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# Fractional Dirac Magnetic Monopole Charges without Observable Singularities

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*Abstract: It is widely believed that Dirac magnetic monopoles and their related electric charges must be quantized, and that any fractional charges one might posit cannot exist without creating forbidden observable singularities. Here, we explicitly present a vector potential for a Dirac monopole with fractional magnetic and electric charges whose curl is a Coulomb magnetic field and which potential has no observable singularities. We then demonstrate how these fractional charges are projected onto  $SO(3)$  from topological covering groups with generators which are the generalized  $m^{\text{th}}$  roots of the  $2 \times 2$  identity matrix  $I$ , situated at various Euler angles on the complex plane of the covering group generators, all without observable singularities. We also show how this projection gives rise to a form of Euclidean transformation between space and time which preserves the invariance of the Minkowski interval  $t^2 - r^2$  in the geodesic coordinates of flat spacetime. Finally, we show that although fractional charges are permitted without observable singularities, these fractional denominators are naturally restricted by helicity considerations to the odd integers 1, 3, 5, 7... and the even integer 2, while other even number denominators 4, 6, 8... are precluded, which is precisely the same charge pattern observed in the Fractional Quantum Hall Effect (FQHE).*

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## 1. Introduction

In 1931 Dirac [1] discovered that if magnetic charges with strength  $g$  were to hypothetically exist, this would imply that the electric charge strength  $e$  must be quantized. The relationship he found, often written as  $2eg = n$  where  $n$  is a positive or negative integer or zero, came to be known as the Dirac Quantization Condition (DQC). In the mid-1970s, to remediate the fiction of Dirac's "nodal lines" which subsequently became known as Dirac strings, Wu and Yang [2], [3] developed an approach which achieves completely equivalent results "without strings," with the only difference being that it is cast in the more-modern language of fiber bundles. In the Wu Yang approach, one uses  $U(1)_{\text{em}}$  gauge theory to obtain the differential equation  $e^{-i\Lambda} de^{i\Lambda} = i2egd\varphi$  (to be derived at (4.2) infra) where  $\Lambda$  is the gauge (really, phase) angle and  $\varphi$  is the geometric azimuth about the z-axis in the three dimensional physical space of the rotation group  $SO(3)$ . This equation is easily seen to be solved for constant electric and magnetic charge strengths by  $\exp(i\Lambda) = \exp(i2eg\varphi)$  (at (4.3) infra).

It has long been believed that the only solution to this latter Wu-Yang equation which is free of observable singularities, is  $2eg = n$  (at (4.7) infra). This is in fact true if (as will be discussed in section 5 infra) one neglects the fact that spinors also change their "version" when rotated over a  $2\pi$  circuit on  $SO(3)$  (see section 41.5 of Misner, Thorne and Wheeler's [4]), and also neglects the existence of roots of unity generators (section 6 infra) which likewise modify the electron version. But if we fully account for these version changes including a careful consideration of roots of unity it becomes possible to expand this solution to include non-singular fractional charges of the form  $2eg = n/m$  where  $m$  is a second integer specifying the fractional charge denominator (at (11.7) infra). This paper will detail how these fractionally-charged monopoles, and their related fractionalized electric charges, may arise without observable singularities. What is especially intriguing is that the singularity-free denominators  $m$  are not permitted to take on any integer value. Rather, considerations of helicity and charge continuity require restriction to the denominators  $m = 1, 2, 3, 5, 7, 9, \dots$ , which are precisely the same fractional charge denominators experimentally observed in the Fractional Quantum Hall Effect (FQHE) [5].

## PART I: A REVIEW OF THE GAUGE THEORY OF DIRAC MONOPOLES

### 2. Local $U(1)_{\text{em}}$ Gauge Transformations, In General

We begin by considering a first electron wavefunction  $\psi_+(x^\mu)$  which is related to a second electron wavefunction  $\psi_-(x^\mu)$  by the local  $U(1)_{\text{em}}$  gauge transformation (throughout, we shall employ natural units  $\hbar = c = 1$ ):

$$\psi_+ \rightarrow \psi'_+ = \exp(i\Lambda)\psi_+ \equiv \psi_-, \quad (2.1)$$

where the phase angle  $\Lambda(x^\mu)$  varies locally as a function of the spacetime coordinates as do the wavefunctions  $\psi(x^\mu)$ . Transformation (2.1) is often written simply as  $\psi \rightarrow \psi' = \exp(i\Lambda)\psi$ , but by placing the label  $\psi_+$  on  $\psi$  and then  $\psi_-$  on  $\psi_- \equiv \psi'_+$ , we lay the foundation for easily introducing the “north” and “south” gauge patches to study monopoles starting in section 3.

Next, we define a gauge potential  $A_{+\mu}(x^\mu)$  to be an electromagnetic vector potential corresponding with the wavefunction  $\psi_+$ , and we then use this to define the gauge-covariant derivative  $D_{+\mu} \equiv \partial_\mu + ieA_{+\mu}$  where  $e$  is the (running) electric charge strength, and where the sign of  $ieA_{+\mu}$  is positive because we are using a Minkowski metric tensor  $\text{diag}(\eta_{\mu\nu}) = (1, -1, -1, -1)$  versus the oppositely-signed convention. Applying this derivative to each side of  $\exp(i\Lambda)\psi_+$  in (2.1), we obtain:

$$\begin{aligned} D_{+\mu}(\exp(i\Lambda)\psi_+) &= (\partial_\mu + ieA_{+\mu})(\exp(i\Lambda)\psi_+) \\ &= i\partial_\mu\Lambda\exp(i\Lambda)\psi_+ + \exp(i\Lambda)\partial_\mu\psi_+ + ieA_{+\mu}\exp(i\Lambda)\psi_+ . \\ &= \exp(i\Lambda)\left[\partial_\mu\psi_+ + [ieA_{+\mu} + i\partial_\mu\Lambda]\psi_+\right] \end{aligned} \quad (2.2)$$

Based on the inner-bracketed expression in the bottom line above, we define a second, transformed gauge potential  $A_{-\mu} \equiv A'_{+\mu}$  corresponding with the wavefunction  $\psi_-$  according to:

$$eA_{-\mu} = eA_{+\mu} + \partial_\mu\Lambda . \quad (2.3)$$

Then, defining a second covariant derivative  $D_{-\mu} \equiv \partial_\mu + ieA_{-\mu}$ , (2.2) simplifies to:

$$D_{+\mu}(\exp(i\Lambda)\psi_+) = \exp(i\Lambda)\left[\partial_\mu + ieA_{-\mu}\right]\psi_+ = \exp(i\Lambda)D_{-\mu}\psi_+ . \quad (2.4)$$

The foregoing represent a fundamental proposition of local gauge theory: the local gauge transformation (2.1) acting on a fermion  $\psi$  must be compensated by the introduction of a gauge fields  $A_\mu$  transforming according to (2.3) in order to maintain gauge invariance of the electrodynamic Lagrangian and its related field equations. The logical consequence of this proposition is Maxwell’s electrodynamics.

The gauge transformation (2.3) may readily be divided through by  $e$  and rewritten using the mathematical identity  $i\partial_\mu\Lambda = e^{-i\Lambda}\partial_\mu e^{i\Lambda}$  as:

$$A_{-\mu} = A_{+\mu} + e^{-i\Lambda}\partial_\mu e^{i\Lambda} / ie . \quad (2.5)$$

Further, one may generally pack a vector potential into the differential one-form  $A = A_\mu dx^\mu$ . Therefore (2.5) compacts and rearranges into:

$$A_- - A_+ = e^{-i\Lambda} de^{i\Lambda} / ie. \quad (2.6)$$

This tells us that these two gauge fields  $A_-$  and  $A_+$  differ from one another by no more than a  $U(1)_{\text{em}}$  gauge transformation, which is apparent because these are just relabeled names for the one-forms  $A$  and  $A'$  transforming according to  $A' = A + e^{-i\Lambda} de^{i\Lambda} / ie$ . Therefore, these two gauge fields are not observably-distinct.

