# Might Baryons be Yang-Mills Magnetic Monopoles?

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## Abstract:

**We demonstrate how the baryons which constitute the vast preponderance of the material universe are no more and no less than Yang-Mills magnetic monopoles, with quarks and gluons confined, and only mesons permitted to net flux in and out.**

### 1. Introduction

In this paper, we pose the following questions:

> Why, theoretically, do there exist in nature, naturally-occurring sources, namely baryons, consisting of exactly three strongly-interacting fermion constituents which we call "quarks"? Why, and by what mechanism, do the massless gauge particles of Quantum Chromodynamics (QCD), which we call gluons, cause these quarks to remain confined within the baryons? How, and why is it, that the interactions between baryons only occur via the exchange of mediating quark / antiquark pairs that we call "mesons," and not through any free gluon exchange? And how, despite the absence of any known symmetry breaking in QCD, and even with the gluons being massless, do these meson mediators obtain their mass?

These are questions of more than passing interest, because two most-common types of baryon, of course, are the proton and neutron, which account for the very vast preponderance of the material universe. It would be good to have a theoretical foundation for understanding what these baryons actually are.

We do know, because there are three quarks per baryon, that it is very helpful and can explain many things about the strong interactions, if we employ the Yang-Mills color group $SU(3)_C$ with a wavefunction $\psi^T = \begin{pmatrix} R & G & B \end{pmatrix}$ in the fundamental representation to ensure Fermi-Pauli-Dirac exclusion, i.e., to make sure that no two fermions in a given system have the exact same set of quantum numbers. But this merely descriptive, and does not explain the underlying question of why there are three quarks per baryon and not some different number, or the even more challenging questions about confinement. If nature were to provide 4 or 7 or 11, for example, then we would merely enforce Fermi-Dirac statistics with SU(4) or SU(7) or SU(11) instead, and would still be asking "why?" there were instead 4 or 7 or 11 quarks per baryon.

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From an historical perspective, Rabi once quipped about the muon, “who ordered this?” Of course, there has been ample experimental evidence for the existence of nucleons since Rutherford and Chadwick respectively discovered the proton and neutron in 1917 and 1933, experimentally. But for these baryons and others, from a theoretical viewpoint, it is still not really understood even to this day, “who ordered this?” Today, we know that baryons contain three quarks, but we don’t know why this number is three. It is still a struggle to understand why and how these quarks remain stubbornly confined, and how an interaction such as SU(3) QCD which relies on massless gauge bosons (gluons) can still give rise to massive quark / antiquark pairs (mesons) which mediate nuclear interactions. Much research has been focused on finding clever ways to “glue” quarks together, but a fundamental understanding of baryons remains elusive. In fact, properly understanding baryons and confinement and massive meson exchange has proved to be so challenging, that it led the Clay Institute to in 2000 to offer a large prize for solving the so-called “mass-gap” problem of Yang-Mills Theory, [1] which today remains unclaimed. And at bottom, the biggest barrier to cracking this puzzle emanates from the fact that to this day, nobody really knows, theoretically, what a baryon is. “Who ordered baryons?” is still very much a live question.

On a seemingly-different front – which this paper will seek to show is not at all a different front – almost as soon as James Clerk Maxwell published his 1873 *A Treatise on Electricity and Magnetism,* questions arose about magnetic monopoles: “Why is there not symmetry between electric and magnetic charges?” “Do magnetic monopoles exist?” “If so, where and how can they be found?” For almost 140 years, those questions have been asked, and many experiments have been done and continue to be done to detect magnetic monopoles. But to date, magnetic charges have never been conclusively detected and they remain one of the deepest and most elusive mysteries of the natural world.

The thesis of this paper is simple: that the magnetic monopoles which come into existence in Yang-Mills theory are synonymous with baryons. Baryons are Yang-Mills magnetic monopoles. Yang-Mills magnetic monopoles contain exactly three confined quarks, tightly bound via massless gluons, with interactions mediated by massive mesons. To the question what is a baryon? the answer is this: a Yang-Mills magnetic monopole. To the question do magnetic monopoles exist and if so where can we find them? the answer is this: yes, they exist, and they are everywhere. We ourselves and everything we see and touch and hear and smell and feel is built predominantly out of Yang-Mills magnetic monopoles. Whenever we talk about a proton or a neutron or any other baryon, we are talking about a Yang-Mills magnetic monopole. We just don’t realize that, yet. A theoretical oddity and orphan child for close to 140 years, magnetic monopoles are in fact the very heart of the material world, but have been hiding in plain sight ever since the time of Maxwell. Nuclear physics, and the physics of confinement and mesons, is the physics of magnetic monopoles, governed classically by Maxwell’s equations plus Yang-Mills, and quantum mechanically by QCD. And to Rabi’s question who ordered this? the answer, for baryons, is this: James Clerk Maxwell, Chen Ning Yang and Robert Mills. They are the theorists who ordered what Rutherford and Chadwick found in their laboratories the better part of a century ago.

## 2. A Basic Review of Classical Electrodynamics and Electric / Magnetic Duality

Maxwell's classical field equations are most often presented in the form of two separate equations for electric and magnetic charge densities:

$$J^{\nu} = \partial_{\mu} F^{\mu\nu}$$
$$P^{\sigma\mu\nu} = \partial^{\sigma} F^{\mu\nu} + \partial^{\mu} F^{\nu\sigma} + \partial^{\nu} F^{\sigma\mu} \quad . \tag{2.1}$$

Taken as is, there is nothing in the above to prevent the existence of a magnetic charge density $P^{\sigma\mu\nu}$ a.k.a. magnetic monopole (which we seek to demonstrate is a baryon density when fully developed in Yang-Mills theory). However, as soon as one defines the field strength density $F^{\mu\nu}$ from the Abelian gauge vector potential $A^{\mu}$ (which in QED represents the photon) using:

$$F^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu} , \tag{2.2}$$

the latter equation (2.1) becomes $P^{\sigma\mu\nu} = 0$, by identity. Thus, the timeless mystery of Maxwell's equations: no magnetic monopoles.

One might think to discard the vector potential $A^{\mu}$ in (2.2) entirely, and specify electrodynamics entirely in terms of the field strength $F^{\mu\nu}$. But as Witten points out: ([2] at page 28)

> "the vector potential is not just a convenience [but] is needed in 20th-century physics for three very good purposes:
>
> - To write a Schrödinger equation for an electron in a magnetic field.
> - To make it possible to derive Maxwell's equations from a Lagrangian.
> - To write anything at all for non-Abelian gauge theory, which – in our modern understanding of elementary particle physics – is the starting point in describing the strong, weak and electromagnetic interactions."

In fact, if one really probes the question, the real issue is not why magnetic charges don't exist, but rather, <u>why electric charges do exist</u>. This is easiest to understand making use of the "duality" formalism (which we will employ quite often in this paper), first developed by Reinich [3] and later elaborated by Wheeler, [4] which uses the Levi-Civita formalism (see [5] at pages 87-89) in which the "dual" $*A^{\sigma\tau}$ of any second-rank antisymmetric tensor $A^{\sigma\tau}$ in four-space $\mathbf{R^4}$ is constructed according to the discrete transformation $*A^{\mu\nu} \equiv \frac{1}{2!}\varepsilon^{\alpha\beta\mu\nu} A_{\alpha\beta}$, and in which first and third-rank (antisymmetric) tensor duals are formed by $*A^{\mu} = \frac{1}{3!}\varepsilon^{\mu\nu\sigma\tau} A_{\nu\sigma\tau}$ and $*A_{\tau\sigma\nu} = \varepsilon_{\gamma\tau\sigma\nu} A^{\gamma}$. Using the known mathematical identify $\frac{1}{2} A^{\sigma\tau}\left(B_{\tau\sigma;\nu} + B_{\sigma\nu;\tau} + B_{\nu\tau;\sigma}\right) - *A_{\nu\sigma} *B^{\tau\sigma}{}_{;\tau} = 0$ for any two antisymmetric tensors $A$ and $B$ ([4] at page 251, note 22), it can readily be shown that first rank and third rank antisymmetric objects are identically self-dual, that is, $*A^{\mu} = A^{\mu}$ and $*A_{\tau\sigma\nu} = A_{\tau\sigma\nu}$. Using duality, one may equivalently write both of Maxwell's equations (2.3) in first rank form:

$$J^{\nu} = \partial_{\mu} F^{\mu\nu}$$
$$P^{\nu} = \partial_{\mu} * F^{\mu\nu} . \tag{2.3}$$

or, alternatively and equivalently, in the third rank form:

$$* J^{\sigma\mu\nu} = \partial^{\sigma} * F^{\mu\nu} + \partial^{\mu} * F^{\nu\sigma} + \partial^{\nu} * F^{\sigma\mu}$$
$$P^{\sigma\mu\nu} = \partial^{\sigma} F^{\mu\nu} + \partial^{\mu} F^{\nu\sigma} + \partial^{\nu} F^{\sigma\mu} , \tag{2.4}$$

with $*F^{\mu\nu} \equiv \frac{1}{2!}\varepsilon^{\alpha\beta\mu\nu} F_{\alpha\beta}$. Whether one uses (2.1), (2.3) or (2.4) is entirely a matter of preference, and depends largely on what will most simplify any given calculation that one is trying to do. In fact, to be fully complete, the final, equivalent pairing one can consider is:

$$* J^{\sigma\mu\nu} = \partial^{\sigma} * F^{\mu\nu} + \partial^{\mu} * F^{\nu\sigma} + \partial^{\nu} * F^{\sigma\mu}$$
$$P^{\nu} = \partial_{\mu} * F^{\mu\nu} . \tag{2.5}$$

This particular pairing in the context of QCD, may be related to so-called "dark matter," which we will return to briefly at the very end of this paper.

We know very well that Maxwell's classical electric charge equation in the first rank form of $J^{\nu} = \partial_{\mu} F^{\mu\nu}$ may be derived from the Lagrangian (density):

$$\mathfrak{L}_e = -\tfrac{1}{4} F^{\mu\nu} F_{\mu\nu} + J^{\mu} A_{\mu} \tag{2.6}$$

via the Euler-Lagrange equation:

$$\partial^{\sigma}\left(\frac{\partial \mathfrak{L}}{\partial(\partial^{\sigma}\phi)}\right) - \frac{\partial \mathfrak{L}}{\partial \phi} = 0 \tag{2.7}$$

for $\phi = A^{\mu}$. As will be reviewed further in section 3, this, of course, is a *classical* field equation, which only applies for high-action physics in which $S(\varphi) = \int d^4x \mathfrak{L}(\varphi) >> \hbar$.

But what about the classical magnetic charge equation $P^{\nu} = \partial_{\mu} * F^{\mu\nu}$? What is its Lagrangian? Well, Witten says we need a vector potential to have a Lagrangian. So, what is the vector potential? Let us posit a vector potential that we will call $M^{\mu}$. Because the field for the magnetic charge $P^{\nu}$ is the dual field $*F^{\mu\nu} \equiv \frac{1}{2!}\varepsilon^{\alpha\beta\mu\nu} F_{\alpha\beta}$, we know right away that we can derive $P^{\nu} = \partial_{\mu} * F^{\mu\nu}$ from a Langrangian $\pm \mathfrak{L}_m = -\frac{1}{4} * F^{\mu\nu} * F_{\mu\nu} + P^{\mu} M_{\mu}$, so long as we define $M^{\mu}$ in terms of $A^{\mu}$ as:

$$\partial^{\mu} M^{\nu} - \partial^{\nu} M^{\mu} \equiv *F^{\mu\nu} = \tfrac{1}{2!}\varepsilon^{\alpha\beta\mu\nu}\left(\partial_{\alpha} A_{\beta} - \partial_{\beta} A_{\alpha}\right). \tag{2.8}$$

This merely constrains $M^\mu$ such that it is not independent of $A^\mu$, but is instead "interleaved" with $A^\mu$ according to the parametric differential equation (2.8). Put differently, because they are not fully independent, $M^\mu$ and $A^\mu$ will share degrees of freedom. It should be clear that a Lagrangian $\pm \mathfrak{L}_m = -\frac{1}{4} * F^{\mu\nu} * F_{\mu\nu} + P^\mu M_\mu$, via the Euler-Lagrange equation (2.7) will yield $P^\nu = \partial_\mu * F^{\mu\nu}$ for either sign, which is why we show a $\pm$. So, which sign do we choose? Because $* F^{\mu\nu} * F_{\mu\nu} = -F^{\mu\nu} F_{\mu\nu}$ by identity, contrasting $\mathfrak{L}_m$ with $\mathfrak{L}_e$ in (2.6), we see that the choice of the positive sign in $\mathfrak{L}_m$ would cause the kinetic energy term $F^{\mu\nu}F_{\mu\nu}$ to entirely vanish from the combined Lagrangian $\mathfrak{L}_e + \mathfrak{L}_m$. This should not happen, so we know that we should choose the *negative* sign. Therefore, we establish:

$$\mathfrak{L}_m = \frac{1}{4} * F^{\mu\nu} * F_{\mu\nu} - P^\mu M_\mu \tag{2.9}$$

as the magnetic monopole Lagrangian necessary to produce $P^\nu = \partial_\mu * F^{\mu\nu}$ and not negate the kinetic energies associates with the electric charge equation $J^\nu = \partial_\mu F^{\mu\nu}$.

So, now all is well, with one exception: Take (2.8) for $* F^{\mu\nu}$, plug it into the third rank electric charge equation $* J^{\sigma\mu\nu} = \partial^\sigma * F^{\mu\nu} + \partial^\mu * F^{\nu\sigma} + \partial^\nu * F^{\sigma\mu}$ from (2.4), and lo and behold, we find that $* J^{\sigma\mu\nu} = 0$, just like the magnetic monopole $P^{\sigma\mu\nu} = 0$, again, by identity. And because the first rank electric charge $J^\mu = \frac{1}{3!}\varepsilon^{\mu\nu\sigma\tau} * J_{\nu\sigma\tau}$, this means that there is no electric charge.

This is not new, but is a well-known problem, and it is why some authors will write about the "source free" Maxwell equations $F^{\mu\nu} = 0$, $* F^{\mu\nu} = 0$, recognizing that while electric sources clearly exist in nature everywhere from lightning to electric currents to the electrons in atoms, the theory required to permit electric sources to exist is still not fully satisfactory. This is because, in a classic case of exposing one's feet when pulling up the sheets to cover one's shoulders, as soon as one creates a Lagrangian for a magnetic charge in order to be able to talk about magnetic charges quantum mechanically, one at the same time forces the electric charges to become zero. That is why we said above that the real issue is not why magnetic charges don't exist, but rather, why electric charges do exist.*

But in Yang-Mills theory, zero charge is not a problem: Magnetic charges exist, as do electric charges. Specifically, as can be found in virtually any elementary textbook on particle physics or quantum field theory e.g., [6] equation IV.5(17) or [7] equation (15.1.13), the field

* The author addresses this problem in the context of U(1) electrodynamics in a 2005 paper at http://arxiv.org/abs/hep-ph/0508257 by imposing a local duality symmetry and then absorbing the local phases (called "complexion angles" by Reinich) into two new gauge particles which must be introduced in addition to the photon in order to maintain duality symmetry. U(1) electrodynamics then immediately reveals a hidden SU(2) symmetry. This symmetry is then broken much as in SU(2)xU(1) electroweak theory to preserve a massless photon, while revealing a very heavy mass in the 2.25 to 2.5 TeV range for the vector mediator $M^\mu$ of magnetic monopole interactions, so that electric and magnetic charges both exist, but the magnetic charge interactions are exceptionally "weak" and cannot be observed at ordinary energies.

strength tensor for a Yang-Mills (non-Abelian) gauge theory is:

$$F^{i\;\mu\nu} = \partial^\mu G^{i\;\nu} - \partial^\nu G^{i\;\mu} + f^{ijk} G_j^{\;\mu} G_k^{\;\nu} \qquad (2.10)$$

where the $G_i^{\;\mu}$ are the gauge bosons (classical potentials) of whatever Yang-Mills group one is using (for instance, weak $SU(2)_W$ or $SU(3)_C$), $f^{ijk}$ are the group structure constants, and the Latin internal symmetry indexes $i, j, k = 1,2,3\ldots N^2 - 1$ for SU(N) are raised and lowered with the unit matrix $\delta_{ij}$.

It often simplifies things to multiply (2.10) through by the group generators $T^i$, and then employ the group structure $f^{ijk} T_i = -i\left[T^j, T^k\right]$ to rewrite (2.10) as:

$$F^{\mu\nu} = \partial^\mu G^\nu - \partial^\nu G^\mu - i\left[G^\mu, G^\nu\right]. \qquad (2.11)$$

where $F^{\mu\nu} \equiv T^i F_i^{\;\mu\nu}$ and $G^\mu \equiv T^i G_i^{\;\mu}$ are NxN matrices for SU(N). In particular, even in Yang-Mill theory, Maxwell's classical equations remain fully intact in the forms (2.1), as we shall review in section 4, and it is only the definition of $F^{\mu\nu}$. They simply migrate over to being an NxN matrix of equations, rather than just a single equation. The differences between Abelian U(1) theory and non-Abelian SU(N) theory all emanate from the extra term $ig\left[G^\mu, G^\nu\right]$ in (2.11), which is non-zero simply because the $G^\mu$ and $G^\nu$, which are now NxN matrices, do not commute, $\left[G^\mu, G^\nu\right] \neq 0$. In short, Yang-Mills theory is merely Maxwell's electrodynamics for non-commuting gauge fields.

