cdot G^2 \right). \quad (7.5)$$

Because the G^i in (7.5) are now SU(2) scalar elements of $G = T^i G^i$, we can commute at will in order to consolidate and reduce terms. The result of this calculation reveals that:

$$|V| = \frac{1}{16} g^4 G^i \cdot G^i G^j \cdot G^j - g^2 p \cdot G^i p \cdot G^i. \quad (7.6)$$

Restoring the previously-suppressed spacetime indexes, (7.4) becomes:

$$V^{-1} = \frac{1}{|V|} \begin{pmatrix} -\frac{1}{4} g^2 G^i{}^\sigma G^i{}_\sigma + gp^\sigma G^3{}_\sigma & gp^\sigma G^1{}_\sigma - igp^\sigma G^2{}_\sigma \\ gp^\sigma G^1{}_\sigma + igp^\sigma G^2{}_\sigma & -\frac{1}{4} g^2 G^i{}^\sigma G^i{}_\sigma - gp^\sigma G^3{}_\sigma \end{pmatrix} \quad (7.7)$$

while (7.6) becomes:

$$|V| = \frac{1}{16} g^4 G^i{}^\sigma G^i{}_\sigma G^j{}^\tau G^j{}_\tau - g^2 p^\sigma G^i{}_\sigma p^\tau G^i{}_\tau. \quad (7.8)$$

This inverse (7.7) will remain finite, i.e., have no poles, even without the $+i\epsilon$ prescription, except where $|V|=0$, i.e., except where $p^\sigma G^i{}_\sigma p^\tau G^i{}_\tau = \frac{1}{16} g^2 G^i{}^\sigma G^i{}_\sigma G^j{}^\tau G^j{}_\tau$.

Now it is time to crank out the mass spectrum. First, we substitute (7.7) into (6.5) to write:

$$\frac{1}{2} \mathcal{K}^{ij} \equiv \frac{1}{|V|} \text{Tr} \left(T^i \begin{pmatrix} -\frac{1}{4} g^2 G^i{}^\sigma G^i{}_\sigma + gp^\sigma G^3{}_\sigma & gp^\sigma G^1{}_\sigma - igp^\sigma G^2{}_\sigma \\ gp^\sigma G^1{}_\sigma + igp^\sigma G^2{}_\sigma & -\frac{1}{4} g^2 G^i{}^\sigma G^i{}_\sigma - gp^\sigma G^3{}_\sigma \end{pmatrix} T^j \right). \quad (7.9)$$

Now we simply substitute the T^i generators of SU(2) into the above in all nine i, j combinations. To show how this works in detail, we show the calculation for \mathfrak{K}^{33} , as such: (the $\frac{1}{4}$ below originates from $\text{Tr}(T^{i^2}) = \frac{1}{2}$)

$$\begin{aligned}
\frac{1}{2}\mathfrak{K}^{33} &= \frac{1}{4|V|} \text{Tr} \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -\frac{1}{4}g^2G^i{}^\sigma G^i{}_\sigma + gp^\sigma G^3{}_\sigma & gp^\sigma G^1{}_\sigma - igp^\sigma G^2{}_\sigma \\ gp^\sigma G^1{}_\sigma + igp^\sigma G^2{}_\sigma & -\frac{1}{4}g^2G^i{}^\sigma G^i{}_\sigma - gp^\sigma G^3{}_\sigma \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \\
&= \frac{1}{4|V|} \text{Tr} \begin{pmatrix} -\frac{1}{4}g^2G^i{}^\sigma G^i{}_\sigma + gp^\sigma G^3{}_\sigma & -gp^\sigma G^1{}_\sigma + igp^\sigma G^2{}_\sigma \\ -gp^\sigma G^1{}_\sigma - igp^\sigma G^2{}_\sigma & -\frac{1}{4}g^2G^i{}^\sigma G^i{}_\sigma - gp^\sigma G^3{}_\sigma \end{pmatrix} \\
&= -\frac{1}{8} \frac{1}{|V|} g^2 G^i{}^\sigma G^i{}_\sigma \\
&= \frac{2g^2 G^i{}^\sigma G^i{}_\sigma}{16g^2 p^\sigma G^i{}_\sigma p^\tau G^i{}_\tau - g^4 G^i{}^\sigma G^i{}_\sigma G^j{}^\tau G^j{}_\tau}
\end{aligned} \tag{7.10}$$