### 3. A Coulomb Magnetic Field which is the Curl of a Vector Potential, i.e., a $U(1)_{\text{em}}$ Magnetic Monopole

The electromagnetic field strength two-form  $F = \frac{1}{2} F_{\mu\nu} dx^\mu dx^\nu$  is generally related to the vector potential by  $F = dA$ , and so is a locally-exact two-form. Extracting the electric / magnetic bivector  $F_{\mu\nu}$ , the space components of the field strength tensor are  $F_{ij} = \partial_i A_j - \partial_j A_i$ . The magnetic field vector  $F_{ij} = -\varepsilon_{ijk} B^k$  where  $\varepsilon_{ijk}$  is the antisymmetric Levi-Civita tensor and  $\varepsilon_{123} = +1$ , and where  $B^k = \mathbf{B} = (B_x, B_y, B_z)$  in Cartesian coordinates. Likewise, using  $\text{diag}(\eta_{\mu\nu}) = (1, -1, -1, -1)$  to lower indexes in  $A^\mu = (\phi, \mathbf{A}) = (\phi, A_x, A_y, A_z)$ , and with  $\partial_i = \nabla = (\partial_x, \partial_y, \partial_z)$ , this means that  $F_{ij} = -\varepsilon_{ijk} B^k = \partial_i A_j - \partial_j A_i$ , or  $\mathbf{B} = \nabla \times \mathbf{A}$ . So whenever we have  $F = dA$  in general for a given potential, the magnetic field  $\mathbf{B}$  will be the curl of the vector potential  $\nabla \times \mathbf{A}$ .

Now, by way of reviewing known monopole physics, let us define the two four-vector potentials in  $A_-$  and  $A_+$  of the last section *such that* these are the potentials for a *Coulomb magnetic field*  $\mathbf{B}$  which is the curl of these vector potentials,  $\mathbf{B} = \nabla \times \mathbf{A}$ , that is, let us now define the gauge potentials for a magnetic monopole. We do this by simply postulating a form for these potentials, then showing that these do in fact reproduce a Coulomb magnetic field with  $\mathbf{B} = \nabla \times \mathbf{A}$ .

We start by positing a (running) magnetic charge strength  $g$  for such a monopole, and then postulate each of the potential one-forms  $A_-$  and  $A_+$  in a spherical coordinate basis to be:

$$\begin{aligned} A_+ &\equiv g (\cos \theta - 1) d\varphi \\ A_- &\equiv g (\cos \theta + 1) d\varphi \end{aligned} \quad (3.1)$$

Confining our domain to  $0 \leq \theta \leq \pi$ ,  $A_+$  is “northerly” because it is defined everywhere except for  $\theta = \pi$ , i.e., except due south of the origin, while  $A_-$  is a “southerly” potential defined everywhere except for  $\theta = 0$ , i.e., except due north of the origin. Often these are referred to as the north and south gauge patches,  $A_N \equiv A_+$  and  $A_S \equiv A_-$ , and we see via (2.6) that these differ from one another simply by a gauge transformation and so are not observably-distinct. We now

show that these will indeed produce a Coulomb magnetic field for which the curl  $\mathbf{B} = \nabla \times \mathbf{A}$  for both of the vector potentials  $\mathbf{A}_+$ ,  $\mathbf{A}_-$ .

First, we hold  $g$  constant,  $dg = 0$ , that is, we do not let  $g$  run over the region of spacetime in question. Now, because differential forms geometry teaches that  $dd = 0$  in general and thus  $dd\varphi = 0$  in this specific setting, this all means that:

$$F = dA_- = dA_+ = gd \cos \theta d\varphi. \quad (3.2)$$

Therefore, for either potential, the magnetic field  $\mathbf{B} = \nabla \times \mathbf{A}_+ = \nabla \times \mathbf{A}_-$  is the curl of the gauge potential, as desired.

Of course,  $dF = ddA_- = ddA_+ = 0$  via the same identity  $dd = 0$ , which means that  $F$  is closed and locally exact. But it is not globally exact. Specifically, if we integrate (3.2) over a closed two-dimensional surface with  $g$  still held constant, and if we also apply Gauss' / Stokes' theorem, then:

$$\iiint dF = \oiint F = \oiint gd \cos \theta d\varphi = g \int_0^\pi d \cos \theta \int_0^{2\pi} d\varphi = g \cos \theta \Big|_0^\pi \varphi \Big|_0^{2\pi} = -4\pi g. \quad (3.3)$$

The fact that we are holding  $g$  constant throughout the spacetime region of the integration is reflected by our having moved  $g$  outside the integral after the third equal sign above. Now let us specifically pinpoint the magnetic field.

To do so, we consider the circumstance under which the electric fields vanish, that is, under which  $F_{0k} = -F_{k0} = \mathbf{E} = 0$ . In this circumstance,  $\oiint F = \oiint \frac{1}{2} F_{\mu\nu} dx^\mu dx^\nu = \oiint \frac{1}{2} F_{ij} dx^i dx^j$ .

Then, using this in (3.3) also in view of  $F_{ij} = -\varepsilon_{ijk} B^k$ , we find that:

$$\oiint F = \oiint \frac{1}{2} F_{\mu\nu} dx^\mu dx^\nu = \oiint F_{12} dx^1 dx^2 + \oiint F_{23} dx^2 dx^3 + \oiint F_{31} dx^3 dx^1 = -\oiint \mathbf{B} \cdot d\mathbf{S} = -4\pi g. \quad (3.4)$$

So from the final equality above, this means that:

$$\oiint \mathbf{B} \cdot d\mathbf{S} = 4\pi g = \mu, \quad (3.5)$$

where  $\mu \equiv 4\pi g$  is defined as the total magnetic flux across the closed surface. Conversely, the magnetic charge strength  $g = \mu / 4\pi$  represents the steradial density of magnetic flux across the closed surface. This, of course, is Gauss' law for magnetism in integral form, but with a non-zero magnetic flux  $\mu$  across the closed surface. Thus, this is the integral formulation of Gauss' law for a non-vanishing magnetic monopole. Because this was arrived at using  $\mathbf{E}=0$  in (3.4), (3.5), there are no *electric* fields induced by this monopole, and as a result, (3.5) describes this magnetic monopole at rest.

Now, in general, Coulomb's law cannot be derived from Gauss' law alone. However, if the magnetic monopole is stationary – which it is because  $\mathbf{E} = 0$  in (3.4) and (3.5) – then the magnetic field  $\mathbf{B}$  in (3.5) will be exactly spherically symmetric. As a result of this spherical symmetry, we may remove  $\mathbf{B}$  from the integrand in (3.5), thus writing:

$$\mathbf{B} \oint \mathbf{dS} = \mathbf{B} \cdot 4\pi r^2 = 4\pi g = \mu. \quad (3.6)$$

Because of the spherical symmetry, only the radial component  $B_r$  of  $\mathbf{B}$  will be non-zero, that is, in spherical coordinates, we will have  $\mathbf{B} = (B_r, B_\phi, B_\theta) = (B_r, 0, 0)$ . Therefore, (3.6) now yields:

$$B_r = \frac{g}{r^2} = \frac{\mu}{4\pi r^2}. \quad (3.7)$$

This is indeed a Coulomb magnetic field which has a (constant) magnetic charge strength  $g$ , and for which the total magnetic flux across any closed surface is  $\mu = 4\pi g$ . Furthermore, this Coulomb magnetic field is the curl of the vector potentials,  $\mathbf{B} = \nabla \times \mathbf{A}_+ = \nabla \times \mathbf{A}_-$ . Consequently, we have completed our review of how the potentials postulated in (3.1) do in fact specify a non-vanishing Coulomb magnetic field with  $\mathbf{B} = \nabla \times \mathbf{A}$ .