Consequently, as soon as one substitutes the non-Abelian (2.11) into Maxwell's equation (2.1) for $P^{\sigma\mu\nu} = \partial^\sigma F^{\mu\nu} + \partial^\mu F^{\nu\sigma} + \partial^\nu F^{\sigma\mu}$, while the terms based on $\partial^\mu G^\nu - \partial^\nu G^\mu$ continue to zero out by identity in the usual way, one nonetheless arrives at a residual non-zero magnetic charge:

$$\begin{aligned} P^{\sigma\mu\nu} &= -i\left(\partial^\sigma\left[G^\mu, G^\nu\right] + \partial^\mu\left[G^\nu, G^\sigma\right] + \partial^\nu\left[G^\sigma, G^\mu\right]\right) \\ &= -i\left(\left[\partial^\sigma G^\mu, G^\nu\right] + \left[G^\mu, \partial^\sigma G^\nu\right] + \left[\partial^\mu G^\nu, G^\sigma\right] + \left[G^\nu, \partial^\mu G^\sigma\right] + \left[\partial^\nu G^\sigma, G^\mu\right] + \left[G^\sigma, \partial^\nu G^\mu\right]\right) \end{aligned}, \qquad (2.12)$$

all because of the fact that $\left[G^\mu, G^\nu\right] \neq 0$. *The thesis of this paper will be to show that these non-zero $P^{\sigma\mu\nu}$ objects are baryons, and that these $\left[G^\mu, G^\nu\right]$ objects are mesons which mediate nuclear and other baryon interactions*. In particular, as we shall later see in, for example, equation (6.20), the three cyclically-symmetric spacetime indexes $\mu, \nu, \sigma$ in $P^{\sigma\mu\nu}$ are indicative of three fermion / anti-fermion currents within $P^{\sigma\mu\nu}$, while the two antisymmetric indexes $\mu, \nu$ in $\left[G^\mu, G^\nu\right]$ are indicative of two currents, one of which is a fermion, and the other of which is an antifermion, hence a meson.

But first, we must keep in mind that $P^{\sigma\mu\nu} = \partial^\sigma F^{\mu\nu} + \partial^\mu F^{\nu\sigma} + \partial^\nu F^{\sigma\mu}$ is a <u>classical</u> field equation, which means that (2.12) is also classical. It is therefore important before proceeding further, to examine the basic differences between classical and quantum electrodynamics, as well as some semi-classical hybrids of the two, so that (2.12) and its offshoots to be developed here are understood in proper context.

## 3. A Brief Review of Path Integration and QED, including Magnetic Monopole Interactions

The path integral formulation of quantum field theory is based upon the path integral:

$$Z = \int D\varphi \exp\left((i/\hbar)\int d^4x \mathfrak{L}(\varphi)\right) \equiv \mathcal{C}\exp(iW(J)), \tag{3.1}$$

together with a suitable Lagrangian density $\mathfrak{L}(\varphi)$ for whatever field $\varphi$ theory is under consideration. In the $\hbar \to 0$ limit, that is, in situations where the relevant action being considered is much greater than $\hbar$, i.e., $S(\varphi) = \int d^4x \mathfrak{L}(\varphi) >> \hbar$, one can use stationary phase (or steepest descent) approximation to derive the Euler-Lagrange equation (2.7) from the above path integral (see, e.g., [6] at 19). Because it is only valid for $S(\varphi) >> \hbar$, the Euler-Lagrange equation is a classical field equation. Therefore, so too are the classical field equations $J^\nu = \partial_\mu F^{\mu\nu}$ and $P^\nu = \partial_\mu {}^*F^{\mu\nu}$ of (2.3) which are derived from the $\mathfrak{L}_e$ of (2.6) and $\mathfrak{L}_m$ of (2.9) using the Euler-Lagrange equation applicable only to high-action physics where $S(\varphi) >> \hbar$. In low-action physics, where $\hbar$ starts to dominate, the Euler-Lagrange equation (2.7) no longer applies, nor do any of the field equations reviewed in section 2, and one must directly deduce $W(J)$ in order to obtain proper mathematical expressions governing the physics of these quantum fields.

As a general mathematical approach, one solves (3.1) to deduce $W(J)$ from a given Lagrangian density $\mathfrak{L}(\varphi)$, using what Zee [6] refers to as the "central identity of quantum field theory":

$$\int D\varphi \exp\left(\tfrac{1}{2}\varphi \cdot K \cdot \varphi - V(\varphi) + J \cdot \varphi\right) = \mathcal{C}\exp\left(-V(\delta/\delta J)\right)\exp\left(-\tfrac{1}{2} J \cdot K^{-1} \cdot J\right), \tag{3.2}$$

with the quadratic terms in (3.2) converted over to $W(J)$ via the Gaussian integral:

$$\int dx \exp\left(\tfrac{1}{2}Ax^2 + Jx\right) = \left(-2\pi/A\right)^{.5} \exp\left(-J^2/2A\right). \tag{3.3}$$

Basically, one starts with (3.2), takes the Lagrangian density $\mathfrak{L}(\varphi)$ of the theory under consideration, applies whatever tricks or resourcefulness one can muster to put at least part of the Lagrangian in the general quadratic form $\tfrac{1}{2}Ax^2 + Jx$, and takes all the remaining terms and puts them into $V(\varphi)$. Then, one uses $\varphi = (\delta/\delta J)J \cdot \varphi$ to express $V(\varphi)$ instead as the operator $V(\delta/\delta J)$, which enables $\exp(-V(\delta/\delta J))$ to be removed to the front of the path integral over $D\varphi$. In essence, this turns the field $\varphi$ into an operator $\delta/\delta J$ that is independent of the variable of integration $\varphi$ so is can be treated as a constant during integration. One then uses (3.3) to evaluate the remaining quadratic $\frac{1}{2}\varphi \cdot K \cdot \varphi + J \cdot \varphi$ still inside the integrand. Finally, as needed, after obtaining the entire right hand side of (3.2), one uses $\exp(-V(\delta / \delta J))$ to operate on $\exp\left(-\frac{1}{2} J \cdot K^{-1} \cdot J\right)$ and thus generate Green's functions and Wick coefficients and generally derive invariant amplitudes including terms of any desired order. This may be converted to Feynman diagrams as desired. The only problem is that while $\exp(-V(\delta / \delta J))\exp\left(-\frac{1}{2} J \cdot K^{-1} \cdot J\right)$ can crank out lots of terms, there is no known surefire way to deduce the underlying function which accommodates all those terms, which is to say, it is difficult or impossible in many situations to express $\exp(-V(\delta / \delta J))\exp\left(-\frac{1}{2} J \cdot K^{-1} \cdot J\right)$ in a fully closed form.

Let's look at QED with boundary terms equal to zero, as a simple example. For QED, one can use the product rule and (2.2) to convert the electric charge Lagrangian (2.6) into:

$$\begin{aligned} \mathfrak{L}_e &= -\tfrac{1}{4} F^{\mu\nu} F_{\mu\nu} + J^\mu A_\mu \\ &= -\tfrac{1}{2}\partial^\mu\left(A^\nu \partial_\mu A_\nu - A^\nu \partial_\nu A_\mu\right) + \tfrac{1}{2} A^\nu \partial^\mu \partial_\mu A_\nu - \tfrac{1}{2} A^\nu \partial^\mu \partial_\nu A_\mu + J^\mu A_\mu \end{aligned}. \tag{3.4}$$

Then, after integrating by parts to zero out the boundary term, $\partial^\mu\left(A^\nu \partial_\mu A_\nu - A^\nu \partial_\nu A_\mu\right) \rightarrow 0$, and with some renaming and raising and lowering of indexes, this becomes:

$$\mathfrak{L}_e = \tfrac{1}{2} A_\nu \left(g^{\mu\nu} \partial^\sigma \partial_\sigma - \partial^\mu \partial^\nu\right) A_\mu + J^\mu A_\mu . \tag{3.5}$$

Now, $\mathfrak{L}_e$ is in precisely the quadratic form needed to evaluate (3.2) via (3.3), and in this simple case, $V(\varphi) = 0$ so $\exp(-V(\delta / \delta J)) = 1$. As is well-known, after converting to momentum space and inverting the configuration space operator via

$$D_{\nu\lambda}\left(g^{\mu\nu} \partial^\sigma \partial_\sigma - \partial^\mu \partial^\nu\right) e^{ik^\alpha x_\alpha} = \delta^\mu{}_\lambda e^{ik^\alpha x_\alpha} , \tag{3.6}$$

(whereby we are essentially deducing $K^{-1}$ in (3.2)), we obtain the momentum space propagator

$$D_{\mu\nu}(k) = \frac{-g_{\mu\nu} + (1-\xi) k_\mu k_\nu / k^\sigma k_\sigma}{k^\sigma k_\sigma + i\varepsilon} , \tag{3.7}$$

along with the well-known result:

$$W\left(J^\nu\right) = -\frac{1}{2}\int \frac{d^4 k}{(2\pi)^4} J^\mu(k) * \frac{-g_{\mu\nu} + (1-\xi) k_\mu k_\nu / k^\sigma k_\sigma}{k^\sigma k_\sigma + i\varepsilon} J^\nu(k) . \tag{3.8}$$

Were we to have retained the boundary term during integration by parts, that term would have gone into $V(\varphi)$ in (3.2), and (3.7) placed into (3.1) would then be operated on from the left by $\exp(-V(\delta / \delta J))$. That is, the non-quadratic boundary terms would then operate on the definite integral obtained from the quadratic terms.*

In momentum space, current conservation $\partial_\mu J^\mu(x)=0$ becomes $k_\mu J^\mu(k)=0$. ([6] at 31) Thus, (3.7) by virtue of conserving the currents reduces immediately to:

$$W(J)=+\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}J^\mu(k)*\frac{1}{k^\sigma k_\sigma+i\varepsilon}J_\mu(k). \tag{3.9}$$

As is well known, the plus sign in front of (3.9) expresses the fundamental result that like electric charges will repel, because the potential energy between the charge densities $J^0(x)$ is positive. (see [6] following I.5(5)) A similar equation in QCD tells us that two like quarks, say a Red and another Red, repel one another, while unlike quarks, say a Red and a Blue, will attract.

There are a few other observations that we now ought to make, as these will be helpful in the ensuing discussion. First, we note that all direct mention of the photon / field $A_\mu$ has been removed from (3.8), because this field itself was the variable of integration in (3.1). Nonetheless, $A_\mu$ is still implicit in (3.8) in two ways. First, of course, (3.7) is the photon propagator. Second, because the photon $A_\mu$ may be expressed in terms of a polarization vector $\varepsilon_\mu$ using $A_\mu=\varepsilon_\mu e^{-ik^\alpha x_\alpha}$, and because the spin sum $-g_{\mu\nu}=\sum_{spin}\varepsilon_\mu*\varepsilon_\nu$ explicitly contains $\varepsilon_\mu$, we can always substitute $\sum_{spin}\varepsilon_\mu*\varepsilon_\nu$ for $-g_{\mu\nu}$ in (3.8) if we wish to see an explicit connection to the photon field, thus, with $k_\mu J^\mu(k)=0$:

$$W(J)=-\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}J^\mu(k)*\frac{\sum_{spin}\varepsilon_\mu*\varepsilon_\nu}{k^\sigma k_\sigma+i\varepsilon}J^\nu(k). \tag{3.10}$$

Look at in this way, (3.10) is the quantum field theory counterpart of Maxwell's classical electric charge equation $J^\nu=\partial_\mu F^{\mu\nu}$ from (2.1) written as:

$$J^\nu(x)=\partial_\mu F^{\mu\nu}=\left(g^{\mu\nu}\partial^\sigma\partial_\sigma-\partial^\mu\partial^\nu\right)A_\mu(x). \tag{3.11}$$

For high-action situations where $S(\varphi)>>\hbar$, we may use the classical equation (3.11). For low action settings where $S(\varphi)\sim\hbar$, Maxwell's (3.11) is no longer valid, and we have to use (3.10) in its place. Both (3.10) and (3.11) emanate from the same $\mathfrak{L}_e=-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}+J^\mu A_\mu$ of (2.6). The classical (3.11) is obtained from $\mathfrak{L}_e$ by applying an $S(\varphi)>>\hbar$ approximation to the path integral (3.1), via the Euler Lagrange equation (2.7). The quantum field expression (3.10) is obtained from $\mathfrak{L}_e$ by directly deducing $W(J)$ from (3.1), without any approximation, and is valid for $S(\varphi)\sim\hbar$.

* The author develops a detailed example which maintains boundary terms, in http://arxiv.org/abs/0911.1081, for QED in curved spacetime.

The second observation is that while the classical (3.11) expresses $J^{\nu}$ as a function of $A_\mu$, it is often desirable to obtain an inverse expression for $A_\mu$ directly in terms of $J^{\nu}$. And in this paper, in section 5, obtaining such an inverse for Yang-Mills fields will be a very central part of the development. Because $D_{\mu\nu}(k)$ in (3.7) also happens to be the momentum space inverse of the $g^{\mu\nu}\partial^\sigma\partial_\sigma - \partial^\mu\partial^\nu$ in (3.11), see (3.6), we can combine the classical (3.11) and the quantum mechanical (3.7) and also use $k_\mu J^\mu(k)=0$ to obtain the <u>semi-classical</u> relationship:

$$A_\mu(x) = D_{\mu\nu}(k)J^\nu(x) = \frac{-g_{\mu\nu}}{k^\sigma k_\sigma + i\varepsilon} J^\nu(x). \tag{3.12}$$

Semi-classical relationships of the form (3.12) will be very important for showing the connection between Yang-Mills magnetic monopoles $P^{\sigma\mu\nu}$ and baryons, again, as will be seen in section 5.

Now let us turn to the magnetic monopoles. Here, we start with the magnetic monopole Lagrangian $\mathfrak{L}_m$ of (2.9). This is identical to the $\mathfrak{L}_e$ of (2.6) but for the simple symbolic substitutions of $J^\mu \to P^\mu$, $A_\mu \to M_\mu$ and $F^{\mu\nu} \to {}^*F^{\mu\nu}$, and the fact that $\mathfrak{L}_m$ needed to have an opposite overall sign from $\mathfrak{L}_e$ in order to prevent a vanishing of the kinetic energy from $\mathfrak{L}_e + \mathfrak{L}_m$ due to the identity ${}^*F^{\mu\nu}\,{}^*F_{\mu\nu} = -F^{\mu\nu}F_{\mu\nu}$. So, the after-integration by parts Lagrangian density for $\mathfrak{L}_m$, contrast (3.5) for $\mathfrak{L}_e$, will be:

$$\mathfrak{L}_m = -\tfrac{1}{2} M_\nu \left(g^{\mu\nu}\partial^\sigma\partial_\sigma - \partial^\mu\partial^\nu\right) M_\mu - P^\mu M_\mu . \tag{3.13}$$

Therefore, in place of (3.3) our guiding Gaussian integral will be

$$\int dx \exp\left(-\tfrac{1}{2}Ax^2 - Jx\right) = (2\pi/A)^{.5} \exp\left(J^2/2A\right). \tag{3.14}$$

The inversion of $g^{\mu\nu}\partial^\sigma\partial_\sigma - \partial^\mu\partial^\nu$ in (3.13) will be identical to that which is shown in (3.6) yielding an identical propagator to (3.7), but with the understanding that this propagator is for the parameterized field $\varphi = M_\mu$ which is now the field of integration, defined in terms of the photon field by the parametric differential equation (2.8). The end result of the path integration, corresponding to (3.8) will be:

$$W(P^\nu) = +\frac{1}{2}\int \frac{d^4k}{(2\pi)^4} P^\mu(k) * \frac{-g_{\mu\nu} + (1-\xi)k_\mu k_\nu / k^\sigma k_\sigma}{k^\sigma k_\sigma + i\varepsilon} P^\nu(k). \tag{3.15}$$

The only real mathematical difference between $W(P^\nu)$ and $W(J^\nu)$ is that in $W(P^\nu)$ the overall sign has flipped, see (3.14) versus (3.3). Conserving this current with $\partial_\mu P^\mu(x) = 0$, which in momentum space is $k_\mu P^\mu(k) = 0$, (3.15) now becomes:

$$\boxed{W\left(P^{\nu}\right)=-\frac{1}{2}\int\frac{d^{4}k}{(2\pi)^{4}}P^{\mu}(k)*\frac{1}{k^{\sigma}k_{\sigma}+i\varepsilon}P_{\mu}(k)}. \tag{3.16}$$

The above now expresses the fundamental result that like *magnetic* charges will *attract*, because the potential energy between the charge densities $P^{0}(x)$ is negative. A similar equation in QCD, once we establish that Yang-Mills magnetic charges are baryons, *would establish the fundamental result of nuclear physics that like baryons attract*. It is certainly known empirically that this is so; to date, there is no theoretical imperative for why this would be so. Showing that Yang-Mills magnetic charges are baryons would via a like equation to (3.16) at the same time provide theoretical imperative to the strongly attractive nature of the nuclear interaction.