The remaining components of \mathfrak{K}^{ij} are calculated exactly as in (7.10), simply using the other generators T^i of SU(2) in place of the T^3 generators shown above. Doing a complete calculation for all of these components yields:

$$\frac{1}{2}\mathfrak{K}^{ij} = \frac{1}{|V|} \begin{pmatrix} -\frac{1}{8}g^2G^i{}^\sigma G^i{}_\sigma & -\frac{1}{2}igp^\sigma G^3{}_\sigma & \frac{1}{2}igp^\sigma G^2{}_\sigma \\ \frac{1}{2}igp^\sigma G^3{}_\sigma & -\frac{1}{8}g^2G^i{}^\sigma G^i{}_\sigma & -\frac{1}{2}igp^\sigma G^1{}_\sigma \\ -\frac{1}{2}igp^\sigma G^2{}_\sigma & \frac{1}{2}igp^\sigma G^1{}_\sigma & -\frac{1}{8}g^2G^i{}^\sigma G^i{}_\sigma \end{pmatrix}. \tag{7.11}$$

Having derived each of the \mathfrak{K}^{ij} , now let's backtrack. First we insert (7.11) into (6.4), and use an explicit vector of $J^j{}_\sigma$ to write:

$$\frac{1}{2}\mathfrak{K} = \frac{1}{|V|} \begin{pmatrix} J^{1\tau} & J^{2\tau} & J^{3\tau} \end{pmatrix} \begin{pmatrix} -\frac{1}{8}g^2G^i{}^\sigma G^i{}_\sigma & -\frac{1}{2}igp^\sigma G^3{}_\sigma & \frac{1}{2}igp^\sigma G^2{}_\sigma \\ \frac{1}{2}igp^\sigma G^3{}_\sigma & -\frac{1}{8}g^2G^i{}^\sigma G^i{}_\sigma & -\frac{1}{2}igp^\sigma G^1{}_\sigma \\ -\frac{1}{2}igp^\sigma G^2{}_\sigma & \frac{1}{2}igp^\sigma G^1{}_\sigma & -\frac{1}{8}g^2G^i{}^\sigma G^i{}_\sigma \end{pmatrix} \begin{pmatrix} J^1{}_\tau \\ J^2{}_\tau \\ J^3{}_\tau \end{pmatrix}. \tag{7.12}$$

Fully multiplied out, this becomes:

$$\frac{1}{2}\mathfrak{K} = \frac{1}{|V|} \begin{pmatrix} J^{1\tau} \left(-\frac{1}{8}g^2G^i{}^\sigma G^i{}_\sigma \right) J^1{}_\tau + J^{1\tau} \left(-\frac{1}{2}igp^\sigma G^3{}_\sigma \right) J^2{}_\tau + J^{1\tau} \left(\frac{1}{2}igp^\sigma G^2{}_\sigma \right) J^3{}_\tau \\ + J^{2\tau} \left(\frac{1}{2}igp^\sigma G^3{}_\sigma \right) J^1{}_\tau + J^{2\tau} \left(-\frac{1}{8}g^2G^i{}^\sigma G^i{}_\sigma \right) J^2{}_\tau + J^{2\tau} \left(-\frac{1}{2}igp^\sigma G^1{}_\sigma \right) J^3{}_\tau \\ + J^{3\tau} \left(-\frac{1}{2}igp^\sigma G^2{}_\sigma \right) J^1{}_\tau + J^{3\tau} \left(\frac{1}{2}igp^\sigma G^1{}_\sigma \right) J^2{}_\tau + J^{3\tau} \left(-\frac{1}{8}g^2G^i{}^\sigma G^i{}_\sigma \right) J^3{}_\tau \end{pmatrix}. \tag{7.13}$$