Now, we begin to examine the full set of conditions under which this Coulomb magnetic monopole with  $\mathbf{B} = \nabla \times \mathbf{A}$  does not give rise to any observable singularities.

#### 4. Conditions under which the $U(1)_{\text{em}}$ Magnetic Monopole has No Observable Singularities: The Standard Dirac Quantization Condition

Returning to (3.1), we first find that the difference:

$$A_- - A_+ = 2gd\phi. \quad (4.1)$$

Combining the above with (2.6) then yields the Wu-Yang [2], [3] differential equation:

$$e^{-i\Lambda} de^{i\Lambda} / ie = 2gd\phi. \quad (4.2)$$

This differential equation is solved for constant  $e$  and constant  $g$ , i.e., for  $de = 0$  and  $dg = 0$  by:

$$\exp(i\Lambda) = \exp(i2eg\phi), \quad (4.3)$$

as is easily seen by plugging (4.3) back into the left hand side of (4.2) then reducing.

We next employ this solution to operate on  $\psi_+$ , which combined with (2.1) yields:

$$\psi_+ \rightarrow \psi'_+ = \psi_- = \exp(i\Lambda)\psi_+ = \exp(i2eg\phi)\psi_+. \quad (4.4)$$

Clearly, for  $\varphi=0$ , we have  $\psi_-(0)=\psi_+$ . Now, let us move this wavefunction through the Coulomb magnetic field of (3.7) around a closed curve in the azimuthal direction, going from  $\varphi=0$  to  $\varphi=2\pi$ . When this single circuit about the monopole is complete, from (4.4) with  $\varphi=2\pi$  we obtain:

$$\psi_+ \rightarrow \psi_- = \psi'_+ = \exp(i\Lambda)\psi_+ = \exp(i4\pi eg)\psi_+. \quad (4.5)$$

Now let's turn to the question of observable singularities.

To avoid observable singularities, it is required that the electron wavefunction at  $\varphi=2\pi$  be the same wavefunction as it is at the geometrically identical azimuth  $\varphi=0$  on SO(3), in other words, that it have the single value  $\psi_+(0) \rightarrow \psi_+(2\pi) = \psi_+(0)$  and not multiple values at the same azimuthal orientation on SO(3). This requirement will be satisfied *if and only if*:

$$\psi_+ \rightarrow \psi_- = \psi'_+ = \exp(i\Lambda)\psi_+ = \exp(i4\pi eg)\psi_+ = 1 \cdot \psi_+ = \exp(i2\pi n)\psi_+, \quad (4.6)$$

which, with  $1 = \exp(i2\pi n)$  where  $n$  is a positive or negative integer or zero, means *if and only if*:

$$2eg = n = 1, 2, 3, 4, \dots \quad (4.7)$$

Using  $g = \mu / 4\pi$ , this may alternatively be expressed as

$$e\mu = 2\pi n. \quad (4.8)$$

These are two different but equivalent ways of stating the standard Dirac Quantization Condition (DQC). From (4.7), we see that the electric charge strength is quantized in units of  $e = \frac{1}{2}n/g$ , and reciprocally, that the magnetic charge strength is quantized in units of  $g = \frac{1}{2}n/e$ .

With condition (4.7) imposed, (4.4) becomes:

$$\psi_+ \rightarrow \psi_- = \psi'_+ = \exp(i\Lambda)\psi_+ = \exp(in\varphi)\psi_+, \quad (4.9)$$

which contains the implied quantized relationship:

$$\Lambda = n\varphi \quad (4.10)$$

between the phase angle  $\Lambda$  and the azimuth angle  $\varphi$ . So during the course of traversing a circuit from  $\varphi=0$  to  $\varphi=2\pi$ , (4.10) tells us that what Dirac often refers to in [1] as the observable “change in phase round” a “closed curve” becomes:



$$\Lambda_{\Delta} = 2\pi n = 2\pi, 4\pi, 6\pi, 8\pi \dots, \quad (4.11)$$

where we use the  $\Delta$  subscript to make clear that this is a *change* in phase, not an absolute phase. In other words, keeping in mind that only a change in phase but not an absolute phase is observable, (4.11) tells us that the change in phase over a single closed azimuthal circuit must be an integral multiple of  $2\pi$  in order to avoid observable singularities. Because all phase angles with  $\Lambda = 2\pi n$  have identical orientation and magnitude in the complex phase space defined by  $\exp(i\Lambda) = \cos \Lambda + i \sin \Lambda \equiv a + ib$ , (4.11) tells us that to avoid observable singularities as specified by (4.6), the phase difference must be  $\Lambda_{\Delta} = 2\pi n$  whenever the electron returns to its original azimuth orientation. For purposes of discussion, we shall refer to this  $\Lambda_{\Delta} = 2\pi n$  phase difference with like-orientation for the initial and final phases as a “return to phase.”

Using the quantization condition (4.7) we may finally return to (3.1) to write the monopole potentials as:

$$\begin{aligned} eA_+ &= \frac{1}{2} n (\cos \theta - 1) d\varphi \\ eA_- &= \frac{1}{2} n (\cos \theta + 1) d\varphi \end{aligned} \quad (4.12)$$

It is sensible that for charge strengths which are quantized, the associated potential energies will likewise be quantized as above.

All of the foregoing summarizes the present-day understanding of  $U(1)_{em}$  magnetic monopoles and the quantization condition  $2eg = n$  of (4.7) which is understood to be required if these monopoles are to exist without observable singularities. But there are other charge conditions which may also exist without observable singularities. These will be the focus of the remainder of this paper.

## **PART II: DIRAC MONOPOLE CHARGES WITH HALF-INTEGER FRACTIONS**

### **5. Tidally-Locked Electron Wavefunctions and Half-Integer Fractional Monopole Charges based on Wavefunction Version**

In the derivation of the Dirac Quantization Condition just reviewed, there is an unstated assumption that the electron wavefunction, over the course of its circuit about the monopole from  $\varphi = 0$  to  $\varphi = 2\pi$ , it not itself undergoing any rotation. But now let us examine what happens if the electron itself rotates in a “tidal lock” with the monopole as it traverses the monopole (as the moon does when it traverses the earth such that the far side of the moon is never visible from earth), so that in the course of traversing from  $\varphi = 0$  to  $\varphi = 2\pi$  about the monopole the electron also rotates through  $2\pi$  via the rotation group of  $SU(2)$  which is the universal cover of  $SO(3)$ . This is important for two reasons. First, this will establish that a half-unit magnetic charge  $2eg = n/2$  can also exist without observable singularities. Second, this will provide the template for showing in sections 10 and 11 how additional fractional charges may also exist without observable singularities.