If we wish for the $\varphi=M_{\mu}$ to make an explicit appearance in (3.15), we may define a new polarization vector $\xi_{\mu}$ via $M_{\mu}=\xi_{\mu}e^{-ik^{\alpha}x_{\alpha}}$, and employ $-g_{\mu\nu}=\sum_{spin}\xi_{\mu}*\xi_{\nu}$ and $k_{\mu}P^{\mu}(k)=0$ to write:

$$W\left(P^{\nu}\right)=+\frac{1}{2}\int\frac{d^{4}k}{(2\pi)^{4}}P^{\mu}(k)*\frac{\sum_{spin}\xi_{\mu}*\xi_{\nu}}{k^{\sigma}k_{\sigma}+i\varepsilon}P^{\nu}(k). \tag{3.17}$$

Viewed in this way, the above is the quantum field theory counterpart of Maxwell's classical magnetic charge equation $P^{\nu}=\partial_{\mu}*F^{\mu\nu}$ from (2.3), written as:

$$P^{\nu}(x)=\partial_{\mu}*F^{\mu\nu}=\left(g^{\mu\nu}\partial^{\sigma}\partial_{\sigma}-\partial^{\mu}\partial^{\nu}\right)M_{\mu}(x), \tag{3.18}$$

contrast the parallel relationship between the quantum and classical electric charge field equations (3.10) and (3.11). Equations (3.10) and (3.17) may be thought of as the quantum field theory version of Maxwell's equations (2.3).

A semi-classical relationship corresponding to (3.12) may similarly be developed, namely:

$$M_{\mu}(x)=D_{\mu\nu}(k)P^{\nu}(x)=\frac{-g_{\mu\nu}}{k^{\sigma}k_{\sigma}+i\varepsilon}P^{\nu}(x). \tag{3.19}$$

This type of relationship comes into play when considering possible dark or "hidden" matter, in conjunction with a third rank current $J^{\sigma\mu\nu}=*J^{\sigma\mu\nu}$ based on (2.5), as will be briefly discussed at the end of this paper.

More importantly, we wish to go back to (3.15), set $k_{\mu}P^{\mu}(k)=0$, and make use of the duality and self-dual relationship $P^{\mu}=*P^{\mu}=\frac{1}{3!}\varepsilon^{\mu\alpha\beta\sigma}P_{\alpha\beta\sigma}$ to write (3.15) as:

$$\boxed{W\left(P^{\sigma\mu\nu}\right)=-\frac{1}{72}\int\frac{d^4k}{(2\pi)^4}P_{\delta\gamma\tau}(k)*\frac{\varepsilon^{\nu\delta\gamma\tau}g_{\mu\nu}\varepsilon^{\mu\alpha\beta\sigma}}{k^\sigma k_\sigma+i\varepsilon}P_{\alpha\beta\sigma}}. \tag{3.20}$$

This is the quantum mechanical counterpart of the classical magnetic monopole field equation $P^{\sigma\mu\nu}=\partial^\sigma F^{\mu\nu}+\partial^\mu F^{\nu\sigma}+\partial^\nu F^{\sigma\mu}$ of (2.1).

We can use this to form a propagator for the mediation of interactions between third rank-magnetic monopole sources with six spacetime indexes, namely:

$$D^{\delta\gamma\tau\alpha\beta\sigma}(k)=\frac{1}{36}\frac{\varepsilon^{\nu\delta\gamma\tau}g_{\mu\nu}\varepsilon^{\mu\alpha\beta\sigma}}{k^\sigma k_\sigma+i\varepsilon}, \tag{3.21}$$

and then write (3.20) in terms of (3.21) as:

$$W\left(P^{\sigma\mu\nu}\right)=-\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}P_{\delta\gamma\tau}(k)D^{\delta\gamma\tau\alpha\beta\sigma}(k)*P_{\alpha\beta\sigma}(k). \tag{3.22}$$

(By convention, we keep the minus sign in the expression for $W\left(P^{\sigma\mu\nu}\right)$ and not the propagator so as to emphasize the attraction between like charges.) These fully quantum relationships (3.20) and alternatively (3.22), (3.21) express the quantum field interactions between two third rank magnetic charges.

Equation (3.20) and its equivalent pair of equations (3.22), (3.21) are extremely important relationships, because these describe the quantum field interaction between third rank magnetic monopoles. <u>To the degree we can show that in Yang-Mills theory, these magnetic charges $P^{\mu\nu\sigma}$ are baryons, the Yang-Mills counterparts of these equations will become the equations governing the nuclear interactions between and among baryons such as protons and neutrons!</u> What is nice about these relationships, is that while the $M_u=\xi_\mu e^{-ip^\alpha x_\alpha}$ field is implicit via the spin sum $-g_{\mu\nu}=\sum_{spin}\xi_\mu*\xi_\nu$, this can all be swept into $g_{\mu\nu}$. Thus, in the foregoing quantum field equations (3.20) and alternatively (3.22), (3.21), we do not have to directly concern ourselves any longer with the parametric relation of (2.8).

Before we proceed to the next section, there are two points to be made following the discussion in this section. First, equation (3.20) shows the power, in particular, of first and third rank duality. One can use duality as a “Trojan horse” of sorts, to perform “difficult” calculations using the “simple” first rank-sources $J^\mu$ and $P^\mu$, and then, after the calculation is done, to convert the resulting expressions over to “richer” and more complex expressions containing the third-rank sources $J^{\sigma\mu\nu}$ and $P^{\sigma\mu\nu}$. One good example of this, is to consider how difficult if not impossible it would be to derive a Lagrangian from $P^{\sigma\mu\nu}=\partial^\sigma F^{\mu\nu}+\partial^\mu F^{\nu\sigma}+\partial^\nu F^{\sigma\mu}$ in (2.1). Duality makes that task easier by working from the equivalent equation $P^\nu=\partial_\mu*F^{\mu\nu}$ in (2.3) with $M^\nu$ defined as in (2.8), and so we can derive the Lagrangian $\mathfrak{L}_m=\frac{1}{4}*F^{\mu\nu}*F_{\mu\nu}-P^\mu M_\mu$

of (2.9) pretty much the same way as we derive the usual $\mathfrak{L}_e = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + J^\mu A_\mu$ of (2.6). Once we have this Lagrangian, we can easily place that back into the Euler-Lagrange equation to get back to the classical $P^\nu = \partial_\mu * F^{\mu\nu}$, then apply duality to go over to $P^{\sigma\mu\nu} = \partial^\sigma F^{\mu\nu} + \partial^\mu F^{\nu\sigma} + \partial^\nu F^{\sigma\mu}$. As a second example, we use the Lagrangian $\mathfrak{L}_m = \frac{1}{4} * F^{\mu\nu} * F_{\mu\nu} - P^\mu M_\mu$ to derive the $W(P)$ of (3.16) using the simpler first rank $P^\mu$, then afterwards we apply duality to get to the more complex and richer expression in (3.20).

Second, and more fundamentally, we will shortly be proceeding to show that a Yang-Mils $P^{\sigma\mu\nu}$ is a baryon. We shall do so by making use of the <u>classical</u> field equation $P^{\sigma\mu\nu} = \partial^\sigma F^{\mu\nu} + \partial^\mu F^{\nu\sigma} + \partial^\nu F^{\sigma\mu}$, and substituting into this equation a semi-classical equation akin to (3.12). Thus, we will be employing a semi-classical set of equations which apply only in the high-action arena $S(\varphi) >> \hbar$ to establish the connection between Yang-Mills magnetic monopoles and baryons. So an obvious question will be: does this result remain valid even under low-action, fully quantum conditions where $S(\varphi) \sim \hbar$? The answer we will posit is: yes!

Why? If we can establish in the semi classical or even classical arena that $P^{\sigma\mu\nu}$ has all the properties of a baryon in circumstances where $S(\varphi) >> \hbar$, then there is no logic to suggest that $P^{\sigma\mu\nu}$ will cease to be a baryon once we consider quantum conditions where $S(\varphi) \sim \hbar$. Once a baryon, always a baryon! What will happen, however, is that once we move into the low-action arena where $S(\varphi) \sim \hbar$, we will have to forego the use of any of the semi-classical equations we have developed, because they will no longer correctly describe, mathematically, the behavior of these baryons in the low action arena. Thus, to describe low action $S(\varphi) \sim \hbar$ baryonic physics with complete mathematical precision, we will have to discard any mathematics based on the classical equation $P^{\sigma\mu\nu} = \partial^\sigma F^{\mu\nu} + \partial^\mu F^{\nu\sigma} + \partial^\nu F^{\sigma\mu}$, and will be required to turn exclusively to a Yang-Mills equation parallel to (3.20) to understand the precise behaviors of a baryon in that low-action arena.

So in a very basic sense, using a "bicycle riding" metaphor, we will use semi-classical extensions of the classical equation $P^{\sigma\mu\nu} = \partial^\sigma F^{\mu\nu} + \partial^\mu F^{\nu\sigma} + \partial^\nu F^{\sigma\mu}$ as "training wheels" to demonstrate that $P^{\sigma\mu\nu}$ is in fact a baryon under classical, high-action conditions. Then, once that is completed, we would remove the training wheel equations, and rely on a fully quantum field equation which is a Yang-Mills cousin to (3.20) for $W(P^{\sigma\mu\nu})$, to tell us how these baryons behave in the quantum arena in which our training wheel equations begin to break down or simply cease to work. But no matter what the action, high or low, the Yang-Mills magnetic monopole $P^{\sigma\mu\nu}$ will still be a baryon! It will just adhere to different mathematical equations in different action arenas.

Finally, in Figure 1 following, is a "map" of all the classical, quantum and semi-classical equations we have reviewed in sections 2 and 3.

Figure 1: Quantum, Classical and Semi-Classical Electrodynamics Equation Map

Path Integral

$$Z = \int D\varphi \exp\left((i/\hbar)\int d^4x \mathfrak{L}(\varphi)\right) = \mathcal{C}\exp(iW(J))$$

Input Equations

Classical

$S(\varphi) >> \hbar$

$F^{\mu\nu} = \partial^{[\mu}A^{\nu]}$

$*F^{\mu\nu} \equiv \frac{1}{2!}\varepsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}$ Fields

$\partial^{[\mu}M^{\nu]} \equiv *F^{\mu\nu} = \frac{1}{2!}\varepsilon^{\alpha\beta\mu\nu}\left(\partial_\alpha A_\beta - \partial_\beta A_\alpha\right),$

Quantum

$S(\varphi) >> \hbar$ and $S(\varphi) \sim \hbar$

$k_\mu J^\mu(k) = 0$ $k_\mu P^\mu(k) = 0$ Conservation

$$\partial^\sigma\left(\frac{\partial\mathfrak{L}}{\partial(\partial^\sigma\phi)}\right) - \frac{\partial\mathfrak{L}}{\partial\phi} = 0$$

$W(J)$

Lagrangians

$\mathfrak{L}_e = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + J^\mu A_\mu$

Electric charge / field

$$J^\nu = \partial_\sigma F^{\sigma\nu} = \left(g^{\mu\nu}\partial^\sigma\partial_\sigma - \partial^\mu\partial^\nu\right)A_\mu$$

$$W(J) = +\frac{1}{2}\int \frac{d^4k}{(2\pi)^4} J^\mu(k) * \frac{1}{k^\sigma k_\sigma + i\varepsilon} J_\mu(k)$$

(like charges repel)

[1] $\mathfrak{L}_m = \frac{1}{4} * F^{\mu\nu} * F_{\mu\nu} - P^\mu M_\mu$

Magnetic charge / field

$$P^\nu = \partial_\mu * F^{\mu\nu}$$

$$W(P) = -\frac{1}{2}\int \frac{d^4k}{(2\pi)^4} P^\mu(k) * \frac{1}{k^\sigma k_\sigma + i\varepsilon} P_\mu(k)$$

(like charges attract)

Third Rank Monopole Conversion

$*P^\mu = P^\mu = \frac{1}{3!}\varepsilon^{\alpha\beta\lambda\mu}P_{\alpha\beta\lambda}$

$*P^{\mu\nu\sigma} = P^{\mu\nu\sigma} = -\varepsilon^{\mu\nu\sigma\tau}P_\tau$

$$P^{\mu\nu\sigma} = \partial^\sigma F^{\mu\nu} + \partial^\mu F^{\nu\sigma} + \partial^\nu F^{\sigma\mu}$$

$$W(P) = -\frac{1}{72}\int \frac{d^4k}{(2\pi)^4} P_{\alpha\beta\lambda}(k) * \frac{\varepsilon^{\alpha\beta\lambda\mu}\varepsilon^{\sigma\tau\delta}{}_\mu}{k^\sigma k_\sigma + i\varepsilon} P_{\sigma\tau\delta}$$

$$J^\nu = \left(g^{\mu\nu}\partial^\sigma\partial_\sigma - \partial^\mu\partial^\nu\right)A_\mu$$

Semi-Classical

Inverse

$$A_\mu(k) = D_{\mu\nu}(k)J^\nu(k)$$

Propagator

$$D_{\mu\nu}(k) = \frac{-g_{\mu\nu} + (1-\xi)k_\mu k_\nu / k^\sigma k_\sigma}{k^\sigma k_\sigma + i\varepsilon}$$

$$A_\mu(k) = \frac{-g_{\mu\nu} + (1-\xi)k_\mu k_\nu / k^\sigma k_\sigma}{k^\sigma k_\sigma + i\varepsilon} J^\nu(k)$$

[1] sign chosen because $*F^{\mu\nu} * F_{\mu\nu} = -F^{\mu\nu}F_{\mu\nu}$ . Opposite sign choice would remove kinetic energy from $\mathfrak{L}_e + \mathfrak{L}_m$ .

## 4. Yang Mills "Classical" Field Equations are the Maxwell Equations

In section 2, we began with the classical field equation $J^\nu = \partial_\mu F^{\mu\nu}$ of (2.1), and then derived a Lagrangian $\mathfrak{L}_e = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + J^\mu A_\mu$ in (2.6) which was designed to reproduce (2.1) via the Euler-Lagrange equation (2.7). Similarly for $P^\nu = \partial_\mu * F^{\mu\nu}$ of (2.3) in relation to the $\mathfrak{L}_m = \frac{1}{4} * F^{\mu\nu} * F_{\mu\nu} - P^\mu M_\mu$ of (2.9). Once known, those Lagrangians then gave us the basis, via the path integral (3.1), to arrive at $W(J)$ in (3.9) and $W(P)$ in (3.16), which are fully quantum field expressions. That is, as illustrated in the equation map of Figure 1, once we have the Lagrangians, we can go in either direction: to the "left branch" to derive a classical field equation, or to the "right branch" derive the quantum amplitudes inherent in $W(J)$ and $W(P)$.

Yang-Mills theory obtains its unique dynamical properties because of the field strength tensor $F^{\mu\nu}$ in (2.11) and particularly the non-commuting term $\left[G^\mu, G^\nu\right]$ which contains NxN matrices for SU(N). It will be helpful when working with this non-Abelian Yang-Mills $F^{\mu\nu}$ to employ a little "trick" which puts $F^{\mu\nu}$ into a form that is far easier to calculate with than the ugly expression (2.10). Specifically, we write (2.11) as:

$$F^{\mu\nu} = \partial^\mu G^\nu - \partial^\nu G^\mu - i\left[G^\mu, G^\nu\right] = \left(\partial^\mu - iG^\mu\right)G^\nu - \left(\partial^\nu - iG^\nu\right)G^\mu = D^\mu G^\nu - D^\nu G^\mu = D^{[\mu}G^{\nu]} . \tag{4.1}$$

where we have defined

$$D^\mu \equiv \partial^\mu - iG^\mu . \tag{4.2}$$

But of course, (4.2) is simply a gauge-covariant derivative. So what (4.1) tells us rather succinctly, is that Yang-Mills (non-Abelian) gauge theory is just Abelian gauge theory in which the gauge-covariant derivative $D^\mu = \partial^\mu - iG^\mu$ is used to form the field strength tensor $F^{\mu\nu}$ of (4.1). This is in the nature of applying gauge theory to gauge theory. This compact expression $F^{\mu\nu} = D^{[\mu}G^{\nu]}$ will serve us well for cleanly carrying out calculations in a variety of situations.

The first thing we will wish to have available in Yang-Mills theory, are classical equations corresponding to Maxwell's $J^\nu = \partial_\mu F^{\mu\nu}$ and $P^{\sigma\mu\nu} = \partial^\sigma F^{\mu\nu} + \partial^\mu F^{\nu\sigma} + \partial^\nu F^{\sigma\mu}$ of (2.1). So to start with we ask: what do Maxwell's classical equations look like for Yang-Mills theory? As it turns out, as we shall very briefly review now without a lot of explicit calculation (it is a good exercise for the reader to confirm this), these equations are exactly the same.