We see, as expected, a total of nine (9) transition currents, however, there are two basic types: those along the diagonal where the magnitude of the mass Kernel term is $-\frac{1}{8}g^2G^i{}^\sigma G^i{}_\sigma/|V|$, and those off diagonal where the magnitude is $\pm\frac{1}{2}igp^\sigma G^i{}_\sigma/|V|$. With higher order groups like SU(3), SU(4), etc., there are additional types of terms that do not appear in

SU(2). In SU(3), for example, there would be a new *type* of term for \mathfrak{K}^{38} which mediates between the two neutral currents, yet another type of term for \mathfrak{K}^{14} which crosses between the SU(2) subgroup and the new elements of SU(3), etc. The \mathfrak{K}^{18} elements of SU(3), in general, also have very interesting features that are not apparent in SU(2).

Now, we want to pick off masses and lifetimes. To do this we go back to the propagator (5.2) for a *massive* vector boson, and insert that into (6.3) to write:

$$\frac{1}{2}\mathfrak{K} = J^\mu D_{\mu\nu} J^\nu = J^\mu \frac{g_{\mu\nu}}{m^2 - i\mathcal{E}} J^\nu = J^\sigma \frac{1}{m^2 - i\mathcal{E}} J_\sigma. \quad (7.14)$$

Then, we compare (7.14) to (7.11) to pick off masses and lifetimes. Let's first look at the diagonal elements of (7.11), because these are all the same for each of the neutral current transitions. Here, comparing (7.11) to (7.14), we have:

$$\frac{1}{m^2 - i\mathcal{E}} = -\frac{1}{|V|} \frac{1}{8} g^2 G^i{}^\sigma G^i{}_\sigma = \frac{2g^2 G^i{}^\sigma G^i{}_\sigma}{16g^2 p^\sigma G^i{}_\sigma p^\tau G^i{}_\tau - g^4 G^i{}^\sigma G^i{}_\sigma G^j{}^\tau G^j{}_\tau}. \quad (7.15)$$

Recalling the discussion in section 5, the term on the very left is the propagator that one is “expecting” for a *massive* vector boson, while the terms on the right are the terms from the *actual* Yang-Mills SU(2) propagator for a *massless* gauge boson (no symmetry has ever been broken here!), with these “pseudo masses” arising solely out of the perturbations. Let us now invert (7.15), to more directly write, for the bosons which mediate the 11, 22 and 33 transitions:

$$m^2 - i\mathcal{E} = \frac{16g^2 p^\sigma G^i{}_\sigma p^\tau G^i{}_\tau - g^4 G^i{}^\sigma G^i{}_\sigma G^j{}^\tau G^j{}_\tau}{2g^2 G^i{}^\sigma G^i{}_\sigma}. \quad (7.16)$$

Because the expression on the right hand side of (7.16) is completely real, this is a completely stable particle, so we can remove the $i\mathcal{E}$, take the square root, and write:

$$m = \pm \sqrt{\frac{16g^2 p^\sigma G^i{}_\sigma p^\tau G^i{}_\tau - g^4 G^i{}^\sigma G^i{}_\sigma G^j{}^\tau G^j{}_\tau}{2g^2 G^i{}^\sigma G^i{}_\sigma}}. \quad (7.17)$$

While we do not at this time know what this “mass” actually is as a number (taking this next step will be reviewed in section 8), we see that this may be a non-zero mass. So long as the term inside the square root is positive, then m will be a positive mass. If the term inside the square root should become exactly equal to zero (which means $|V| = 0$ and so $V^{-1} = \infty$), then even though we have a non-invertible perturbation matrix V , this simply means that the masses are all zero. Finally, if term inside the square root should become negative, then m becomes imaginary, which means that this is a massless, unstable particle with finite lifetime. *Equation (7.17) above shows that we have indeed filled the “mass gap” by generating non-zero vector boson masses from a theory with unbroken symmetry in which the gauge boson masses remain at zero.*