We start with the three 2x2 Pauli matrices  $\sigma_i$  of SU(2), posit three associated angles  $\theta_i$  in the physical space of spacetime, and form the matrices  $U_i = \exp(i\sigma_i\theta_i/2)$  which are unitary,  $U^\dagger U = 1$ , given that  $\sigma_i^\dagger = \sigma_i$  are Hermitian, which  $U_i$  matrices describe rotations through respective angles  $\theta_i = \theta_x, \theta_y, \theta_z$  about each of the x, y, z axes. It is well-known how to use the series  $\exp(ix) = 1 + ix - \frac{1}{2!}x^2 - i\frac{1}{3!}x^3 + \frac{1}{4!}x^4 \dots$  together with the fact that  $\sigma_i^{2n} = I_i$  and  $\sigma_i^{2n+1} = \sigma_i$  to flesh out these unitary matrices, each of which has  $\det U_i = 1$  into the well-known:

$$\begin{aligned}
 U_1 &= \exp\left(i\sigma_1 \frac{\theta_1}{2}\right) = \begin{pmatrix} \cos(\theta_1/2) & i \sin(\theta_1/2) \\ i \sin(\theta_1/2) & \cos(\theta_1/2) \end{pmatrix} \\
 U_2 &= \exp\left(i\sigma_2 \frac{\theta_2}{2}\right) = \begin{pmatrix} \cos(\theta_2/2) & \sin(\theta_2/2) \\ -\sin(\theta_2/2) & \cos(\theta_2/2) \end{pmatrix} \\
 U_3 &= \exp\left(i\sigma_3 \frac{\theta_3}{2}\right) = \begin{pmatrix} \cos(\theta_3/2) + i \sin(\theta_3/2) & 0 \\ 0 & \cos(\theta_3/2) - i \sin(\theta_3/2) \end{pmatrix} \\
 &= \begin{pmatrix} \exp(i\theta_3/2) & 0 \\ 0 & \exp(-i\theta_3/2) \end{pmatrix}
 \end{aligned} \tag{5.1}$$

Continuing with the natural units  $\hbar = c = 1$  let us next consider an electron traveling with velocity  $\beta = v$  along the z axis and thus the Lorentz contraction factor  $\gamma = 1/\sqrt{1-v^2}$ . As is often done, we may then define the boost parameters  $\cosh \chi \equiv \gamma$  and  $\sinh \chi \equiv \gamma\beta$ , and write the Lorentz transformation as the hyperbolic “rotation”:

$$\begin{pmatrix} t \\ z \end{pmatrix} \rightarrow \begin{pmatrix} t' \\ z' \end{pmatrix} = \begin{pmatrix} \cosh \chi & \sinh \chi \\ \sinh \chi & \cosh \chi \end{pmatrix} \begin{pmatrix} t \\ z \end{pmatrix}. \tag{5.2}$$

Several of the points to now be developed are found in [6], amidst pages 36 to 42.

The electron wavefunction  $\psi$  is a four-component Dirac spinor which we can denote by  $\psi^T = (\xi^T, \eta^T)$ , where  $\xi$  and  $\eta$  are each two-component spinors with all components interrelated via Dirac’s equation  $(i\gamma^\mu \partial_\mu - m)\psi = 0$ . Under a transformation (5.2) defined by the Lorentz group SO(1,3), which includes a general boost  $\chi$  and spatial rotation through  $\theta$  on SO(1,3), these spinor components will transform on SL(2,C) according to:

$$\psi = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \rightarrow \psi' = \begin{pmatrix} \xi' \\ \eta' \end{pmatrix} = \begin{pmatrix} \exp(i\sigma \cdot (\theta - i\chi)/2) & 0 \\ 0 & \exp(i\sigma \cdot (\theta + i\chi)/2) \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \tag{5.3}$$

where  $\sigma = \sigma_i$  are the 2x2 Pauli matrices. So for a non-relativistic electron with  $\chi \rightarrow 0$  undergoing simply a rotation without boost, this simplifies to:

$$\psi = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \rightarrow \psi' = \begin{pmatrix} \xi' \\ \eta' \end{pmatrix} = \begin{pmatrix} \exp(i\sigma \cdot \theta / 2) & 0 \\ 0 & \exp(i\sigma \cdot \theta / 2) \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = I_{(2)} \otimes \exp\left(i\sigma \cdot \frac{\theta}{2}\right) \psi = I_{(2)} U \psi, \quad (5.4)$$

where  $I_{(2)}$  is a 2x2 identity matrix. For an azimuthal rotation through  $\theta_3 = \varphi$  about the z axis only, this becomes  $\psi \rightarrow \psi' = I_{(2)} \otimes \exp(i\sigma_3 \varphi / 2) \psi = I_{(2)} \otimes U_3 \psi$ , for which the unitary matrix  $U_3$  is explicitly given by the third relation in (5.1) with  $\theta_3$  replaced by  $\varphi$ . Thus, (5.4) for an azimuthal rotation only will operate identically upon each of the two-spinors  $\xi, \eta$ . So for  $\xi$ :

$$\xi' = U_3 \xi = \exp\left(i\sigma_3 \frac{\varphi}{2}\right) \xi = \begin{pmatrix} \cos(\varphi/2) + i \sin(\varphi/2) & 0 \\ 0 & \cos(\varphi/2) - i \sin(\varphi/2) \end{pmatrix} \begin{pmatrix} \xi_A \\ \xi_B \end{pmatrix}, \quad (5.5)$$

and likewise for when  $\eta$  replaces  $\xi$ . The need to consider the spinors  $\xi$  and  $\eta$  together in the four-component Dirac wavefunction  $\psi^T = (\xi^T, \eta^T)$  arises because these are interchanged  $\xi \leftrightarrow \eta$  under parity, but when the boost is removed the overall  $\psi$  as well as each of  $\xi$  and  $\eta$  will transform in identical fashion. We finally consolidate the operation (5.5) on both  $\xi, \eta$  into one expression, using the  $\psi \rightarrow \psi_+$  labelling of (2.1), as such:

$$\begin{aligned} \psi_+ = \begin{pmatrix} \xi_+ \\ \eta_+ \end{pmatrix} \rightarrow \psi'_+ = \begin{pmatrix} \xi'_+ \\ \eta'_+ \end{pmatrix} &= I_{(2)} \otimes \exp\left(i\sigma_3 \frac{\varphi}{2}\right) \psi_+ = I_{(2)} \otimes U_3 \psi_+ \\ &= \begin{pmatrix} \cos(\varphi/2) \pm i \sin(\varphi/2) & 0 \\ 0 & \cos(\varphi/2) \pm i \sin(\varphi/2) \end{pmatrix} \begin{pmatrix} \xi_+ \\ \eta_+ \end{pmatrix} = \begin{pmatrix} \exp(\pm i\varphi/2) & 0 \\ 0 & \exp(\pm i\varphi/2) \end{pmatrix} \begin{pmatrix} \xi_+ \\ \eta_+ \end{pmatrix}, \end{aligned} \quad (5.6)$$

with the  $\pm$  signs denoting the respective operations on each component of  $\xi_{+T} = (\xi_{+A}, \xi_{+B})$  and  $\eta_{+T} = (\eta_{+A}, \eta_{+B})$ . This is a more explicit form of (5.4) for an azimuthal rotation with  $\sigma \cdot \theta = \sigma_3 \theta_3 = \sigma_3 \varphi$ , also adopting the labelling of (2.1).

Now, let us return to the gauge transformation  $\psi_+ \rightarrow \psi'_+ = \exp(i\Lambda) \psi_+$  of (2.1) and contrast this against (5.6). As already noted, now quoting Dirac from page 63 of [1], “the value of [the phase] at a particular point has no physical meaning and only the difference between the values of [the phase] at two different points is of any importance.” So, if we are comparing phases as between two different azimuthal points (for the non-relativistic electron presently under examination), then we should also inquire whether the electron has been rotated at all when moving from one such point to the next. If the electron has not rotated but the phase has changed, then the transformation will be  $\psi_+ \rightarrow \psi'_+ = \exp(i\Lambda) \psi_+$  from (2.1). Conversely, if the electron has rotated but the phase has not changed, then the transformation will be

$\psi_+ \rightarrow \psi'_+ = I_{(2)} \exp(i\sigma_3\varphi/2)\psi_+$  from (5.6). But, if both the phase has changed *and* the electron has rotated, then the complete transformation will be a combination of the operations from both (2.1) and (5.6), namely:

$$\psi_+ \rightarrow \psi'_+ = I_{(2)} \otimes U_3 \exp(i\Lambda)\psi_+ = I_{(2)} \otimes \exp\left(i\sigma_3 \frac{\varphi}{2}\right) \exp(i\Lambda)\psi_+ = I_{(2)} \otimes \exp i\left(\sigma_3 \frac{\varphi}{2} + \Lambda\right)\psi_+. \quad (5.7)$$

With (5.7) we are now equipped to ask what happens if the electron makes a complete circuit about the monopole through a  $2\pi$  azimuth *and* simultaneously does so in a tidal lock with the monopole and so also rotates through  $2\pi$ , all on SO(3).