The Yang-Mills counterparts to (2.1) are derived directly from a Yang-Mills Lagrangian, via the Euler-Lagrange equation (2.7). The customary Yang-Mills Lagrangian corresponding to (2.6) for non-Abelian field $G^\nu$ and "electric charge" sources $J^\mu$, which uses our trick (4.1), is:

$$\mathfrak{L} = \mathrm{Tr}\left(-\tfrac{1}{2} F^{\mu\nu} F_{\mu\nu} + 2J^\mu G_\mu\right) = \mathrm{Tr}\left(-D^\mu G^\nu D_{[\mu} G_{\nu]} + 2J^\mu G_\mu\right). \tag{4.3}$$

The factor of 2 arises simply because the group generators for SU(N) are normalized to

$\mathrm{Tr}\left(T^iT^j\right)=\frac{1}{2}\delta^{ij}$, and the trace arises because $G_\mu$ and $J^\mu$ are NxN matrices.

To find the classical $S(\varphi)=\int d^4x\mathfrak{L}(\varphi)>>\hbar$ field equations, following the left branch of Figure 1, we can use the right side of (4.3) in the Euler-Lagrange equation (2.7). Although the calculation is more involved than that for QED, all of the new non-linear terms in $\partial\mathfrak{L}/\partial G^\sigma$ cancel identically because of the index commutators, so that $\partial\mathfrak{L}/\partial G^\sigma=-2\mathrm{Tr}J_\tau$. And, all the non-linear terms in $\partial\mathfrak{L}/\partial\left(\partial^\sigma G^\tau\right)$ consolidate into $\partial\mathfrak{L}/\partial\left(\partial^\sigma G^\tau\right)=-2\mathrm{Tr}F_{\sigma\tau}$ based on the new non-commuting terms in the Yang-Mills $F^{\mu\nu}$. Therefore, the Euler-Lagrange equation solves to $-\mathrm{Tr}J_\tau=-\mathrm{Tr}\left(\partial^\sigma F_{\sigma\tau}\right)$. With the trace removed and revised indexing, the Yang-Mills "classical" field equation is just that of Maxwell:

$$J^\nu=\partial_\mu F^{\mu\nu} \tag{4.4}$$

By the same logic, based on a magnetic charge Lagrangian:

$$\mathfrak{L}=\mathrm{Tr}\left(\tfrac{1}{2}*F^{\mu\nu}*F_{\mu\nu}-2P^\mu M_\mu\right)=\mathrm{Tr}\left(D^\mu G^\nu D_{[\mu}M_{\nu]}-2P^\mu M_\mu\right), \tag{4.5}$$

we expect that the magnetic equation will also be unchanged form that of Maxwell, giving us $P^\nu=\partial_\mu *F^{\mu\nu}$, and therefore via duality:

$$P^{\sigma\mu\nu}=\partial^\sigma F^{\mu\nu}+\partial^\mu F^{\nu\sigma}+\partial^\nu F^{\sigma\mu}. \tag{4.6}$$

Of course, in Yang-Mills theory, the magnetic gauge bosons $M^\mu$ appearing in (4.5) need to be parameterized to the electric gauge bosons $G^\mu$ by a non-Abelian version of (2.8). Specifically, with (2.8) and (2.11) as a reference point, we *define* the $M^\mu$ in terms of $*F^{\mu\nu}$ as:

$$\partial^\mu M^\nu-\partial^\nu M^\mu-i\left[M^\mu,M^\nu\right]\equiv *F^{\mu\nu}=\tfrac{1}{2!}\varepsilon^{\alpha\beta\mu\nu}\left(\partial_\alpha G_\beta-\partial_\beta G_\alpha-i\left[G^\mu,G^\nu\right]\right). \tag{4.7}$$

This means that with the Yang-Mills field $F^{\mu\nu}=D^{[\mu}G^{\nu]}$ together with (4.7) above, we can use Maxwell's equations to explore Yang-Mills theory in high-action $S(\varphi)>>\hbar$ physics. Confirming that these classical field equations take on an identical form in Yang-Mills theory, as we just have, is an important step for our overall development.

## 5. A Classical Yang Mills Inverse

We start the next stage of development by using the trick of (4.1) in Maxwell's charge equation (4.4) to obtain:

$$J^\nu=\partial_\mu F^{\mu\nu}=\partial_\mu D^{[\mu}G^{\nu]}=\partial_\mu D^\mu G^\nu-\partial_\mu D^\nu G^\mu=\left(g^{\mu\nu}\partial_\sigma D^\sigma-\partial^\mu D^\nu\right)G_\mu. \tag{5.1}$$

Contrasting the above to (3.11), we now want to obtain the inverse expression for $G_\mu$ in terms of $J^\nu$, similarly to what was done in (3.12). Thus, we need the inverse $I_{\mu\nu}$ of the configuration space operator $g^{\mu\nu}\partial_\sigma D^\sigma - \partial^\mu D^\nu$, defined such that $G_\nu \equiv I_{\sigma\nu} J^\sigma$. We will not for the moment call this inverse a propagator $D_{\mu\nu}$ because technically propagators are derived via the path integral. Of course, in QED the propagator that emerges from path integration happens to be identical to the inverse, see the "observation" made in the discussion of (3.12). But for Yang-Mills theory, we ought not assume *a priori* that this identity between propagator and inverse will continue to be the case. So to make clear this distinction, we are naming this inverse $I_{\mu\nu}$.

Similarly to (3.6), but using the configuration space operator $g^{\mu\nu}\partial_\sigma D^\sigma - \partial^\mu D^\nu$, we now wish to obtain:

$$I_{\nu\lambda}\left(g^{\mu\nu}\partial^\sigma D_\sigma - \partial^\mu D^\nu\right)e^{ik^\alpha x_\alpha} = I_{\nu\lambda}\left(g^{\mu\nu}\left(\partial^\sigma\partial_\sigma - \partial^\sigma G_\sigma\right) - \left(\partial^\mu\partial^\nu - \partial^\mu G^\nu\right)\right)e^{ik^\alpha x_\alpha} = \delta^\mu{}_\lambda e^{ik^\alpha x_\alpha} \qquad (5.2)$$

The presence in the above of the terms such as $\partial^\mu G^\nu$ which are derivatives of fields introduces a complexity that is not encountered in U(1) Abelian gauge theory. This added complexity occurs because these derivatives in $\partial^\mu G^\nu$ do not directly operate on the Fourier kernel $e^{ik^\alpha x_\alpha}$ but instead operate on the gauge field $G^\nu$. Because the field $G^\nu = G^\nu\left(x^\sigma\right)$ is a function of spacetime, we may make use of the commutator relationship:

$$\partial^\sigma G^\mu = i\left[k^\sigma, G^\mu\right] \qquad (5.3)$$

to replace then various $\partial^\sigma G^\mu$ which appear in (5.2). The space components of this relationship, $\partial^a A^b = i\left[k^a, A^b\right]$ for the photon field are used in Dirac theory to derive the electron magnetic moment, see, for example, [8], just after equation (2.964).* The time component of the above, $\partial^0 G^\mu = i\left[k^0, G^\mu\right]$ is a variant of Heisenberg's equation of motion, see for example [9], equation (3.61), which also uses this four-dimensional expression.

So, we substitute (5.3) into (5.2), and with some renaming of indexes to get a $\delta^\mu{}_\nu$ on the right, we obtain:

$$I_{\sigma\nu}\left(-g^{\mu\sigma}\left(k^\alpha k_\alpha + i\left[k^\alpha, G_\alpha\right] - m^2\right) + k^\mu k^\sigma + i\left[k^\mu, G^\sigma\right]\right) = \delta^\mu{}_\nu\,. \qquad (5.4)$$

Before we try to calculate this inverse, knowing that this might have no inverse (see [6], chapter

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* One can see how this operates as a derivative by considering the very simple example $(\partial/\partial x)x^2 = 2x$. The canonical Heisenberg commutator in the space dimensions is $\left[x^i, p^j\right] = i\hbar g^{ij}$. If we apply this to $\left[x^i x^k, p^j\right]$, we find that $\left[x^i x^k, p^j\right] = 2i\hbar g^{ij} x^k$, which we can write as $\partial^j\left(x^i x^k\right) = 2\hbar g^{ij} x^k = -i\left[x^i x^k, p^j\right]$. This is just a fancy way of writing $(\partial/\partial x)x^2 = 2x$. But it turns that this works like a derivative for any order in *x*, i.e., $(\partial/\partial x)x^n = nx^{n-1}$, etc., so that any time we have a field $A^i(x)$, we can apply $\partial^j A^i(x) = i\left[p^j, A^i(x)\right]$.

III.4), let us add a square mass term $m^2$ by hand in the usual way. Also, let us require that this configuration space operator be symmetric under $\mu \leftrightarrow \sigma$ interchange by symmetrizing the above expression using an index anticommutator $\frac{1}{2}k^{\{\mu}, G^{\sigma\}}$. Thus, we re-specify (5.4) as:

$$I_{\sigma\nu}\left(-g^{\mu\sigma}\left(k^\alpha k_\alpha + i\left[k^\alpha, G_\alpha\right] - m^2\right) + k^\mu k^\sigma + \tfrac{1}{2}i\left[k^{\{\mu}, G^{\sigma\}}\right]\right) = \delta^\mu{}_\nu \tag{5.5}$$

Finally, let us also require that $I_{\sigma\nu}$ be symmetric under $\sigma \leftrightarrow \nu$ interchange, by writing this in general form for three unknowns *A*, *B* and *C* as:

$$I_{\sigma\nu} \equiv A g_{\sigma\nu} + B k_\sigma k_\nu + \tfrac{1}{2}Ci\left[k_{\{\sigma}, G_{\nu\}}\right]. \tag{5.6}$$

Finally, we plug this into (5.5). *We now need to solve the expression*:

$$\left(A g_{\sigma\nu} + B k_\sigma k_\nu + \tfrac{1}{2}Ci\left[k_{\{\sigma}, G_{\nu\}}\right]\right)\left(-g^{\mu\sigma}\left(k^\alpha k_\alpha + i\left[k^\alpha, G_\alpha\right] - m^2\right) + k^\mu k^\sigma + \tfrac{1}{2}i\left[k^{\{\mu}, G^{\sigma\}}\right]\right) = \delta^\mu{}_\nu\,. \tag{5.7}$$

It is *very important* as we proceed, to keep in mind that the $G^\sigma$ is an NxN matrix for the Yang-Mills gauge group SU(N). Thus, any expressions which put $G^\sigma$ into a denominator have to be understood as requiring the formation of a *Yang-Mills matrix inverse*. So that the expressions we develop have a similar "look" to familiar expressions from QED, we will generally use a "quoted denominator" notation $1/"M" \equiv M^{-1}$ to designate a Yang-Mills matrix inverse. Thus, $G^{\sigma^{-1}} = 1/"G^\sigma"$, etc.

As we start to solve (5.7) in the usual way, we first determine that:

$$A = -\frac{1}{"k^\alpha k_\alpha + i\left[k^\alpha, G_\alpha\right] - m^2"} = -\left(k^\alpha k_\alpha + i\left[k^\alpha, G_\alpha\right] - m^2\right)^{-1}, \tag{5.8}$$

where as stated we use the quotes to denote a matrix inverse. Putting this back into (5.7), and after absorbing out the metric tensor, we find ourselves now left with the expression:

$$\begin{aligned}
&\frac{k^\mu k_\nu + \tfrac{1}{2}i\left[k^{\{\mu}, G_{\nu\}}\right]}{"k^\alpha k_\alpha + i\left[k^\alpha, G_\alpha\right] - m^2"} \\
&= -\left(Bk^\mu k_\nu + \tfrac{1}{2}Ci\left[k^{\{\mu}, G_{\nu\}}\right]\right)\left(k^\alpha k_\alpha + i\left[k^\alpha, G_\alpha\right] - m^2\right) \\
&\quad + \tfrac{1}{2}Bk_\sigma k_\nu i\left[k^{\{\mu}, G^{\sigma\}}\right] - \tfrac{1}{4}C\left[k_{\{\sigma}, G_{\nu\}}\right]\left[k^{\{\mu}, G^{\sigma\}}\right] + \tfrac{1}{2}Ci\left[k_{\{\sigma}, G_{\nu\}}\right]k^\mu k^\sigma + Bk_\sigma k_\nu k^\mu k^\sigma
\end{aligned}, \tag{5.9}$$

Observing that the top line term has a numerator $k^\mu k_\nu + \frac{1}{2}i\left[k^{\{\mu}, G_{\nu\}}\right]$ and the second line term contains $Bk^\mu k_\nu + \frac{1}{2}Ci\left[k^{\{\mu}, G_{\nu\}}\right]$, we see that these numerators can be cancelled out if we set *B*=*C*, and if the terms on the third line can somehow be zeroed out. In fact, to be able to form this inverse at all, that is exactly what we are *required* to do. So, we now set *B*=*C*, and we also set the entire third line to zero, which as we shall momentarily review, amounts to a gauge fixing condition. We then do some reduction and consolidation to obtain:

$$B = C = -\frac{1}{\frac{"k^{\alpha}k_{\alpha} + i\left[k^{\alpha}, G_{\alpha}\right] - m^{2}"}{"k^{\alpha}k_{\alpha} + i\left[k^{\alpha}, G_{\alpha}\right] - m^{2}"}}, \tag{5.10}$$

subject to the condition:

$$\boxed{\left(k_{\sigma}k_{\nu} + \tfrac{1}{2}i\left[k_{\{\sigma}, G_{\nu\}}\right]\right)\left(k^{\mu}k^{\sigma} + \tfrac{1}{2}i\left[k^{\{\mu}, G^{\sigma\}}\right]\right) = 0}. \tag{5.11}$$

Again, these result from setting *B*=*C* and then setting the third line of (5.9) to zero.

Note, again, that we were required to make these selections (5.10) and (5.11) in order to form an inverse. Gauge-fixing methods such as Faddeev-Popov are about understanding the conditions required to obtain defined inverses, and as we shall momentarily see, (5.11) is a gauge fixing condition. So we now plug (5.8) and (5.10) with *B*=*C* into (5.6) in the gauge (5.11), to obtain the inverse:

$$\boxed{I_{\sigma\nu} = \frac{-g_{\sigma\nu} + \dfrac{k_{\sigma}k_{\nu} + \frac{1}{2}i\left[k_{\{\sigma}, G_{\nu\}}\right]}{"m^{2} - k^{\alpha}k_{\alpha} - i\left[k^{\alpha}, G_{\alpha}\right]"}}{"k^{\alpha}k_{\alpha} - m^{2} + i\left[k^{\alpha}, G_{\alpha}\right]"}}. \tag{5.12}$$

We may also use (5.3) and $k_{\sigma}k_{\nu} \to -\partial_{\sigma}\partial_{\nu}$ to convert this inverse fully back into configuration space, thus:

$$I_{\sigma\nu} = \frac{-g_{\sigma\nu} + \dfrac{-\partial_{\sigma}\partial_{\nu} + \frac{1}{2}\partial_{\{\sigma}G_{\nu\}}}{"m^{2} + \partial^{\alpha}\partial_{\alpha} - \partial^{\alpha}G_{\alpha}"}}{"-\partial^{\alpha}\partial_{\alpha} - m^{2} + \partial^{\alpha}G_{\alpha}"}. \tag{5.13}$$

Note that the term $\left[k^{\alpha}, G_{\alpha}\right] = \partial^{\alpha}G_{\alpha}$ appears in two places in the above, but we do not set this to zero here because we are using different gauge fixing conditions, namely, those of (5.11).

Now, we look at some special cases of (5.13). First, we compare (5.12) to the usual, well-known propagator for a massive vector boson in QED, which is:

$$D_{\sigma\nu} = \frac{-g_{\sigma\nu} + \dfrac{k_{\sigma}k_{\nu}}{m^{2}}}{k^{\alpha}k_{\alpha} - m^{2} + i\varepsilon}. \tag{5.14}$$

In the case where $G^{\sigma} \to 0$, we no longer need to take any matrix inverses, and (5.12) reduces to:

$$I_{\sigma\nu} = \frac{-g_{\sigma\nu} + \dfrac{k_\sigma k_\nu}{m^2 - k^\alpha k_\alpha}}{k^\alpha k_\alpha - m^2}. \tag{5.15}$$

This closely resembles (5.14), sans the $+i\varepsilon$, and also with $m^2 - k^\alpha k_\alpha$ rather than just $m^2$ appearing in the denominator of the right hand term in the numerator.

But, in comparing (5.12) to (5.14), we see two substantial differences. First, the denominators in (5.12) are actually matrix inverses because they include the NxN Yang-Mills matrices $G^\sigma$ for SU(N). Second, <u>and this is an absolutely fundamental point</u>, consider what happens to (5.12) and (5.14) when we set the mass term $m^2 = 0$. In (5.14) for the usual propagator, the term $k_\sigma k_\nu / m^2 \to \infty$ because of the $m^2$ in that denominator. This originates in the fact that the QED configuration space operator $g^{\mu\nu}\partial_\sigma\partial^\sigma - \partial^\mu\partial^\nu$ has no inverse. So the massless propagator becomes infinite! This is what leads to the need for gauge fixing techniques such as Faddeev-Popov, whereby we end up with the massless propagator (3.7). We cannot just set $m^2 = 0$ in (5.14) and keep a finite expression.