One may ask, wait a moment, terms of the form $G^\sigma G_\sigma$, although they can be summed to yield a spacetime scalar mass “number,” are not reliable, because this scalar mass number will

change under a gauge transformation. That is true in general, but that brings us back to the real point of section 4 dealing with the similarity of the transformations of the perturbation V and the gravitational field H . Referring to $-V_{\mu\nu} = \frac{1}{2}g(\{p_\mu, G_\nu\} + \{G_\mu, p_\nu\}) + \frac{1}{2}g^2\{G_\mu, G_\nu\}$ from (4.2), although a term such as $\{G_\mu, G_\nu\}$ (and thus its contracted form $G^\sigma G_\sigma$) will not *by itself* be reliable under a gauge transformation, the overall combination of terms in V is reliable. This is because at the end of the day, the overall combination of terms in $V_{\mu\nu}$ transforms according to (4.4), and as we later saw following (4.17), a *gauge transformation on the combination of terms in (4.4) is synonymous with no more and no less than the freedom to make a general coordinate transformation*. Thus, because (7.17) contains a *combination* of terms that arises out of the perturbation, any gauge transformation will leave (7.17) completely intact, and will do nothing other than transform (7.17) into a new set of coordinates! The mass magnitude numbers which emerge from (7.17) will be unaffected by any change of gauge.

Next, let us take a look at the off-diagonal elements of (7.11), for the 12, 13, 23, 21, 31 and 32 transition currents. For all of these, the relationship corresponding to (7.15) is:

$$\frac{1}{m^2 - i\varepsilon} = \pm \frac{1}{|V|} \frac{1}{2} igp^\sigma G^i_\sigma = \pm i \frac{8gp^\sigma G^i_\sigma}{16g^2 p^\sigma G^i_\sigma p^\tau G^i_\tau - g^4 G^i_\sigma G^i_\sigma G^j_\tau G^j_\tau}, \quad (7.18)$$

with G^i_σ in the numerator such that $i = 1, 2$ or 3 , is whatever number is *not* in the i, j transition current, see (7.11). The inverse of this relationship akin to (7.16) is:

$$m^2 - i\varepsilon = \pm i \frac{16g^2 p^\sigma G^i_\sigma p^\tau G^i_\tau - g^4 G^i_\sigma G^i_\sigma G^j_\tau G^j_\tau}{8gp^\sigma G^i_\sigma}. \quad (7.19)$$

There are actually two ways to approach this. The first approach is to note that the term on the right is imaginary, so we can set $m=0$ on the left and simply write:

$$\varepsilon = \pm \frac{16g^2 p^\sigma G^i_\sigma p^\tau G^i_\tau - g^4 G^i_\sigma G^i_\sigma G^j_\tau G^j_\tau}{8gp^\sigma G^i_\sigma}. \quad (7.20)$$

Referring to [1] after (6.130), the time-dependence of a particle goes as:

$$\text{time dependence} \sim \exp(-imt)\exp(-\varepsilon t/2). \quad (7.21)$$

So, if $m=0$, because ε has both $+$ and $-$ values, the time dependency will go as $\exp(\pm \varepsilon t/2)$, for positive ε . The $-\varepsilon$ dependency is fine, because over time the particle becomes less likely to still be alive. But the $+\varepsilon$ dependency is problematic, because the particle becomes exponentially *more likely* to still be alive as time passes. As much as we might prefer a world in which life expectancy exponentially *increases* the older one gets, that is not reality for either people or particles. So this first approach does not work.