To avoid observable singularities, as in section 4, we must still have a single-valued electron after the full  $2\pi$  circuit is complete, that is, we must still have  $\psi'_+ = \psi_+(2\pi) = \psi_+(0)$ . But now, the condition required to avoid a singularity will be given by  $\psi'_+ = \psi_+$  with  $\varphi = 2\pi$  in (5.7), that is, by:

$$\psi_+ \rightarrow \psi'_+ = I_{(2)} \otimes \exp(i\sigma_3\pi) \exp(i\Lambda)\psi_+ = I_{(2)} \otimes \exp i(\sigma_3\pi + \Lambda)\psi_+ = I_{(4)}\psi_+ = I_{(4)} \exp(i2\pi n)\psi_+. \quad (5.8)$$

This will be recognized as an eigenvalue equation  $(I_{(2)} \otimes \exp i(\sigma_3\pi + \Lambda) - I_{(4)} \exp(i2\pi n))\psi_+ = 0$  for the phase difference  $\Lambda_\Delta$ , where  $I_{(4)}$  is the 4x4 identity matrix, and of course  $1 = \exp(i2\pi n)$  for all integers  $n$ .

But in this particular case we can deduce from (5.5) with  $\varphi/2 = \pi$  that  $\exp(i\sigma_3\pi) = -I$  so that  $I \exp(i\sigma_3\pi) = -I_{(4)}$  which produces a sign reversal. Therefore, with  $\varphi = 0$  and  $\varphi = 2\pi$  explicitly denoted, (5.8) can be reduced to:

$$\psi_+(0) \rightarrow \psi'_+ = \psi_+(2\pi) = -\exp(i\Lambda)\psi_+ = \psi_+(0). \quad (5.9)$$

So to avoid an observable singularity when the electron traverses the monopole in a tidal lock, given, mathematically, that  $-1 = \exp(i\pi(2n-1))$ , and in contrast to (4.6), we must now have:

$$\exp(i\Lambda)\psi_+ = -\psi_+ = \exp(i\pi(2n-1))\psi_+. \quad (5.10)$$

As a result, for the tidal-locked electron, (5.10) tells us that after a single  $2\pi$  circuit the change in phase will be:

$$\Lambda_\Delta = \pi(2n-1) = \pi, 3\pi, 5\pi, 7\pi, \dots, \quad (5.11)$$

which is an odd-integer multiple of  $\pi$ , contrast (4.11) which is an even-integer multiple of  $\pi$ .

Most importantly, if combine (5.10) with the Wu-Yang equation (4.4) also obtained from a single  $2\pi$  circuit about the monopole, that is, if we combine (5.10) with (4.4) for  $\varphi = 2\pi$ , we now obtain:

$$\exp(i\Lambda)\psi_+ = \exp(i\pi(2n-1))\psi_+ = \exp(i2eg\varphi)\psi_+ = \exp(i4\pi eg)\psi_+. \quad (5.12)$$

From  $\exp(i\pi(2n-1))\psi_+ = \exp(i4\pi eg)\psi_+$  above, we may finally extract  $\pi(2n-1) = 4\pi eg$  which reduces to:

$$2eg = n - \frac{1}{2} = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \dots, \quad (5.13)$$

for  $n = 1, 2, 3, 4, 5, \dots$ . Contrasting the usual DQC  $2eg = n$  of (4.7), we see that to avoid observable singularities for a tidally-locked electron which rotates in synchronization with its circuit about the monopole, we must now have a Dirac quantization condition for which the charges are *half-integer charge fractions that skip over the whole integer charges*. If we then merge (4.7) for an electron that does not rotate (no tidal lock) with (5.13) for an electron that does maintain a tidal lock during its circuit about the monopole, we may combine these together to write a fractional DQC which includes half-integer electric and magnetic charges, given by:

$$2eg = \frac{n}{m}; \quad n = 0, 1, 2, 3, 4, \dots; \quad m = 1, 2, \quad (5.14)$$

precisely as was stated in the opening paragraph of this section. Now, let's step back and discuss what has happened here.

We see from (5.4) that a Dirac electron houses two spinors  $\xi$ ,  $\eta$  which transform identically under pure rotations sans boost. It is well-known, as elaborated by Misner, Thorne and Wheeler (MTW) in their classic exposition at section 41.5 of [4] that a spinor changes its sign every time it undergoes a  $2\pi$  rotation, and only regains its original sign after a  $4\pi$  rotation. We discuss this by saying that an electron changes its “version” after a  $2\pi$  rotation and only recovers its original version after  $4\pi$ . MTW analogize this version change to the macroscopic and entirely classical “orientation-entanglement” phenomenon wherein an object connected to its environment by a set of threads will only regain its original state of entanglement after it is rotated twice over  $4\pi$ , but will have an opposite entanglement following only a  $2\pi$  rotation. But it is not necessary here to use this macroscopic analogy. Equation (5.9) makes clear that at the quantum level, the sign of the electron wavefunction will invert following a  $2\pi$  rotation and only be restored after  $4\pi$ , and this is because a  $\theta$ -angle rotation on  $SO(3)$  is implicitly accompanied by a half angle  $\theta/2$  rotation on  $SU(2)$ . (This exact connection will be reviewed in section 8, see (8.10) through (8.12) supra.) This change in version is seen most directly by using  $U_3$  with  $\theta_3 = \varphi$  to operate on  $\xi$  as such:

$$U_3\xi = \begin{pmatrix} \exp(i\varphi/2) & 0 \\ 0 & \exp(-i\varphi/2) \end{pmatrix} \begin{pmatrix} \xi_A \\ \xi_B \end{pmatrix} = \begin{pmatrix} \exp(i\varphi/2)\xi_A \\ \exp(i\varphi/2)\xi_B \end{pmatrix}, \quad (5.15)$$

so that when  $\varphi = 2\pi$  this will become  $U_3(2\pi)\xi = -\xi$  but when  $\varphi = 4\pi$  this will become  $U_3(4\pi)\xi = \xi$ . Numerically, this is encoded in the denominator of 2 first appearing in (5.3), due to the projective two-to-one, double-covering, homomorphic mapping  $\pi : SU(2) \rightarrow SO(3)$ , and when the complete Dirac theory is taken into account, likewise to the mapping  $\pi : SL(2, C) \rightarrow SO(1, 3)$ . This double covering produces a two-valuedness in the sign of a positively-signed electron wavefunction rotated from  $\varphi = 0$  to  $\varphi = 2\pi k$  depending on whether there are an even number of cycles  $k = 2, 4, 6, 8, \dots$  in which case the sign remain positive, or an odd number of cycles  $k = 1, 3, 5, 7, \dots$  in which case the sign flips to being negative.