But in (5.12) or (5.15) we <u>can</u> set $m^2 = 0$ with impunity. That is, we can make the gauge boson $G^\sigma$ mass $m = 0$ without causing the inverse to become infinite. In fact, if we do set $m^2 = 0$, (5.12) simply becomes a Yang-Mills massless particle propagator:

$$I_{\sigma\nu} = \frac{-g_{\sigma\nu} - \dfrac{k_\sigma k_\nu + \frac{1}{2} i \left[k_{\{\sigma}, G_{\nu\}}\right]}{"k^\alpha k_\alpha + i\left[k^\alpha, G_\alpha\right]"}}{"k^\alpha k_\alpha + i\left[k^\alpha, G_\alpha\right]"}. \tag{5.16}$$

This a perfectly finite expression! No express gauge fixing was required (though setting $\left(k_\sigma k_\nu + \frac{1}{2} i \left[k_{\{\sigma}, G_{\nu\}}\right]\right)\left(k^\mu k^\sigma + \frac{1}{2} i \left[k^{\{\mu}, G^{\sigma\}}\right]\right) = 0$ in (5.11) to obtain the inverse (5.12) implicitly did all the required gauge fixing). But, most importantly: <u>*we have revealed a vector boson "mass" without having ever engaged in spontaneous symmetry breaking. This is a new mechanism for generating a vector boson mass, even with massless m=0 gauge bosons $G^\sigma$!*</u>

Specifically, comparing the bottom "denominators" of the usual inverse (5.14) for a massive propagator and the Yang-Mills inverse (5.16), which denominators are where we expect to find the mass of a vector boson, we find the correspondence:

$$\boxed{\frac{1}{k^\alpha k_\alpha - m^2 + i\varepsilon} \leftrightarrow \frac{1}{"k^\alpha k_\alpha + i\left[k^\alpha, G_\alpha\right]"} = \left(k^\alpha k_\alpha + i\left[k^\alpha, G_\alpha\right]\right)^{-1}}. \tag{5.17}$$

Most precisely – and this is very important to fully understand – if the interaction under consideration, say QCD, contains massless gauge bosons because we have not broken any symmetry to give rise to gauge bosons masses as we do, for example, in $SU(2)_W \times U(1)_Y$, one will be "expecting" a massless propagator of the usual form (3.8) that is used for the massless photon of QED. But in fact, as seen in (5.17), when observing vector particles (such as the $\pi$ mesons), one will be "observing" masses which originate from the massless propagator denominator / inverse $\left(k^\alpha k_\alpha + i\left[k^\alpha, G_\alpha\right]\right)^{-1}$ for a massless gauge boson in Yang-Mills theory. Not knowing about this $\left(k^\alpha k_\alpha + i\left[k^\alpha, G_\alpha\right]\right)^{-1}$ denominator / inverse, one will compare one's observations to the denominator $k^\alpha k_\alpha - m^2 + i\varepsilon$ which is known for a massive vector boson, and will conclude that the (inverted) $k^\alpha k_\alpha + i\left[k^\alpha, G_\alpha\right]$ is actually the (inverted) $k^\alpha k_\alpha - m^2 + i\varepsilon$ term that is expected from the known massive boson propagator (5.14). And so, the observer will conclude that there are massive vector bosons, despite the fact that all the gauge bosons are massless, and will wonder how this can occur and maybe even call this a "mass gap" and offer a reward for figuring out how this can happen.

This is how it happens: The non-linear interactions of Yang-Mills theory give rise to a "pseudo mass"* which arises from the observables of mass dimension -2 on the right hand side of the $\leftrightarrow$ in (5.17) being mistaken for observables of mass dimension -2 on the left hand side. A person who is "confused" in this way will wonder why there appear to be non-zero rest masses when in fact the gauge bosons have zero mass and the symmetry of the Yang-Mills theory has never been broken. Thus, there will appear to be particles with masses and defined half-lives such as the $\pi$ mesons, *even if the gauge bosons are massless* (which they are because we have set $m = 0$ to get to (5.16) / (5.17)). <u>*We have therefore "revealed" a "mass" even while the Yang-Mills gauge bosons have remained massless.*</u> This is similar to how after ordinary spontaneous symmetry breaking such as that used in electroweak theory, one finds terms of the form $\frac{1}{2}\left(\frac{1}{2}vg\right)^2 B^\sigma B_\sigma$ in the Lagrangian where one expects to see $\frac{1}{2}m^2 B^\sigma B_\sigma$, and so associates $m = \frac{1}{2}vg$ with the mass of the boson $B^\sigma$, that is $\frac{1}{2}m^2 B^\sigma B_\sigma \leftrightarrow \frac{1}{2}\left(\frac{1}{2}vg\right)^2 B^\sigma B_\sigma$. This is the approach that one uses to fill the so-called "mass gap"!

Note that when it comes to actually calculating masses, the correspondence (5.17) will yield some rich mass spectra, particularly because any calculation will require taking SU(N) matrix inverses first. That is, the NxN matrix $k^\alpha k_\alpha + i\left[k^\alpha, G_\alpha\right]$ for SU(N) must first be inverted, *and then and only then* will the reciprocals of the numeric results that emerge correspond to an observed boson mass. Imagine calculating $\left(k^\alpha k_\alpha + i\left[k^\alpha, G_\alpha\right]\right)^{-1}$ in SU(3) for example, and all of the complicated real and imaginary and complex terms that will emerge, and then using a transition amplitude to pick off masses from the denominators of the resultant expressions. That, in effect, is how these masses are generated to fill the mass gap, and how a detailed calculation of meson mass spectra would occur. (Keep in mind that this is all based on *classical* high-action field equations, so in fact (5.17) will be modified once quantum fields are accounted for. But the basic idea imparted by (5.17) will remain intact despite the particular expression that emerges from the fully-quantum version of the foregoing.)

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* and also a finite lifetime because a complex mass value simply indicates a massive particle with a defined half-life while an imaginary mass indicates a massless particle of defined half-life, see [10] at 150.

Having looked at $m$=0, let us move on to consider the special case where both $m$=0 and $G^\sigma \to 0$. Here there is no longer the need to take any matrix inverses, so we remove the inversion quotes, and (5.16) becomes:

$$I_{\sigma\nu} = \frac{-g_{\sigma\nu} - k_\sigma k_\nu / k^\alpha k_\alpha}{k^\alpha k_\alpha} . \tag{5.18}$$

This is just the massless vector boson propagator (3.7) sans $+i\varepsilon$, forced into the gauge $\xi = 2$, and bypassing entirely Faddeev-Popov and the usual approaches to gauge fixing. Although we have referred to $I_{\sigma\nu}$ as an inverse, here it truly is a known propagator as well.

Finally, let us return to (5.12), and consider a particle that is "on mass shell," with either $k^\alpha k_\alpha - m^2 = 0$ for a massive particle or $k^\alpha k_\alpha = 0$ for a massless particle. For an on-shell particle, the usual propagator (5.14) becomes:

$$D_{\sigma\nu} = \frac{-g_{\sigma\nu} + \dfrac{k_\sigma k_\nu}{k^\alpha k_\alpha}}{+i\varepsilon} . \tag{5.19}$$

But from (5.12), with either with either $k^\alpha k_\alpha - m^2 = 0$ for a massive particle or $k^\alpha k_\alpha = 0$ for one that is massless, the result is the same, and (5.12) becomes:

$$I_{\sigma\nu} = \frac{-g_{\sigma\nu} + \dfrac{k_\sigma k_\nu + \frac{1}{2} i\left[k_{\{\sigma}, G_{\nu\}}\right]}{"-i\left[k^\alpha, G_\alpha\right]"}}{"+i\left[k^\alpha, G_\alpha\right]"} = \frac{ig_{\sigma\nu}}{"\left[k^\alpha, G_\alpha\right]"} + \frac{k_\sigma k_\nu + \frac{1}{2} i\left[k_{\{\sigma}, G_{\nu\}}\right]}{"\left[k^\alpha, G_\alpha\right]\left[k^\beta, G_\beta\right]"} . \tag{5.20}$$

This is a "naturally-occurring" form of $+i\varepsilon$ based on Yang-Mills interactions, with the term $+i\left[k^\alpha, G_\alpha\right] = \partial^\alpha G_\alpha$ (again, which we do <u>not</u> set to zero here because we are using different gauge fixing conditions here) playing a role identical to $+i\varepsilon$ to avoid poles for on shell particles. The "confused" observer, who is "expecting" a $+i\varepsilon$ term and instead observes a $+i\left[k^\alpha, G_\alpha\right]$ term, will simply calculate the lifetime parameter $1/\varepsilon$ based on what is produced by $\left[k^\alpha, G_\alpha\right]^{-1}$.

So, the Yang-Mills inverse (5.12) steers around all the usual problems with propagators and inverses. Not only does it explain how vector "pseudo masses" will come into existence even if the gauge bosons of the underlying theory remain massless, but it has no problem with becoming undefined (infinite) for a massless boson, and it does not require using the $+i\varepsilon$ prescription to avoid infinite poles, because it produces fully finite results under all the usual scenarios.

Now, one may ask, how did we get to a massless vector particle inverse (5.18) forced into the $\xi = 2$ gauge without any apparent gauge fixing? The answer is that we did in fact fix a gauge back in (5.11). Equation (5.11) is to be regarded as the gauge condition which, in Yang-

Mills theory, is required to be able to form a matrix inverse for the (classical, high action, $S(\varphi) >> \hbar$) configuration space operator $g^{\mu\nu}\partial_\sigma D^\sigma - \partial^\mu D^\nu$ in the Yang-Mills-Maxwell field equation (5.1). And, in the process of this, we have been forced to fix the Faddeev-Popov gauge to $\xi = 2$, see (5.18), and to forego the usual covariant gauge condition $\partial^\alpha G_\alpha = 0$. That (5.11) is in the nature of a deep gauge covariant condition becomes most striking if we also convert (5.11) back into configuration space, as we did with the inverse in (5.13). Doing so yields the rather fascinating operator equation:

$$\boxed{\begin{aligned}&\left(\partial_\sigma\partial_\nu - \tfrac{1}{2}\partial_{\{\sigma}G_{\nu\}}\right)\left(\partial^\mu\partial^\sigma - \tfrac{1}{2}\partial^{\{\mu}G^{\sigma\}}\right)\\ &= \partial_\sigma\partial_\nu\partial^\mu\partial^\sigma - \tfrac{1}{2}\partial_\sigma\partial_\nu\partial^{\{\mu}G^{\sigma\}} - \tfrac{1}{2}\partial_{\{\sigma}G_{\nu\}}\partial^\mu\partial^\sigma + \tfrac{1}{4}\partial_{\{\sigma}G_{\nu\}}\partial^{\{\mu}G^{\sigma\}} = 0\end{aligned}}. \tag{5.21}$$

This is the spacetime equivalent of the gauge fixing condition that is required to form an inverse for the Yang-Mills configuration space operator $g^{\mu\nu}\partial_\sigma D^\sigma - \partial^\mu D^\nu$ in (5.1). This is a sixteen component mixed equation in ${}^\mu{}_\nu$ indexes, and when raised or lowered into contravariant or covariant form it is not symmetric under $\mu \leftrightarrow \nu$ transposition unless one takes additional steps to symmetrize this relationship. While most physics usually stops at two derivatives from the fields (or three if one counts the conservation of sources, $\partial^\mu J_\mu = 0$ and $\partial^\mu T_{\mu\nu} = 0$), this relationship contains <u>fourth</u> derivatives $\partial_\sigma\partial_\nu\partial^\mu\partial^\sigma$, as well as third derivatives including a $\partial^{\{\mu}G^{\sigma\}}$, and finally the term $\partial_{\{\sigma}G_{\nu\}}\partial^{\{\mu}G^{\sigma\}}$ which is second order in symmetric field derivatives (contrast the antisymmetric term $\partial_{[\sigma}G_{\tau]}\partial^{[\tau}G^{\sigma]}$ that appears in Lagrangians). The above (5.21) replaces any and all of the usual gauge conditions that are used in QED, and all those other gauge conditions, most notably $\partial^\alpha G_\alpha = 0$, must <u>not</u> be used here.

Now that we have the inverse and the gauge conditions required to produce that inverse, let us return to where we started, and make use of this inverse in $G_\nu = I_{\sigma\nu}J^\sigma$ to specify $G_\nu$ as a function of $J^\sigma$. Using (5.12) in $G_\nu = I_{\sigma\nu}J^\sigma$ we first obtain:

$$G_\nu = I_{\sigma\nu}J^\sigma = \frac{-g_{\sigma\nu} + \dfrac{k_\sigma k_\nu + \frac{1}{2}i\left[k_{\{\sigma}, G_{\nu\}}\right]}{"m^2 - k^\alpha k_\alpha - i\left[k^\alpha, G_\alpha\right]"}}{"k^\alpha k_\alpha - m^2 + i\left[k^\alpha, G_\alpha\right]"}J^\sigma . \tag{5.22}$$

However, as noted earlier back at equation (3.9), in momentum space, current conservation $\partial_\mu J^\mu(x) = 0$ becomes $k_\mu J^\mu(k) = 0$. This modifies (5.22) in two respects. First, the term $k_\sigma k_\nu J^\sigma = 0$. Secondly, and of special interest because it breaks a symmetry, the term $\frac{1}{2}i\left[k_{\{\sigma}, G_{\nu\}}\right]J^\sigma = \frac{1}{2}i\left[k_\nu, G_\sigma\right]J^\sigma$. That is, one of the two terms in the anticommutator zeros out, but the second term does not. Given that $I_{\sigma\nu}$ was designed to be symmetric under transposition of the $\sigma \leftrightarrow \nu$ indexes, that symmetry is broken in (5.22). So with those reductions, (5.22) becomes:

$$\boxed{G_\nu = I_{\sigma\nu} J^\sigma = \frac{-g_{\sigma\nu} + \frac{\frac{1}{2} i[k_\nu, G_\sigma]}{"m^2 - k^\alpha k_\alpha - i[k^\alpha, G_\alpha]"}}{"k^\alpha k_\alpha - m^2 + i[k^\alpha, G_\alpha]"} J^\sigma}. \tag{5.23}$$

One can follow the same path outlined above, to derive this inverse for the various special cases already explored: low-perturbation where $G^\sigma \to 0$ (5.14); massless boson $m = 0$ (5.16); both $G^\sigma \to 0$ and $m = 0$ (5.18); and on shell $k^\alpha k_\alpha - m^2 = 0$ for a massive or $k^\alpha k_\alpha = 0$ for a massless particle (5.20).

This inverse expression (5.23) is what we set out to derive at the start of this section, and it will play a very central role in helping us to establish that the Yang-Mills magnetic charge $P^{\sigma\mu\nu}$ is in fact a baryon. With all of the preliminary groundwork now laid, and with the understanding developed in section 3 that by using field equations such as $J^\nu = \partial_\mu F^{\mu\nu}$ in (4.4) and $P^{\sigma\mu\nu} = \partial^\sigma F^{\mu\nu} + \partial^\mu F^{\nu\sigma} + \partial^\nu F^{\sigma\mu}$ we are exploring the high-action realm in which $S(\varphi) = \int d^4 x \mathfrak{L}(\varphi) >> \hbar$, it is time to discover the underlying theoretical basis for the baryons that constitute the very nuclear heart of the material universe.

## 6. The Baryon and Meson Structure of Yang Mills Magnetic Monopoles

In (4.6), we established that the Maxwell equation $P^{\sigma\mu\nu} = \partial^\sigma F^{\mu\nu} + \partial^\mu F^{\nu\sigma} + \partial^\nu F^{\sigma\mu}$ for a magnetic monopole carries over intact to Yang-Mills theory for high-action arenas where the action $S(\varphi) >> \hbar$. Therefore, we can now carry forward on the basis of our earlier equation (2.12), which was derived by the simple substitution the Yang Mills field density $F^{\mu\nu} = \partial^\mu G^\nu - \partial^\nu G^\mu - i[G^\mu, G^\nu]$ of (2.11) into the $P^{\sigma\mu\nu} = \partial^\sigma F^{\mu\nu} + \partial^\mu F^{\nu\sigma} + \partial^\nu F^{\sigma\mu}$ of (2.1), which remains the "classical" magnetic charge equation for Yang-Mills theory.

The first thing we do is substitute $\partial^\sigma G^\mu = i[k^\sigma, G^\mu]$ from (5.3) into (2.12) to yield:

$$P^{\sigma\mu\nu} = ([[k^\sigma, G^\mu], G^\nu] + [G^\mu, [k^\sigma, G^\nu]] + [[k^\mu, G^\nu], G^\sigma] + [G^\nu, [k^\mu, G^\sigma]] + [[k^\nu, G^\sigma], G^\mu] + [G^\sigma, [k^\nu, G^\mu]]). \tag{6.1}$$

If we expand the commutators in the above, terms of the form $G^\mu k^\sigma G^\nu - G^\mu k^\sigma G^\nu$ appear throughout, so that all terms with $k^\sigma$ sandwiched between the two $G^\mu$ drop out. Then, re-consolidating the commutators, (6.1) reduces to:

$$P^{\sigma\mu\nu} = -([[G^\mu, G^\nu], k^\sigma] + [[G^\nu, G^\sigma], k^\mu] + [[G^\sigma, G^\mu], k^\nu]). \tag{6.2}$$

This will be our starting point for exploring the baryonic properties of $P^{\sigma\mu\nu}$.