Instead, let us now entirely discard the $+i\varepsilon$ prescription which was developed for the purpose of averting propagator poles, because as we have shown here, there is no problem with

propagator poles in Yang-Mills theory. Instead, let us use only the m^2 in (7.19), and take the square root, to write:

$$m = \sqrt{\pm i} \sqrt{\frac{16g^2 p^\sigma G^i_\sigma p^\tau G^i_\tau - g^4 G^i_\sigma G^i_\sigma G^j_\tau G^j_\tau}{8gp^\sigma G^i_\sigma}}. \quad (7.22)$$

The term $\sqrt{\pm i}$ raises some very interesting issues. For the + sign in the square root, we have $i^{.5} = \pm \frac{1}{\sqrt{2}}(1+i)$. For the - sign, we have $\sqrt{-i} = i \cdot i^{.5} = i^{1.5} = i^{-.5} = \pm \frac{1}{\sqrt{2}}(1-i)$. These half-integer exponents cycle every two exponents, i.e., $i^{.5} = i^{2.5} = i^{4.5} \dots$ and $i^{-.5} = i^{1.5} = i^{3.5} \dots$, etc.

Thus, (7.22) can take on either of two values:

$$m = \left. \begin{array}{l} \pm \frac{1}{\sqrt{2}}(1+i) \\ \pm \frac{1}{\sqrt{2}}(1-i) \end{array} \right\} \times \sqrt{\frac{16g^2 p^\sigma G^i_\sigma p^\tau G^i_\tau - g^4 G^i_\sigma G^i_\sigma G^j_\tau G^j_\tau}{8gp^\sigma G^i_\sigma}}. \quad (7.23)$$

This is now a *complex* mass magnitude number, and so assuming the large square root term on the right is positive-valued, *this mass will be both non-zero, and will have a finite lifetime*. If the square root term on the right is negative valued, then the factors in front are multiplied by i , and this simply exchanges these factors, because of the cyclic properties noted just above. Thus, a negative number inside the square root yields the exact same result as an equal-magnitude positive number.

While factors $\frac{1}{\sqrt{2}}(1+i)$ and $\frac{1}{\sqrt{2}}(1-i)$ are somewhat “boring” in that the mass magnitude is equal to the lifetime magnitude, this is merely what happens in SU(2). For larger groups starting with SU(3), one encounters more varied factors of the form $A + Bi$ where we do not have simply $A = B = 1/\sqrt{2}$ as in (7.23) above. That is, the mass / lifetime relationships are much more varied for larger gauge groups.

Finally, we do observe that the above expression (7.17) and (7.23) admits both positive and negative masses for the mediators of transition currents. Yang-Mills apparently has nothing directly to say about this, so one would need to appeal to Feynman-Stückelberg and regard the negative energy solutions to be indicative of positive energy antiparticles. Indeed, ever since Dirac, every theory has had negative energy solutions, and these are always interpreted to be positive energy antiparticles. Such an interpretation is clearly borne out by observed reality.

8. Conclusion

One of the shortcomings of existing perturbation theory is that this theory is approximately accurate only for small perturbations, and breaks down entirely for large perturbations. The propagators (3.15) and (3.16) developed here are in no way limited to small perturbation, because they were developed without in any way splitting apart the Lagrangian term $\mathcal{L} = -\frac{1}{2} \text{Tr}(F^{\mu\nu} F_{\mu\nu})$ for the Yang-Mills field, see (2.17). Nor was Lorentz symmetry sacrificed by using a lattice. Because gravitational theory does not in any way limit the strength of a gravitational field – witness black holes and gravitational collapse and expansion – to the degree that perturbations are related to gravitational fields via $kH_{\mu\nu} = -V_{\mu\nu}$, general relativistic

gravitational theory itself gives us the vehicle, in principle, to do calculations for perturbations of unlimited magnitude. Finally, as explained starting in section 5, the inverse perturbation matrix V^{-1} provides a possible foundation to filling the “mass gap” with observed strong interaction boson mass and lifetime spectra.

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