So what is really happening with the odd-integer times  $\pi$  phase change of (5.11) and with the half integer Dirac charges of (5.13), is that the phase change must compensate for any wavefunction version change which might take place as the electron traverses a circuit about the magnetic monopole, such that when the electron returns to its original azimuth by going from  $\varphi = 0$  to  $\varphi = 2\pi$ , there are no observable singularities. Because a tidally-locked electron wavefunction will invert its sign after a  $2\pi$  circuit about the monopole, and because we require a single-valued wavefunction as between  $\varphi = 0$  and  $\varphi = 2\pi$ , we must compensate this version change with a shift in the phase change in order to ensure that there are no observable singularities. Thus, the phase change shifts by half a cycle, from  $\Lambda_\Delta = 2\pi, 4\pi, 6\pi, 8\pi, \dots$  in (4.11) for an electron which does not rotate during its circuit, to  $\Lambda_\Delta = \pi, 3\pi, 5\pi, 7\pi, \dots$  in (5.11) for an electron which does rotate in a tidal lock with the monopole. Then, because the Wu-Yang solution  $\exp(i\Lambda) = \exp(i2eg\varphi)$  of (4.4) in turn relates this phase change to the charge strength product  $2eg$ , the net consequence is that the electric and magnetic charge strengths now become quantized in the half-integer units  $2eg = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \dots$  of (5.13), rather than the whole integer units  $2eg = 1, 2, 3, 4, \dots$  of the standard Dirac condition (4.7). This sort of “*phase / version synchronization,*” wherein the phase must synchronize itself to compensate any wavefunction version changes in order to avoid observable singularities, then becomes the foundation of fractionalized electric and magnetic Dirac charges. In essence, this synchronization compensates for the two-valuedness of the double cover  $\pi : SU(2) \rightarrow SO(3)$  to ensure that there are no observable singularities after any given  $2\pi$  cycle, which in turn owes to the fact that  $SU(2)$  is a simple group while the double-covered  $SO(3)$  is not.

The example just shown with half-integer electric and magnetic charge fractions is just that: an example. As stated at the start of this section, this establishes that half-unit magnetic charges  $2eg = n/2$  can exist without observable singularities, and this will later provide the template for showing in sections 10 and 11 how additional fractional charges may also exist without observable singularities. As we shall now start to demonstrate, this phase / version synchronization can be generalized to permit fractionalized Dirac charges without observable singularities for *any integer denominator*  $m = 1, 2, 3, 4, 5, \dots$ , and not merely for the  $m = 1, 2$  denominators of (5.14), but considerations of helicity and charge continuity will restrict these denominators to only *odd* integers in addition to the even integer 2 already shown above.

## PART III: ROOT OF UNITY TRANSFORMATIONS ON SO(2), THEIR TOPOLOGICAL MAPPING ONTO SO(3), AND THEIR CORRESPONDENCE TO THE HEISENBERG EQUATIONS OF MOTION

### 6. Euler Angles in the Complex Plane, and Root-of-Unity Generators and their Primary Properties

As we shall now demonstrate, the 2x2 Pauli matrices  $\sigma_i$  of the simple universal covering group SU(2) of SO(3) represent a special case of an infinite set of 2x2 generator matrix triplets we denote as  $\tau_i$ ;  $i=1,2,3$  associated with covering groups we denote by  $\tilde{G}(2)$ . Specifically, in the same way that the Pauli matrices are constructed such that  $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = I$  and therefore may be thought of as the “square roots”  $\sigma_i = (I_i)^{1/2}$  of a triplet of 2x2 identity matrices  $I_i$ , these  $\tau_i$  may be thought of as the generalized  $m^{\text{th}}$  roots of the identity triplet  $I_i$ , namely,  $(I_i)^{n/m}$  for  $1 \leq n \leq m$ , where  $n$  and  $m$  are integers. Of course, for  $n=m$  we will recover  $I_i = (I_i)^{m/m}$  which is the identity matrix triplet, and for  $n > m$  we simply recycle *ad infinitum* through the  $(I_i)^{n/m}$  for which  $1 \leq n \leq m$ . If we then utilize the left scripts  ${}^n_m\tilde{G}(2)$  to denote the covering group  $\tilde{G}(2)$  associated with any given set of  $(I_i)^{n/m}$ , and if we likewise denote  ${}^n_m\tau_i \equiv (I_i)^{n/m}$ , then according to these notational conventions the Pauli matrices  $\sigma_i = {}^1_2\tau_i = (I_i)^{1/2}$  and the universal covering group  $SU(2) = {}^1_2\tilde{G}(2)$ .

The starting point for developing these “root of unity” generators  ${}^n_m\tau_i = (I_i)^{n/m}$  emanates from pure mathematics, via the Euler relation for the generalized  $m^{\text{th}}$  roots of unity:

$$\begin{aligned} \sqrt[m]{1^n} &= (1)^{n/m} = \exp i\vartheta = \exp\left(i2\pi \frac{n}{m}\right) = \cos\left(2\pi \frac{n}{m}\right) + i \sin\left(2\pi \frac{n}{m}\right). \\ &= \exp(i2\pi\mathbb{Q}) = \cos(2\pi\mathbb{Q}) + i \sin(2\pi\mathbb{Q}) \end{aligned} \quad (6.1)$$

Above:

$$\vartheta = 2\pi \frac{n}{m} = 2\pi\mathbb{Q} \quad (6.2)$$

is the Euler angle in the complex plane and we use  $\mathbb{Q} = n/m$  to denote a number selected from the infinite set of rational numbers, i.e., a quotient  $n/m$ . We shall also find it convenient to represent the infinite set of irrational numbers as  $\overline{\mathbb{Q}}$ .

What we now need to do, mathematically, is really very simple: Whereas (6.1) represents the roots of the scalar number 1, the desired  ${}_m^n \tau_i = \sqrt[m]{I_i^n} = (I_i)^{n/m}$  represent roots of the 2x2 identity matrix  $\text{diag}(I) = (1,1)$ . So these are just an extension of (6.1) for the number 1, to the triplet  $I_i$  of 2x2 identity matrices, whereby we require the correspondence  $\sigma_i = \frac{1}{2} \tau_i = (I_i)^{1/2}$ . Now, let us find  ${}_m^n \tau_i = \sqrt[m]{I_i^n} = (I_i)^{n/m}$ , explicitly.

Starting with (5.1), it happens that with a judicious choice of these angles  $\theta_i$  we can cause each of these  $U_i$  to be identical to the corresponding  $\sigma_i$  up to an overall constant factor. Specifically, if we choose each of these angles such that  $\theta_i = \pi$ , we readily see that:

$$\begin{aligned} U_1(\pi/2) &= \exp\left(i\sigma_1 \frac{\pi}{2}\right) = \begin{pmatrix} \cos(\pi/2) & i \sin(\pi/2) \\ i \sin(\pi/2) & \cos(\pi/2) \end{pmatrix} = i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = i\sigma_1 \\ U_2(\pi/2) &= \exp\left(i\sigma_2 \frac{\pi}{2}\right) = \begin{pmatrix} \cos(\pi/2) & \sin(\pi/2) \\ -\sin(\pi/2) & \cos(\pi/2) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i\sigma_2 \\ U_3(\pi/2) &= \exp\left(i\sigma_3 \frac{\pi}{2}\right) = \begin{pmatrix} \cos(\pi/2) + i \sin(\pi/2) & 0 \\ 0 & \cos(\pi/2) - i \sin(\pi/2) \end{pmatrix} = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i\sigma_3 \end{aligned} \quad (6.3)$$

Consolidating, we see that  $U_i(\pi/2) = \exp(i\sigma_i \pi/2) = i\sigma_i$  in general, which we rewrite as:

$$\sigma_i = -i \exp(i\sigma_i \pi/2) = -i [\cos(\sigma_i \pi/2) + i \sin(\sigma_i \pi/2)]. \quad (6.4)$$