First, we insert the hard-won Yang-Mills inverse (5.23) for $G_\nu$ into (6.2). Keep in mind that we have done nothing to break the symmetry of the Yang-Mills theory and so the gauge bosons must be presumed to be massless. Nonetheless, we will carry the mass term in these equations, so whatever we derive is perfectly general. If we want to explore the special case for $m=0$ we can always do so by zeroing out the mass at the time, but at the outset, we ought not limit ourselves in this way.

Also, to maintain full generality at the outset, because there are six different appearances of $G_\nu$ in (6.2), there will be six independent substitutions of (5.23) into (6.2). To track this, we will use the first six letters of the Greek alphabet $\alpha,\beta,\gamma,\delta,\varepsilon,\zeta$ to carry out the internal index summations within each of the six substitutions of the inverse (5.23). While it ordinarily does not matter what letters one chooses to do summations, the summation index will in this case double as a label so we can quickly ascertain where any term originated from, as we progress with our development. And more importantly, while $k^\alpha k_\alpha = k^\beta k_\beta$ where the momenta are equal, $k^\alpha = k^\beta$, in the event that $k^\alpha \neq k^\beta$ – for example if these are momentum vectors for two different particles – then $k^\alpha k_\alpha \neq k^\beta k_\beta$. So we are using this index convention to simultaneously label the momenta and to avoid making any *a priori* assumptions about the actual physical values and meanings of the $k^\alpha$ in each of the six inverse substitutions we are making. Similarly, substituting (5.23) into each of the $G_\nu$ in (6.2) introduces six mass numbers *m*. Here too, we wish to avoid assuming anything *a priori*. So, we similarly label each mass with one of the $\alpha,\beta,\gamma,\delta,\varepsilon,\zeta$, and so regard these at least at the outset, as six different, independent mass numbers. Thus, the expression below in (6.3) will contain six momenta $k^\alpha,k^\beta,k^\gamma,k^\delta,k^\varepsilon,k^\zeta$ which may or may not be different form one another, as well as six labeled masses $m_{(\alpha)},m_{(\beta)},m_{(\gamma)},m_{(\lambda)},m_{(\varepsilon)},m_{(\zeta)}$ which also may or may not be different from each other and may also be zero or non-zero. This provides complete generality and maximum flexibility to explore.

Finally, prior to substituting this inverse (5.23) into (6.2), for the three $G^\mu,G^\nu,G^\sigma$ in the left hand side of the commutators in (6.2) we have arranged for the free indexes $\mu,\nu,\sigma$ to be in the right hand position of metric tensor $g^{\alpha\mu}=g^{\mu\alpha}$ of (5.23). Conversely, for the three $G^\mu,G^\nu,G^\sigma$ from the right hand side of the commutators, we have arranged for the free indexes $\mu,\nu,\sigma$ to be in the left hand position in $g^{\alpha\mu}=g^{\mu\alpha}$ (5.23). We may do this because $g^{\alpha\mu}=g^{\mu\alpha}$ is a symmetric tensor and the indexes can thus be disposed in either order, and the order makes no mathematical difference. But, when one draws a Feynman diagram for a $g^{\alpha\mu}=g^{\mu\alpha}$, the directional arrows are established based on a right to left reading, so that $g^{\alpha\mu}$ has a line $\alpha\leftarrow\mu$, while $g^{\mu\alpha}$ has a line $\mu\leftarrow\alpha$. This choice of index placement will ensure that all the directional arrows for any given terms are lined up in the same direction when Feynman diagrams are drawn, just to provide a consistent set of drawing conventions.

With all of the foregoing, finally substituting the inverse equation (5.23) for $G_\nu$ into (6.2) yields:

$$P^{\sigma\mu\nu} = -\left( \begin{array}{l} \left[\left[\dfrac{-g^{\alpha\mu} + \dfrac{\frac{1}{2}i\left[k^{\mu},G^{\alpha}\right]}{"m_{(\alpha)}{}^{2} - k^{\alpha}k_{\alpha} - i\left[k^{\alpha},G_{\alpha}\right]"}}{"k^{\alpha}k_{\alpha} - m_{(\alpha)}{}^{2} + i\left[k^{\alpha},G_{\alpha}\right]"} J_{\alpha}, \dfrac{-g^{\nu\beta} + \dfrac{\frac{1}{2}i\left[k^{\nu},G^{\pi}\right]}{"m_{(\beta)}{}^{2} - k^{\beta}k_{\beta} - i\left[k^{\beta},G_{\beta}\right]"}}{"k^{\beta}k_{\beta} - m_{(\beta)}{}^{2} + i\left[k^{\beta},G_{\beta}\right]"} J_{\beta}\right], k^{\sigma}\right] \\ + \left[\left[\dfrac{-g^{\gamma\nu} + \dfrac{\frac{1}{2}i\left[k^{\nu},G^{\gamma}\right]}{"m_{(\gamma)}{}^{2} - k^{\gamma}k_{\gamma} - i\left[k^{\gamma},G_{\gamma}\right]"}}{"k^{\gamma}k_{\gamma} - m_{(\gamma)}{}^{2} + i\left[k^{\gamma},G_{\gamma}\right]"} J_{\gamma}, \dfrac{-g^{\sigma\delta} + \dfrac{\frac{1}{2}i\left[k^{\sigma},G^{\delta}\right]}{"m_{(\delta)}{}^{2} - k^{\delta}k_{\delta} - i\left[k^{\delta},G_{\delta}\right]"}}{"k^{\delta}k_{\delta} - m_{(\delta)}{}^{2} + i\left[k^{\delta},G_{\delta}\right]"} J_{\delta}\right], k^{\mu}\right] \\ + \left[\left[\dfrac{-g^{\varepsilon\sigma} + \dfrac{\frac{1}{2}i\left[k^{\sigma},G^{\varepsilon}\right]}{"m_{(\varepsilon)}{}^{2} - k^{\varepsilon}k_{\varepsilon} - i\left[k^{\varepsilon},G_{\varepsilon}\right]"}}{"k^{\varepsilon}k_{\varepsilon} - m_{(\varepsilon)}{}^{2} + i\left[k^{\varepsilon},G_{\varepsilon}\right]"} J_{\varepsilon}, \dfrac{-g^{\mu\zeta} + \dfrac{\frac{1}{2}i\left[k^{\mu},G^{\zeta}\right]}{"m_{(\zeta)}{}^{2} - k^{\zeta}k_{\zeta} - i\left[k^{\zeta},G_{\zeta}\right]"}}{"k^{\zeta}k_{\zeta} - m_{(\zeta)}{}^{2} + i\left[k^{\zeta},G_{\zeta}\right]"} J_{\zeta}\right], k^{\nu}\right] \end{array} \right). \qquad (6.3)$$

This, of course, is a rather complicated expression, so let's for now just look at the lowest order terms for which $G^{\mu} \to 0$. With that one change, we can remove all the "quoted" inverses and many other terms, and (6.3) becomes:

$$P^{\sigma\mu\nu} = -\left( \begin{array}{l} \left[\left[\dfrac{g^{\alpha\mu}}{k^{\alpha}k_{\alpha} - m_{(\alpha)}{}^{2}} J_{\alpha}, \dfrac{g^{\nu\beta}}{k^{\beta}k_{\beta} - m_{(\beta)}{}^{2}} J_{\beta}\right], k^{\sigma}\right] \\ + \left[\left[\dfrac{g^{\gamma\nu}}{k^{\gamma}k_{\gamma} - m_{(\gamma)}{}^{2}} J_{\gamma}, \dfrac{g^{\sigma\delta}}{k^{\delta}k_{\delta} - m_{(\delta)}{}^{2}} J_{\delta}\right], k^{\mu}\right] \\ + \left[\left[\dfrac{g^{\varepsilon\sigma}}{k^{\varepsilon}k_{\varepsilon} - m_{(\varepsilon)}{}^{2}} J_{\varepsilon}, \dfrac{g^{\mu\zeta}}{k^{\zeta}k_{\zeta} - m_{(\zeta)}{}^{2}} J_{\zeta}\right], k^{\nu}\right] \end{array} \right). \qquad (6.4)$$

This is clearly a much simpler expression than (6.3). While both (6.3) and (6.4) are classical insofar as they depend upon an action $S(\varphi) >> \hbar$, (6.4) lays out the basic structure of $P^{\sigma\mu\nu}$, while (6.3) shows what happens then the $G^{\mu}$ come into play and start to exert a dominant role. In QCD parlance, we will come to see that (6.4) describes the interactions inside a baryon in the low perturbation regime often referred to as "asymptotic freedom," while (6.3) may describe the "infrared slavery" or "confinement" regime where the gluon $G^{\mu}$ interactions become very dominant. But we need to take first things first, and the lowest order comes first so we now explore the $G^{\mu} \to 0$ regime for $S(\varphi) >> \hbar$ which is specified in (6.4). The terms in (6.3) involving $\left[k^{\nu},G^{\tau}\right]$ will generate higher-order interactions via (6.3), but (6.4) is the "skeleton" of $P^{\sigma\mu\nu}$ which reveals the underlying structural characteristic of $P^{\sigma\mu\nu}$ in the lowest order. So we will now explore this structural equation (6.4) in earnest, to see what it tells us about what is going on inside of the magnetic charges $P^{\sigma\mu\nu}$.

It should be noted first of all that (6.4) is something of a “chameleon equation,” because depending on how one manipulates this equation, one may highlight the currents / fermions, one may highlight the gauge bosons, and one may explore both currents and gauge bosons in a mixed view. In the gauge boson view, one leaves the $g^{\mu\nu}$ showing explicitly in the above, which thereby displays complete boson propagators. In the current view, the $g^{\mu\nu}$ are absorbed into the currents via $J^{\nu} = g^{\nu\beta} J_{\beta}$. In “mixed” view, we have little of each. We start with the current / fermion view, by applying $J^{\mu} = g^{\alpha\mu} J_{\alpha}$ to (6.4) thus:

$$P^{\sigma\mu\nu} = -\left( \begin{array}{l} \left[\left[\dfrac{J^{\mu}}{k^{\alpha}k_{\alpha} - m_{(\alpha)}{}^{2}}, \dfrac{J^{\nu}}{k^{\beta}k_{\beta} - m_{(\beta)}{}^{2}}\right], k^{\sigma}\right] \\ + \left[\left[\dfrac{J^{\nu}}{k^{\gamma}k_{\gamma} - m_{(\gamma)}{}^{2}}, \dfrac{J^{\sigma}}{k^{\delta}k_{\delta} - m_{(\delta)}{}^{2}}\right], k^{\mu}\right] \\ + \left[\left[\dfrac{J^{\sigma}}{k^{\varepsilon}k_{\varepsilon} - m_{(\varepsilon)}{}^{2}}, \dfrac{J^{\mu}}{k^{\zeta}k_{\zeta} - m_{(\zeta)}{}^{2}}\right], k^{\nu}\right] \end{array} \right). \tag{6.5}$$

The reader should pause at this point to compare this closely, term by term, with (6.2). It was to get from (6.5) to (6.2), that we went through all the work in section 5 to develop the inverse $I_{\sigma\nu}$ of (5.12).

Next, the $J^{\mu}$ above are all NxN matrices for SU(N), and the internal symmetries of these groups are hidden inside just to keep notation compact and easy for the calculations we have done thus far. Now, however, it is time to start bringing the internal symmetry explicitly into the picture, so we use $J^{\mu} = T^{i} J_{i}{}^{\mu}$, $i = 1,2,3 \ldots N^{2} - 1$, and similar carefully-indexed expressions for the other five currents in (6.5). After some renaming of the summed internal symmetry indexes, we obtain:

$$P^{\sigma\mu\nu} = -\left( \begin{array}{l} \left[\left[\dfrac{T^{i}J_{i}{}^{\mu}}{k^{\alpha}k_{\alpha} - m_{(\alpha)}{}^{2}}, \dfrac{T^{j}J_{j}{}^{\nu}}{k^{\beta}k_{\beta} - m_{(\beta)}{}^{2}}\right], k^{\sigma}\right] \\ + \left[\left[\dfrac{T^{i}J_{i}{}^{\nu}}{k^{\gamma}k_{\gamma} - m_{(\gamma)}{}^{2}}, \dfrac{T^{j}J_{j}{}^{\sigma}}{k^{\delta}k_{\delta} - m_{(\delta)}{}^{2}}\right], k^{\mu}\right] \\ + \left[\left[\dfrac{T^{i}J_{i}{}^{\sigma}}{k^{\varepsilon}k_{\varepsilon} - m_{(\varepsilon)}{}^{2}}, \dfrac{T^{j}J_{j}{}^{\mu}}{k^{\zeta}k_{\zeta} - m_{(\zeta)}{}^{2}}\right], k^{\nu}\right] \end{array} \right). \tag{6.6}$$

The group structure matrices $T^{i}$ and their associated commutator may be factored out of this entire expression (the reader can check this by expanding all commutators, factoring these out, and then reconsolidating), so as to write:

$$P^{\sigma\mu\nu} = -\left[T^i, T^j\right] \left( \begin{array}{l} \left[ \left( \dfrac{J_i{}^{\mu}}{k^{\alpha}k_{\alpha} - m_{(\alpha)}{}^2} \dfrac{J_j{}^{\nu}}{k^{\beta}k_{\beta} - m_{(\beta)}{}^2} \right), k^{\sigma} \right] \\ + \left[ \left( \dfrac{J_i{}^{\nu}}{k^{\gamma}k_{\gamma} - m_{(\gamma)}{}^2} \dfrac{J_j{}^{\sigma}}{k^{\delta}k_{\delta} - m_{(\delta)}{}^2} \right), k^{\mu} \right] \\ + \left[ \left( \dfrac{J_i{}^{\sigma}}{k^{\varepsilon}k_{\varepsilon} - m_{(\varepsilon)}{}^2} \dfrac{J_j{}^{\mu}}{k^{\zeta}k_{\zeta} - m_{(\zeta)}{}^2} \right), k^{\nu} \right] \end{array} \right). \tag{6.7}$$

The group structure constants $f^{ijk}$ in $if^{ijk}T_k = \left[T^i, T^j\right]$ maintain the commutation position of each of the $J_i{}^{\mu}$, that is, $\left[T^i, T^j\right] J_i{}^{\mu} J_j{}^{\nu} = \left[J^{\mu}, J^{\nu}\right]$. This expression is perfectly symmetrical in appearance as between currents $J_i{}^{\mu}$, but now we will take a simple step to break this symmetry: we will simply move both currents into the right hand numerators. Thus, we simply rewrite the above as:

$$P^{\sigma\mu\nu} = -\left[T^i, T^j\right] \left( \begin{array}{l} \left[ \left( \dfrac{1}{k^{\alpha}k_{\alpha} - m_{(\alpha)}{}^2} \dfrac{J_i{}^{\mu} J_j{}^{\nu}}{k^{\beta}k_{\beta} - m_{(\beta)}{}^2} \right), k^{\sigma} \right] \\ + \left[ \left( \dfrac{1}{k^{\gamma}k_{\gamma} - m_{(\gamma)}{}^2} \dfrac{J_i{}^{\nu} J_j{}^{\sigma}}{k^{\delta}k_{\delta} - m_{(\delta)}{}^2} \right), k^{\mu} \right] \\ + \left[ \left( \dfrac{1}{k^{\varepsilon}k_{\varepsilon} - m_{(\varepsilon)}{}^2} \dfrac{J_i{}^{\sigma} J_j{}^{\mu}}{k^{\zeta}k_{\zeta} - m_{(\zeta)}{}^2} \right), k^{\nu} \right] \end{array} \right). \tag{6.8}$$

It is worth noting by the way, that the six currents may be referred to and distinguished by 6=3x2 combinations of the spacetime indexes $\mu, \nu, \sigma$ and internal symmetry indexes $i, j$.