So we can now square this expression, and because  $\sigma_i^2 = I_i$ , we can write the identity matrix triplet  $I_i$  as:

$$I_i = (-i)^2 \exp(i\sigma_i \pi) = (-i)^2 [\cos(\sigma_i \pi) + i \sin(\sigma_i \pi)]. \quad (6.5)$$

We deliberately do *not* turn  $(-i)^2$  into -1, because when we later take square roots of this, we want to recover  $-i$  alone, and not extraneously introduce a two-valued  $\pm i = \sqrt{-1}$ . Of course, the identity matrix taken to any integer power  $n$  is still the identity matrix  $(I_i)^n = I_i$ , so the most general expression for this triplet of identity matrices is:

$$I_i = (I_i)^n = (-i)^{2n} \exp(i\sigma_i \pi n) = (-i)^{2n} [\cos(\sigma_i \pi n) + i \sin(\sigma_i \pi n)]. \quad (6.6)$$

Now that we have the identity matrices represented in this form, it is an easy matter to obtain their generalized  $m^{\text{th}}$  roots,  $(I_i)^{n/m}$ . There are simply:



$${}_m^n \tau_i = (I_i)^{\frac{n}{m}} = (-i)^{\frac{2n}{m}} \exp\left(i\sigma_i \pi \frac{n}{m}\right) = (-i)^{\frac{2n}{m}} \left[ \cos\left(\sigma_i \pi \frac{n}{m}\right) + i \sin\left(\sigma_i \pi \frac{n}{m}\right) \right]. \quad (6.7)$$

This is the explicit expression for the  ${}_m^n \tau_i = (I_i)^{n/m}$  that we sought to obtain.

It is also helpful to use the Euler formulation  $-i = \exp(i3\pi/2)$  to write:

$$(-i)^{2n/m} = \exp(i3\pi n/m), \quad (6.8)$$

and then use this in (6.7) to write:

$${}_m^n \tau_i = \exp\left(i3\pi \frac{n}{m}\right) \exp\left(i\sigma_i \pi \frac{n}{m}\right) = \exp\left(i3I_i \pi \frac{n}{m}\right) \exp\left(i\sigma_i \pi \frac{n}{m}\right) = \exp\left(i\pi \frac{n}{m} (3I_i + \sigma_i)\right). \quad (6.9)$$

After the second equal sign we simply introduce the 2x2 identity matrices  $I_i$  into  $\exp(i3\pi n/m)$  which multiplies the 2x2  $\exp(i\sigma_i \pi n/m)$ . Because  $\sigma_i$  and  $I_i$  are both matrices, we must be attentive to their commutation, because the Baker–Campbell–Hausdorff and Zassenhaus formulae inform us that  $e^{A(X+Y)} = e^{AX} e^{AY} e^{-(A^2/2)[X,Y]} e^{(A^3/6)(2[Y,[X,Y]]+[X,[X,Y]])} \dots$  as a general rule whenever  $X$  and  $Y$  are matrices. Here, however,  $I_i$  is the identity matrix, so  $[I_i, \sigma_j] = 0$ , and thus the simple rule  $e^{AX} e^{AY} = e^{A(X+Y)}$  for scalar exponents may be applied. This is what enables us to obtain the sum  $3I_i + \sigma_i$  in the final expression.

As a test, to confirm that (6.7) correctly corresponds to the square roots of  $I_i$ , we may set  $m=2$  in (6.7) to obtain:

$${}_2^n \tau_i = (-i)^n \exp(i\sigma_i \pi n/2) = (-i)^n \left[ \cos(\sigma_i \pi n/2) + i \sin(\sigma_i \pi n/2) \right]. \quad (6.10)$$

Referring to (6.6) we see that  ${}_2^n \tau_i = I_i$  for  $n=0, 2, 4, 6, \dots$ , which recovers the identity matrices  $I_i$  which trivially represent one set of square roots of the  $I_i$  themselves. And referring to (6.4), we see that  ${}_2^1 \tau_i = \sigma_i$  for  $n=1$ , while (6.7) shows that for successive  $n=3, 5, 7, \dots$  the sign flip in  $(-i)^n$  is precisely offset by a flip in  $\exp(i\sigma_i \pi n/2)$ , so that  ${}_2^n \tau_i = \sigma_i$  for  $n=1, 3, 5, 7, \dots$  generally.

We now make use of the Euler angle  $\vartheta = 2\pi n/m = 2\pi\mathbb{Q}$  of (6.2) to reparameterize (6.9) in terms of  $\vartheta$ , and so may write:

$$\begin{aligned}\tau_i(\vartheta) &= {}^n\tau_i = (I_i)^{\frac{n}{m}} = \exp\left(i\frac{3\vartheta}{2}\right)\exp\left(i\sigma_i\frac{\vartheta}{2}\right) = \exp\left(i\frac{\vartheta}{2}(3I_i + \sigma_i)\right) \\ &= \cos\left(\frac{\vartheta}{2}(3I_i + \sigma_i)\right) + i\sin\left(\frac{\vartheta}{2}(3I_i + \sigma_i)\right)\end{aligned}\tag{6.11}$$

So the roots of unity are  $(1)^{n/m} = \exp i\vartheta$  as in (6.1) while the roots of the 2x2 identity triplet, in contrast, are  $(I_i)^{n/m} = \exp(i(\vartheta/2)(3I_i + \sigma_i))$  as seen in the above.

The universal cover SU(2) has the generators  $\sigma_i = \frac{1}{2}\tau_i$  just discussed following (6.10). If we want an easy way to think about this, we can simply use  $n=1$  and  $m=2$  in (6.2) to find that  $\vartheta = \pi$ , in other words, that  $\sigma_i = \tau_i(\pi)$ . So in terms of  $\vartheta$ , the SU(2) generators are spotted at  $\vartheta = \pi$  in the complex plane, and we immediately know that when we square these generators, we will double the angle, and thereby end up with a triplet of identity matrices  $I_i$  spotted at  $\vartheta = 2\pi$ . Then, when thinking about the other root generators, it is easiest to simply think about the angle at which those generators are disposed. The non-trivial cubed-root generators, for example, will be at  $\vartheta = 120^\circ, 240^\circ$ , so that when cubed they will yield either  $\vartheta = 360^\circ, 720^\circ$  which in either case are a triplet  $I_i$  of identity matrices. For the fourth root the non-trivial generators will be spotted at  $\vartheta = 90^\circ, 180^\circ, 270^\circ$  and when raised to the fourth power will yield  $\vartheta = 360^\circ, 720^\circ, 1080^\circ$  with a  $I_i$  triplet. The pentuple generators will be at  $\vartheta = 72^\circ, 144^\circ, 216^\circ, 288^\circ$  and when raised to the fifth power will again recover an integer multiple of  $360^\circ$  with  $I_i$ . And so on.

Consequently, the Euler angle  $\vartheta = 2\pi n/m$  when used in (6.11) provides a very powerful vehicle to cut through all the algebra of these root covering groups, and think about these groups and their operations very simply in terms of orientations and rotations of the Euler angle  $\vartheta$  on the unit circle in the complex plane in which the generators  ${}^n\tau_i = \tau_i(\vartheta)$  are spotted. From this view, the SU(2) group of  $\sigma_i$  is a universal cover because any other set of generators including the unity matrices  $I_i$  can be obtained merely by rotating the angle of these generators from  $\vartheta = \pi$  to the pertinent rational multiple of  $360^\circ$ , i.e., to any and all  $\vartheta = 2\pi n/m = 2\pi\mathbb{Q}$ .