For a next step, we drill down even further, by employing $J_i{}^{\mu} = \overline{\psi} T_i \gamma^{\mu} \psi$ and the like to introduce fermion wavefunctions. So now we have:

$$P^{\sigma\mu\nu} = -\left[T^i, T^j\right] \left( \begin{array}{l} \left[ \left( \dfrac{1}{k^{\alpha}k_{\alpha} - m_{(\alpha)}{}^2} \dfrac{\overline{\psi} T_i \gamma^{\mu} \psi \overline{\psi} T_j \gamma^{\nu} \psi}{k^{\beta}k_{\beta} - m_{(\beta)}{}^2} \right), k^{\sigma} \right] \\ + \left[ \left( \dfrac{1}{k^{\gamma}k_{\gamma} - m_{(\gamma)}{}^2} \dfrac{\overline{\psi} T_i \gamma^{\nu} \psi \overline{\psi} T_j \gamma^{\sigma} \psi}{k^{\delta}k_{\delta} - m_{(\delta)}{}^2} \right), k^{\mu} \right] \\ + \left[ \left( \dfrac{1}{k^{\varepsilon}k_{\varepsilon} - m_{(\varepsilon)}{}^2} \dfrac{\overline{\psi} T_i \gamma^{\sigma} \psi \overline{\psi} T_j \gamma^{\mu} \psi}{k^{\zeta}k_{\zeta} - m_{(\zeta)}{}^2} \right), k^{\nu} \right] \end{array} \right). \tag{6.9}$$

Now, the next steps are important, so let's walk them through carefully. We first write the two back-to-back wavefunctions $\psi\overline{\psi}$ using $\psi = u(p)e^{-ip_1{}^{\alpha}x_{\alpha}}$ and $\overline{\psi} = \overline{u}(p)e^{p_2{}^{\alpha}x_{\alpha}}$. But because these are back to back, $p_2{}^{\alpha} = p_1{}^{\alpha}$, and so $\psi\overline{\psi} = u\overline{u}$. Keep in mind, because we are working with SU(N) in Yang-Mills theory, that $u\overline{u}$ is an NxN SU(N) matrix, in addition to having the usual 4x4 Dirac structure. So if some variant of $u\overline{u}$ finds its way into any denominators as it momentarily will, we have to take an SU(N) *matrix inverse,* and not just write an ordinary denominator. Also taking the sum over all spins, we know that $\sum_{spins} u\overline{u} = \not{p} + m$. But in addition, $(\not{p}+m)/\left(p^{\beta}p_{\beta} - m^2\right) = 1/(\not{p} - m)$. So suddenly, we find that terms which started as vector boson propagators in (6.4) are turning, chameleon-like in (6.9), into a fermion propagator, complete with a "revealed" mass for the fermion. For example, in the top line of (6.9), we make the following progression of substitutions:

$$\frac{\overline{\psi}T_i\gamma^{\mu}\psi\overline{\psi}T_j\gamma^{\nu}\psi}{k^{\beta}k_{\beta} - m_{(\beta)}{}^2} = \frac{\overline{\psi}T_i\gamma^{\mu}\sum u\overline{u}T_j\gamma^{\nu}\psi}{k^{\beta}k_{\beta} - m_{(\beta)}{}^2} = \frac{\overline{\psi}T_i\gamma^{\mu}(\not{p}+m)T_j\gamma^{\nu}\psi}{k^{\beta}k_{\beta} - m_{(\beta)}{}^2} = \frac{\overline{\psi}T_i\gamma^{\mu}\left(\not{p}+m_{(\beta)}\right)T_j\gamma^{\nu}\psi}{p^{\beta}p_{\beta} - m_{(\beta)}{}^2} = \frac{\overline{\psi}T_i\gamma^{\mu}T_j\gamma^{\nu}\psi}{"\not{p} - m_{(\beta)}"} \quad .(6.10)$$

$$= \overline{\psi}T_i\gamma^{\mu}T_j\gamma^{\nu}\psi \times \left(\not{p} - m_{(\beta)}\right)^{-1}$$

First, we use $\psi\overline{\psi} = u\overline{u}$ and sum over all spin states, and because this is SU(N), over all internal symmetry states, then setting $\sum u\overline{u} = \not{p} + m$. Next, we take the *affirmative step* (which as we will discuss shortly requires some accounting for degrees of freedom and so will render the gauge bosons massless) of setting the rest mass in the resultant $\not{p} + m$ to be equal to (one and the same as) the labeled mass $m_{(\beta)}$ in the denominator, that is, we now set $m = m_{(\beta)}$. (This $m_{(\beta)}$, of course, started out in (6.4) as a gauge boson mass in a gauge boson propagator denominator – more chameleon-like behavior! In a moment, we will discuss how to account for degrees of freedom to make this all work properly.) And we simultaneously promote $k^{\beta} \to p^{\beta}$ into the momentum four-vector for an actual fermion. And finally, we set:

$$\frac{\not{p} + m_{(\beta)}}{p^{\beta}p_{\beta} - m_{(\beta)}{}^2} = \frac{1}{"\not{p} - m_{(\beta)}"} = \left(\not{p} - m_{(\beta)}\right)^{-1} \qquad (6.11)$$

in recognition of the fact, which was discussed at length in section 5, that whenever an SU(N) matrix (including $\sum u\overline{u} = \not{p} + m$) needs to go into a "denominator," we must form its inverse. So, these fermion rest masses $m_{(\beta)}$, etc., such as they are, will be obtained via SU(N) matrix inversion. To maintain a clear visual comparison with familiar equation forms, we will continue to use the "quoted denominators" to designate inverses.

So, we use (6.10) to rewrite (6.9) as:

$$P^{\sigma\mu\nu} = -\left[T^i, T^j\right]\left(\begin{array}{l}\left[\left(\dfrac{g^{\alpha\mu}}{k^\alpha k_\alpha - m_{(\alpha)}{}^2}\dfrac{\overline{\psi} T_i \gamma_\alpha T_j \gamma^\nu \psi}{" \not{p} - m_{(\beta)} "}\right), k^\sigma\right] \\ + \left[\left(\dfrac{g^{\alpha\nu}}{k^\gamma k_\gamma - m_{(\gamma)}{}^2}\dfrac{\overline{\psi} T_i \gamma_\alpha T_j \gamma^\sigma \psi}{" \not{p} - m_{(\delta)} "}\right), k^\mu\right] \\ + \left[\left(\dfrac{g^{\alpha\sigma}}{k^\varepsilon k_\varepsilon - m_{(\varepsilon)}{}^2}\dfrac{\overline{\psi} T_i \gamma_\alpha T_j \gamma^\mu \psi}{" \not{p} - m_{(\zeta)} "}\right), k^\nu\right]\end{array}\right). \tag{6.12}$$

where we have also lowered the index on the left-hand vertices in order to reintroduce the $g^{\alpha\mu}$ to the left-hand terms which once again display explicitly, the appearance of a gauge boson propagator. This "chameleon equation" is now in a fully-mixed fermion / boson view, because we now see three fermion propagators and three gauge boson propagators. And, we see how simply moving both currents into the right hand numerators in (6.8) broke the initial symmetry, gave us both fermion and boson propagators in each term of (6.12), and turned three of the six masses $m_{(\beta)}, m_{(\delta)}, m_{(\zeta)}$ into fermion masses while leaving the other three masses $m_{(\alpha)}, m_{(\gamma)}, m_{(\varepsilon)}$ intact as boson masses. What we have done here, is break a mass symmetry that started out with all boson masses, into a mass asymmetry containing both boson and fermion masses.

But there is one final piece of the puzzle that is required to make this all work properly, which is to account for the degrees of freedom in what we just did to turn (6.9) into (6.12). In going from (6.9) to (6.12), (or even from (6.4) to (6.12) where this is even more evident), we started with six presumed massive vector bosons with masses $m_{(\alpha)}, m_{(\beta)}, m_{(\gamma)}, m_{(\lambda)}, m_{(\varepsilon)}, m_{(\zeta)}$. A massive vector boson has three degrees of freedom, so the six bosons we started with in (6.4) brought 3x6=18 degrees of freedom into $P^{\sigma\mu\nu}$. But then between (6.9) and (6.12) we took three of those masses and turned them into fermion masses. Massive fermions, however, have <u>four</u> degrees of freedom, not three. So for us to promote a massive boson mass into a fermion mass, we must transfer one degree of freedom over from the boson to the fermion. So, by associating $m_{(\beta)}, m_{(\lambda)}, m_{(\zeta)}$ in (6.12) with fermion masses, we are required to steal one degree of freedom from each remaining vector gauge boson. So, now these bosons must drop down to <u>two</u> degrees of freedom apiece and must become massless, which means that all of $m_{(\alpha)}, m_{(\gamma)}, m_{(\varepsilon)}$ now must be set to zero. Now, the 18 degrees of freedom that initially belonged three apiece to six massive vector bosons have been redistributed: 12 of these now belong to the 3 fermions, and only 6 belong to the 3 remaining bosons. This should seem very familiar, as this is the same way in which massless gauge bosons first become massive by swallowing a degree of freedom from a scalar field via the Goldstone mechanism. So, to balance the degrees of freedom to account for what we just did, me must now set all of the remaining $m_{(\alpha)}, m_{(\gamma)}, m_{(\varepsilon)} = 0$, and raising the index on the currents once again, (6.12) now becomes:

$$P^{\sigma\mu\nu} = -\left[T^i, T^j\right]\left(\left[\left(\frac{1}{k^\alpha k_\alpha}\frac{\overline{\psi} T_i \gamma^\mu T_j \gamma^\nu \psi}{" \not{p} - m_{(\beta)} "}\right), k^\sigma\right] + \left[\left(\frac{1}{k^\gamma k_\gamma}\frac{\overline{\psi} T_i \gamma^\nu T_j \gamma^\sigma \psi}{" \not{p} - m_{(\delta)} "}\right), k^\mu\right] + \left[\left(\frac{1}{k^\varepsilon k_\varepsilon}\frac{\overline{\psi} T_i \gamma^\sigma T_j \gamma^\mu \psi}{" \not{p} - m_{(\zeta)} "}\right), k^\nu\right]\right). \tag{6.13}$$

The above, (6.13), can now be said to be equal to (6.9) in all respects, including a proper degrees of freedom accounting.

Now we see that $P^{\sigma\mu\nu}$ contains three fermions, with terms $\overline{\psi} T_i \gamma_\alpha T_j \gamma^\nu \psi \times \left( \not{p} - m_{(\beta)} \right)^{-1}$ that look exactly like the expressions for the Compton scattering of a fermion with a gauge boson, such as $\gamma e \to \gamma e$ of an electron with a photon in QED. (e.g., [10] at 141) As we now see clearly from (6.13), <u>$P^{\sigma\mu\nu}$ *naturally contains three fermions, just like a baryon, along all the machinery for fermion propagation, right alongside of propagators for associated, now massless, gauge bosons*</u>. As a result, for the first time, we will stop referring to this as a Yang-Mills magnetic charge, and think of it as a true "baryon candidate." Now we need to show that this really has all the required formal characteristics to be a real, physical baryon.

Proceeding apace, the commutator $\left[ T^i , T^j \right]$ is still sitting out front of (6.13), so let now work with that. The $\left[ T^i , T^j \right]$ operates to commute the vertices $\left( T_i \gamma^\mu \right)\left( T_j \gamma^\nu \right)$, and in particular, the operation it now performs on each term in the current / fermion portion of (6.13) is:

$$\left[ T^i , T^j \right] \overline{\psi} \left( T_i \gamma^\mu \right)\left( T_j \gamma^\nu \right) \psi = \overline{\psi} \left[ \gamma^\mu , \gamma^\nu \right] \psi , \tag{6.14}$$

which is the same commutation $\left[ G^\mu , G^\nu \right]$ of free indexes with which everything started back in (6.2), and even further back, in the underlying field density $F^{\mu\nu} = \partial^\mu G^\nu - \partial^\nu G^\mu - i \left[ G^\mu , G^\nu \right]$ of (2.12) which is the heart of Yang-Mills theory. So, using the above in (6.13), now yields:

$$P^{\sigma\mu\nu} = - \left( \left[ \left( \frac{1}{k^\alpha k_\alpha} \frac{\overline{\psi} \left[ \gamma^\mu , \gamma^\nu \right] \psi}{\text{"} \not{p} - m_{(\beta)} \text{"}} \right), k^\sigma \right] + \left[ \left( \frac{1}{k^\gamma k_\gamma} \frac{\overline{\psi} \left[ \gamma^\nu , \gamma^\sigma \right] \psi}{\text{"} \not{p} - m_{(\delta)} \text{"}} \right), k^\mu \right] + \left[ \left( \frac{1}{k^\varepsilon k_\varepsilon} \frac{\overline{\psi} \left[ \gamma^\sigma , \gamma^\mu \right] \psi}{\text{"} \not{p} - m_{(\zeta)} \text{"}} \right), k^\nu \right] \right) . \tag{6.15}$$

Now, we also know that baryons interact with one another not via massless gauge bosons i.e., gluons, but via quark-antiquark exchanges, i.e., massive meson exchanges (which as we explored in (5.17), may actually be particles that do not have a formal mass but appear to have a pseudo mass by virtue of confusing $1/\left( k^\alpha k_\alpha - m^2 + i\varepsilon \right)$ with $\left( k^\alpha k_\alpha + i \left[ k^\alpha , G_\alpha \right] \right)^{-1}$). So we should expect mesons to make an explicit appearance somewhere. Let's see. . . Using the first term of (6.15) for an example, let us first expand the commutator:

$$\overline{\psi} \left[ \gamma^\mu , \gamma^\nu \right] \psi = \overline{\psi} \gamma^\mu \gamma^\nu \psi - \overline{\psi} \gamma^\nu \gamma^\mu \psi . \tag{6.16}$$

Now let's look at the charge conjugation (antiparticles) of the above. Using the Dirac relationships $\psi_C = C \overline{\psi}^T$, $\overline{\psi}_C = -\psi^T C^{-1}$, $\gamma^\nu C = C \left( -\gamma^\nu \right)^T$, and $C^{-1} \gamma^\mu C = \left( -\gamma^\mu \right)^T$, we obtain:

$$\overline{\psi}_C \gamma^\mu \gamma^\nu \psi_C = -\psi^T C^{-1} \gamma^\mu \gamma^\nu C \overline{\psi}^T = -\psi^T \left( -\gamma^\mu \right)^T \left( -\gamma^\nu \right)^T \overline{\psi}^T = -\overline{\psi} \gamma^\nu \gamma^\mu \psi . \tag{6.17}$$

This means that (6.16) may be rewritten as:

$$\overline{\psi}\left[\gamma^{\mu},\gamma^{\nu}\right]\psi=\overline{\psi}\gamma^{\mu}\gamma^{\nu}\psi-\overline{\psi}\gamma^{\nu}\gamma^{\mu}\psi=\overline{\psi}\gamma^{\mu}\gamma^{\nu}\psi+\overline{\psi}_{C}\gamma^{\mu}\gamma^{\nu}\psi_{C}\,. \tag{6.18}$$

The commutator – which is central to Yang-Mills theory – naturally pairs a particle with an antiparticle to produce a meson! So we go back to (6.15), and now write:

$$P^{\sigma\mu\nu}=-\left(\begin{array}{l}\left[\left(\dfrac{1}{k^{\alpha}k_{\alpha}}\dfrac{\overline{\psi}\gamma^{\mu}\gamma^{\nu}\psi+\overline{\psi}_{C}\gamma^{\mu}\gamma^{\nu}\psi_{C}}{"\not{p}-m_{(\beta)}"}\right),k^{\sigma}\right]\\+\left[\left(\dfrac{1}{k^{\gamma}k_{\gamma}}\dfrac{\overline{\psi}\gamma^{\nu}\gamma^{\sigma}\psi+\overline{\psi}_{C}\gamma^{\nu}\gamma^{\sigma}\psi_{C}}{"\not{p}-m_{(\delta)}"}\right),k^{\mu}\right]\\+\left[\left(\dfrac{1}{k^{\varepsilon}k_{\varepsilon}}\dfrac{\overline{\psi}\gamma^{\sigma}\gamma^{\mu}\psi+\overline{\psi}\gamma^{\sigma}\gamma^{\mu}\psi}{"\not{p}-m_{(\zeta)}"}\right),k^{\nu}\right]\end{array}\right). \tag{6.19}$$

The above also tells us that the antifermions have the same masses as the fermions, because they are all over a common propagator denominator / inverse.

All that now remains in (6.19) is the final commutator with momentum terms such as $k^{\sigma}$. Going back to (5.3), which tells us that commuting a spacetime field with $k^{\sigma}$ is just a clever way to take its derivatives, we can write that in general, for a second rank tensor field $M^{\mu\nu}$:

$$\partial^{\sigma}M^{\mu\nu}=i\left[k^{\sigma},M^{\mu\nu}\right]. \tag{6.20}$$

With this, (6.19) above may finally be expressed without any commutators, as:

$$\boxed{P^{\sigma\mu\nu}=-i\left(\begin{array}{l}\partial^{\sigma}\left(\dfrac{1}{k^{\alpha}k_{\alpha}}\dfrac{\overline{\psi}\gamma^{\mu}\gamma^{\nu}\psi+\overline{\psi}_{C}\gamma^{\mu}\gamma^{\nu}\psi_{C}}{"\not{p}-m_{(\beta)}"}+\ldots\right)\\+\partial^{\mu}\left(\dfrac{1}{k^{\gamma}k_{\gamma}}\dfrac{\overline{\psi}\gamma^{\nu}\gamma^{\sigma}\psi+\overline{\psi}_{C}\gamma^{\nu}\gamma^{\sigma}\psi_{C}}{"\not{p}-m_{(\delta)}"}+\ldots\right)\\+\partial^{\nu}\left(\dfrac{1}{k^{\varepsilon}k_{\varepsilon}}\dfrac{\overline{\psi}\gamma^{\sigma}\gamma^{\mu}\psi+\overline{\psi}\gamma^{\sigma}\gamma^{\mu}\psi}{"\not{p}-m_{(\zeta)}"}+\ldots\right)\end{array}\right)}. \tag{6.21}$$

In the above, we have now also added a $+\ldots$, because going back to (6.3), we see that these are the lowest order terms in this candidate baryon. No matter what other interactions may take place, and even as we start to consider quantum fields where the classical field equations no longer apply, these basic, zero-order terms will always remain. Different conditions and special cases may and will change the higher order terms, but what appears in (6.21) will always remain the fundamental backbone of a baryon.

Comparing the first term in (6.19) with the like term in (6.2) also yields one other very important result, which will be used momentarily to formally show that mesons are the only particles allowed to leave a baryon, thus confining quarks and gluons. Specifically, this comparison yields:

$$\boxed{\left[G^{\mu},G^{\nu}\right]=-\frac{1}{k^{\alpha}k_{\alpha}}\frac{\overline{\psi}\gamma^{\mu}\gamma^{\nu}\psi+\overline{\psi}_{C}\gamma^{\mu}\gamma^{\nu}\psi_{C}}{"\not{p}-m_{(\beta)}"}+\ldots}. \tag{6.22}$$

Before concluding, there is one final point to note, dealing generally with Yang-Mills theory, and not specifically with baryons or QCD. The commutator $\left[G^{\mu},G^{\nu}\right]$ in (6.22) above is central to Yang-Mills theory. In fact, it appears in the very foundational equation of Yang-Mills theory, namely, (2.11). So this often-seen equation can be written in a totally novel form, as:

$$\boxed{F^{\mu\nu}=\partial^{\mu}G^{\nu}-\partial^{\nu}G^{\mu}-i\left[G^{\mu},G^{\nu}\right]=\partial^{\mu}G^{\nu}-\partial^{\nu}G^{\mu}+i\left(\frac{1}{k^{\alpha}k_{\alpha}}\frac{\overline{\psi}\gamma^{\mu}\gamma^{\nu}\psi+\overline{\psi}_{C}\gamma^{\mu}\gamma^{\nu}\psi_{C}}{"\not{p}-m_{(\beta)}"}+\ldots\right)}. \tag{6.23}$$

One may use this to go back to all the equations of Yang-Mills theory, make use of the field strength in the form of (6.23), and see what sorts of new insights emerge. Keep in mind, one can exercise this chameleon-like expression for $\left[G^{\mu},G^{\nu}\right]$ into a variety of other forms as well, including backtracking through the development in this section. Those chameleon exercises are also very helpful if one wishes to draw Feynman diagrams for baryons and mesons, and they lead to term combinations we have not elaborated here because they were not essential to the main line of development.

## 7. Confinement, and Meson Interaction

Now let's use the language of differential forms to show confinement, which helps to establish our "candidate" baryons and mesons as true, physical baryons and mesons. For the field strength $F^{\mu\nu}=\partial^{\mu}G^{\nu}-\partial^{\nu}G^{\mu}-i\left[G^{\mu},G^{\nu}\right]$, we multiply through by $dx_{\mu}dx_{\nu}$, and use the forms $G=G^{\mu}dx_{\mu}$, $F=F^{\mu\nu}dx_{\mu}dx_{\nu}$, $G^{2}=\left[G^{\mu},G^{\nu}\right]dx_{\mu}dx_{\nu}$, and $dG=\left(\partial^{\mu}G^{\nu}-\partial^{\nu}G^{\mu}\right)dx_{\mu}dx_{\nu}$, in a well-known fashion, to compact this to (see [6], Chapter (4.5)):

$$F=dG-iG^{2}. \tag{7.1}$$

For $P^{\sigma\mu\nu}$ we use the magnetic three-form $P=P^{\sigma\mu\nu}dx_{\sigma}dx_{\mu}dx_{\nu}$, as well as $dF=\left(\partial^{\sigma}F^{\mu\nu}+\partial^{\mu}F^{\nu\sigma}+\partial^{\nu}F^{\sigma\mu}\right)dx_{\sigma}dx_{\mu}dx_{\nu}$ and $dG^{2}=\left(\partial^{\sigma}\left[G^{\mu},G^{\nu}\right]+\partial^{\mu}\left[G^{\nu},G^{\sigma}\right]+\partial^{\nu}\left[G^{\sigma},G^{\mu}\right]\right)dx_{\sigma}dx_{\mu}dx_{\nu}$, to multiply $P^{\sigma\mu\nu}=\partial^{\sigma}F^{\mu\nu}+\partial^{\mu}F^{\nu\sigma}+\partial^{\nu}F^{\sigma\mu}$ through by $dx_{\sigma}dx_{\mu}dx_{\nu}$ and then express this in the compacted form:

$$P=dF=d\left(dG-iG^{2}\right)=-idG^{2}. \tag{7.2}$$

This includes the well-known application of $dd=0$: the exterior derivative of an exterior derivative is zero. This is what made the QED magnetic charge vanish back in (2.1) and (2.2).

Similarly, the chromoelectric charge equation is:

$$*J = d*F = d*\left(dG - iG^2\right). \tag{7.3}$$

Now, we apply Gauss' law to (7.3), to write:

$$\oiiint *J = \oiiint d*F = \oiiint d*\left(dG - iG^2\right) = \oiint *F = \oiint *\left(dG - iG^2\right). \tag{7.4}$$

and most importantly, to (7.2) to write:

$$\boxed{\oiiint P = \oiiint dF = \oiiint d\left(dG - iG^2\right) = -i\oiiint dG^2 = \oiint F = \oiint dG - i\oiint G^2 = -i\oiint G^2}. \tag{7.5}$$

These are the "Maxwell's equations" in integral form for "classical," i.e., high action $S >> \hbar$ <u>chromodynamics</u>, (and indeed, for any "classical" Yang-Mills theory) and they mirror the usual Maxwell equations of electrodynamics:

$$\oiiint *J = \oiiint d*F = \oiint *F = \oiint *dA. \tag{7.6}$$

and

$$\oiiint P = \oiiint dF = \oiiint ddG = \oiint F = \oiint dA = 0. \tag{7.7}$$

In (7.5), $\oiiint P$ describes a three dimensional volume which contains the three-fermion / antifermion object $P^{\sigma\mu\nu}$ of (6.21) which is our candidate baryon. But while Maxwell's (7.7), particularly $\oiint F = 0$, tells us that nothing flows out of a volume which contains a magnetic charge (because there are no magnetic charges due to the Abelian theory), equation (7.5) for Yang-Mills theory says something very different. The crux of (7.5) is the part that reads:

$$\boxed{\oiiint P = \oiint F = -i\oiint G^2}. \tag{7.8}$$

This says that across any closed two-dimensional surface surrounding a three-dimensional volume which contains a magnetic charge *P* as developed in (6.21), <u>there is a net field flux</u>, and it is a net flux $-i\oiint G^2$ of $G^2 = \left[G^\mu, G^\nu\right]dx_\mu dx_\nu$ objects. But what are these objects? From (6.22), we learn that they are quark and antiquarks pairs! They are mesons, and nothing else! The interactions of a Yang-Mills magnetic charge *P* are mediated by fermion / anti-fermion pairs known as mesons! No individual quarks may flow across any closes surface. No individual gluons may flow. Nowhere is there any non-zero term with $\oiint *dG$ as there is in a non-zero $\oiint *dA$ in electrodynamics. All that may flow are mesons. These, indeed, are the hallmarks of confinement, which further advances the hypothesis that *P* is a baryon and $G^2$ is a meson.

## 8. Summary and Conclusion

As a result of all the foregoing, combining (6.21) and (6.22) with (7.8) and $P = P^{\sigma\mu\nu} dx_\sigma dx_\mu dx_\nu$ and $G^2 = [G^\mu, G^\nu] dx_\mu dx_\nu$, we may now conclude and summarize the entire thesis of this paper in the single equation:

$$\boxed{\begin{aligned} \oiiint P &= \oiiint P^{\sigma\mu\nu} dx_\sigma dx_\mu dx_\nu = -i \oiiint \begin{pmatrix} \partial^\sigma \left( \dfrac{1}{k^\alpha k_\alpha} \dfrac{\overline{\psi}\gamma^\mu\gamma^\nu\psi + \overline{\psi}_C\gamma^\mu\gamma^\nu\psi_C}{"\not{p} - m_{(\beta)}"} \right) + \dots \\ + \partial^\mu \left( \dfrac{1}{k^\gamma k_\gamma} \dfrac{\overline{\psi}\gamma^\nu\gamma^\sigma\psi + \overline{\psi}_C\gamma^\nu\gamma^\sigma\psi_C}{"\not{p} - m_{(\delta)}"} \right) + \dots \\ + \partial^\nu \left( \dfrac{1}{k^\varepsilon k_\varepsilon} \dfrac{\overline{\psi}\gamma^\sigma\gamma^\mu\psi + \overline{\psi}\gamma^\sigma\gamma^\mu\psi}{"\not{p} - m_{(\zeta)}"} \right) + \dots \end{pmatrix} dx_\sigma dx_\mu dx_\nu \\ &= -i \oiint G^2 = -i \oiint [G^\mu, G^\nu] dx_\mu dx_\nu = i \oiint \left( \dfrac{1}{k^\alpha k_\alpha} \dfrac{\overline{\psi}\gamma^\mu\gamma^\nu\psi + \overline{\psi}_C\gamma^\mu\gamma^\nu\psi_C}{"\not{p} - m_{(\beta)}"} + \dots \right) dx_\mu dx_\nu \end{aligned}} . \tag{8.1}$$

Yang-Mills magnetic charges *P* are indeed the three-quark objects we call baryons, though we see that quarks remain tightly knitted with antiquarks in the form of mesons even inside the enclosed three dimensional surface of the baryon. If the quarks acquire mass, then gluons must be massless to properly account for all degrees of freedom. All that is permitted to net flow across a closed two-dimensional surface are the quark / anti-quark objects we call mesons. Gluons, and individual quarks not paired with an antiquark, can never show a net flux over any closed surface in isolation. Interactions between baryons occur only via meson exchange. This is confinement, the *P* are baryons, and the $G^2$ are mesons!

The above achieves confinement in a manner analogous to the so-called "MIT Bag Model" [11], [12] by paying close attention – very properly so – to what does and does not flow across a closed surface around a baryon. But (8.1) works without any backpressure or other ad-hoc contrivances, and in a way that explains why the nuclear interaction is mediated by mesons.

While exploring these baryons in quantum field theory via the path integral is topic for another paper, one thing that should be clear is this: Whatever the specific details of the path integration, we know from (3.16) that since baryons are magnetic charges, like baryons will *attract*, which is precisely what they need to do to hold together the atomic nuclei. This is yet another indication that *P* represents a true, physical baryon.

Because *P* is a three- fermion system, we must of course because of Fermi-Dirac statistics make certain that no two fermions in this system have the same quantum numbers. So now, for the first time, we formally may select the gauge group SU(3) as the Yang-Mills gauge group that applies to (8.1), assign each of the fermions wavefunctions in (8.1) to one of three color eigenstates $\psi^T = (R \quad G \quad B)$, and thereby enforce an exclusion principle. And in the process, we have answered the very first question we posed: "Why, theoretically, do there exist in nature, naturally-occurring sources, namely baryons, consisting of exactly three strongly-interacting fermion constituents which we call 'quarks'?" And, for reasons developed to go from (6.12) to (6.13), we do not break any symmetries for this group, now formally $SU(3)_C$, but maintain the eight gauge bosons – now gluons – as massless.

Having fully developed the baryon and the mesons according to (8.1), another point should now be made for future consideration, which brings us back to the very beginning of this paper. Equation (8.1) is no more and no less than the logical result of combining the two classical Maxwell equations $J^\nu = \partial_\mu F^{\mu\nu}$ and $P^{\sigma\mu\nu} = \partial^\sigma F^{\mu\nu} + \partial^\mu F^{\nu\sigma} + \partial^\nu F^{\sigma\mu}$ of (2.1) for Yang-Mills gauge theory based on the $F^{\mu\nu} = \partial^\mu G^\nu - \partial^\nu G^\mu - i\left[G^\mu, G^\nu\right]$ of (2.11). Simply find the inverse of $J^\nu = \partial_\mu F^{\mu\nu}$, plug it into $P^{\sigma\mu\nu} = \partial^\sigma F^{\mu\nu} + \partial^\mu F^{\nu\sigma} + \partial^\nu F^{\sigma\mu}$, do the calculations, and arrive at (8.1). *In short, (8.1) is what one obtains when Maxwell's two equations in the context of Yang-Mills theory are merged together into a single equation.* Think about this again: <u>both</u> of Maxwell's equations are embedded in (8.1), i.e., (8.1) is what one inexorably gets from joining together both of Maxwell's equations in Yang-Mills theory. No more, no less. That simple! For anyone who has ever wondered what Maxwell equations would look like if they were all one equation rather than two, (8.1) is the answer! Maxwell's equations, for non-commuting fields, when combined into one, are the classical equations of nuclear physics!

But by duality, what is sauce for the goose is sauce for the gander. So, going back to (2.5), we can repeat the entire course of development instead putting together $P^\nu = \partial_\mu {}^*F^{\mu\nu}$ and ${}^*J^{\sigma\mu\nu} = J^{\sigma\mu\nu} = \partial^\sigma {}^*F^{\mu\nu} + \partial^\mu {}^*F^{\nu\sigma} + \partial^\nu {}^*F^{\sigma\mu}$, to develop a $J^{\sigma\mu\nu}$ that looks in form, just like the $P^{\sigma\mu\nu}$ in (8.1). That is, there is no reason why, if we have developed the "P-Baryons" of (8.1), we cannot also develop a similar J-Baryon that looks like (8.1) with "electric" simply exchanged with "magnetic" under the $\mathbf{E} \to \mathbf{B}, \mathbf{B} \to -\mathbf{E}$ symmetry of duality. So, broken or unbroken symmetry, *there does exist a dual for (8.1).* If the P-Baryons are the baryons we observe as our protons and neutrons and ourselves and all of the matter in our world, then J-Baryons, as a form of "duality matter," can, and likely must, exist also. The only difference would be, because of (3.16) versus (3.9), that J-Baryons *repel* one another, just as do like electric charges. So, where are these dual baryons, and their dual matter with a repulsive nuclear interaction?

One explanation for the non-appearance of J-Baryons may be that the duality symmetry is broken, [*] and that this inverse duality matter only comes into view at much higher energies. If that is so, then there is potentially a whole universe of dual J-Baryon matter that is not accessible to our senses but nonetheless exists and affects the mass of the universe, perhaps to the point where the majority of the matter in the universe remains hidden from view in the form of J-Baryons. Another explanation, which is not exclusive of the first explanation, is that with such heavy J-Baryons exhibiting nuclear repulsion, they would not be clustered together, but in fact would tend to distribute themselves as far from one another as possible in a homogenous manner and so would be very difficult to detect in the same way as ordinary matter. They would certainly contribute to an acceleration of universal expansion, tending to evenly disperse

[*] As explained by this author in http://arxiv.org/abs/hep-ph/0508257.

throughout the cosmos, and would also be capable if "push comes to shove" to possibly counteract any possible gravitational collapse. Thus, the "dual" of the results developed here, particularly with a proper breaking of duality symmetry, may provide a foundation for a serious, mathematically-rigorous understanding of hypothesized (by some) so-called "dark" or hidden matter. But that is a topic for another day. It is enough for the moment, to have shown that baryons are magnetic monopoles.

One final, overarching point, which returns us to section 3. As made clear throughout, (8.1) is a classical equation, valid for high-action $S(\varphi) >> \hbar$. This means that (8.1) (and even the more general equation developed from (6.3) with $G^{\mu} \neq 0$) will become inexact in the quantum arena. Does this mean that $P^{\sigma\mu\nu}$ will stop being a baryon? Of course not. It merely means that we will be using different equations, derived via path integration, in order to describe the behaviors of these baryons in the low-action arena. It merely means that the higher order terms will change and may vary. But the lowest-order, fully structural terms in (8.1) will always remain intact.

So to conclude: the long-sought and pursued and ever-elusive magnetic monopole, in Yang-Mills theory, is a baryon, and it exists everywhere and anywhere that there is matter in the universe, hiding in plan sight!

**Acknowledgement**

The author wishes to express grateful thanks and appreciation to Dr. Andrej (Andy) Inopin of Kharkov, Ukraine for all of his feedback in helping the author develop this manuscript. In 2005, Andy first got me interested in the confinement problem. While I first started to believe and advocate that Yang-Mills magnetic charges are baryons back in early 2007, the formal proof was elusive, particularly because it was unclear at the time how to best derive the inverses that are developed in section 5 here. That, plus a desire to clearly understand where classical theory ended and quantum theory began in terms of evaluating magnetic charges, which is the subject of section 3 here, delayed further progress for about five years.

In recent weeks, Andy pushed me to develop a completely mathematical proof of my magnetic charge / baryon hypothesis, without using Feynman diagrams as a crutch. And he also pushed me in recent days to fully develop the $u\bar{u}$ into formal fermion propagators in the equations, and not merely use them as such more loosely in Feynman drawings. While this work is my own and I take full responsibility for any errors, oversights or omissions, Andy's gentle prodding and correcting and suggesting at various points along the way contributed significantly to turning the thesis that baryons are Yang-Mills magnetic charges, from very strong suspicion and belief, into, perhaps, mathematically-rigorous physical reality.

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