When we wish to explicitly calculate  $\tau_i(\vartheta)$  for a specific  $\vartheta = 2\pi n/m$ , the formulation  $\tau_i(\vartheta) = \exp(i3\vartheta/2)\exp(i\sigma_i\vartheta/2)$  in (6.11) provides the simplest path to do so. The matrix  $\exp(i\sigma_i\vartheta/2)$  is easily calculated by replacing each of the  $\theta_i$  in (5.1) with  $\vartheta$ . Then, we simply multiply the  $\exp(i\sigma_i\vartheta/2)$  result by  $\exp(i3\vartheta/2)$  for all of the  $\tau_i$  to obtain:

$$\begin{aligned}
 \tau_1(\vartheta) &= \exp\left(i\frac{3\vartheta}{2}\right)\exp\left(i\sigma_1\frac{\vartheta}{2}\right) = \exp\left(i\frac{3\vartheta}{2}\right)\begin{pmatrix} \cos(\vartheta/2) & i\sin(\vartheta/2) \\ i\sin(\vartheta/2) & \cos(\vartheta/2) \end{pmatrix} \\
 \tau_2(\vartheta) &= \exp\left(i\frac{3\vartheta}{2}\right)\exp\left(i\sigma_2\frac{\vartheta}{2}\right) = \exp\left(i\frac{3\vartheta}{2}\right)\begin{pmatrix} \cos(\vartheta/2) & \sin(\vartheta/2) \\ -\sin(\vartheta/2) & \cos(\vartheta/2) \end{pmatrix}. \\
 \tau_3(\vartheta) &= \exp\left(i\frac{3\vartheta}{2}\right)\exp\left(i\sigma_3\frac{\vartheta}{2}\right) = \begin{pmatrix} \exp(i2\vartheta) & 0 \\ 0 & \exp(i\vartheta) \end{pmatrix}
 \end{aligned} \tag{6.12}$$

To obtain explicit roots of unity, for simple angles such as  $\vartheta = 2\pi n/3 = n \cdot 120^\circ$  with  $m=3$  or  $\vartheta = 2\pi n/4 = n \cdot 90^\circ$  with  $m=4$  one can draw suitable triangles and obtain the requisite sines and cosines in terms of roots of integers. But as the fractional  $m$  in  $\vartheta = 2\pi n/m$  become larger integers, it becomes difficult, and in many cases impossible, to draw a regular polygon and then start manipulating subset triangles. The preferred general approach, which can be used for *any* fraction  $m$ , is to instead write these roots as  $x^m = 1$  i.e., as the polynomial equation  $x^m - 1 = 0$ , and then to find each of the  $m$  values of  $x$  which are roots of this polynomial. Of course, one of these  $m$  roots is always 1 itself, so  $x - 1 = 0$  can always be factored out. It is then readily seen with this factorization that this polynomial may be rewritten as:

$$x^m - 1 = (x^{m-1} + x^{m-2} + x^{m-3} \dots + x^3 + x^2 + x + 1)(x - 1) = (x - 1) \sum_{i=0}^{m-1} x^i = 0. \tag{6.13}$$

So the  $m-1$   $m^{\text{th}}$  roots of unity aside from 1 itself are generally found by solving the polynomial:

$$\sum_{i=0}^{m-1} x^i = 0. \tag{6.14}$$

Of course, for large  $m$ , this is not a trivial polynomial to solve. But in principle, this makes it possible to find any and all roots that may be desired. So, for example, for the cubed roots of unity, the polynomial (here, quadratic) is  $x^2 + x + 1 = 0$ , which is readily solved as  $x = (-1 \pm \sqrt{1-4})/2 = (-1 \pm i\sqrt{3})/2$ . Via (6.1), the real part of whatever roots are obtained gives the cos, while the imaginary part gives the sin. These in turn are readily inserted into (6.12) to yield explicit matrices for any specified  $m$ .

Now let's explore the primary properties of these root of unity generators  $\tau_i(\vartheta)$  in (6.12), specifically, trace, Hermiticity, unitarity, determinants, and commutators.

We see that in general the traces of (6.12) are:

$$\begin{aligned}
 \text{Tr}(\tau_1) &= \text{Tr}(\tau_2) = 2\exp(i3\vartheta/2)\cos(\vartheta/2) \\
 \text{Tr}(\tau_3) &= \exp(i2\vartheta) + \exp(i\vartheta)
 \end{aligned} \tag{6.15}$$

As just reviewed, the SU(2) generators  $\sigma_i = \tau_i(\pi)$ , and indeed, the above become traceless, i.e., equal to zero, *if and only if* we set  $\vartheta = \pi$ , and more generally,  $\vartheta = (2n-1)\pi$  for integer  $n$ .

Given  $\sigma_i^\dagger = \sigma_i$  and of course  $I_i = I_i^\dagger$ , the Hermitian conjugate obtained from (6.11) is:

$$\tau_i^\dagger(\vartheta) = \exp\left(-i\frac{\vartheta}{2}(3I_i + \sigma_i)\right), \quad (6.16)$$

which shows the  $\tau_i$  in general are not Hermitian,  $\tau_i \neq \tau_i^\dagger$ . More explicitly, from (6.12) we find:

$$\begin{aligned} \tau_1^\dagger(\vartheta) &= \exp\left(-i\frac{3\vartheta}{2}\right) \begin{pmatrix} \cos(\vartheta/2) & -i\sin(\vartheta/2) \\ -i\sin(\vartheta/2) & \cos(\vartheta/2) \end{pmatrix} \\ \tau_2^\dagger(\vartheta) &= \exp\left(-i\frac{3\vartheta}{2}\right) \begin{pmatrix} \cos(\vartheta/2) & -\sin(\vartheta/2) \\ -\sin(\vartheta/2) & \cos(\vartheta/2) \end{pmatrix}. \\ \tau_3^\dagger(\vartheta) &= \begin{pmatrix} \exp(-i2\vartheta) & 0 \\ 0 & \exp(i\vartheta) \end{pmatrix}. \end{aligned} \quad (6.17)$$

Comparing with (6.12), and given that  $\sin(-\theta) = -\sin\theta$ , these will be Hermitian,  $\tau_i = \tau_i^\dagger$  for all  $\vartheta = -\vartheta$ , that is, *if and only if*  $\vartheta = \pi n$  for integer  $n$ .

It is, however, also easily seen from (6.12) and (6.17) that these  $\tau_i$  are unitary,

$$\tau_i^\dagger \tau_i = I_i. \quad (6.18)$$

Only in the special case  $\vartheta = \pi$  in the first cycle, and for generally  $\vartheta = (2n-1)\pi$ , are these  $\tau_i$  both Hermitian and traceless. As already seen,  $\tau_i(\pi) = \sigma_i$  are simply the Pauli matrices.

As to the determinants of (6.12), it is easy to find that for all the  $\tau_i$ :

$$\det \tau_i(\vartheta) = \exp(i3\vartheta) = \exp(i2\pi(3n/m)). \quad (6.19)$$

So for  $\sigma_i = \tau_i(\pi)$ , setting  $\vartheta = \pi$  above yields  $\det \tau_i = \exp(i3\pi) = \exp(i\pi) = -1$ , as is of course also to be expected. Noting that the unitary rotation matrices (5.1) for which  $U_i^\dagger U_i = 1$ , also have  $\det U_i = 1$ , we may also inquire what  $\tau_i$  have  $\det \tau_i = 1$ . It will be readily seen that  $\det \tau_i = 1$  *if and only if*  $\vartheta = 2\pi n/3 = n \cdot 120^\circ$ , i.e.,  $\mathbb{Q} = n/3$ , i.e.,  $m=3$ , which corresponds with

the cubed roots of unity. Specifically, for  $m=3$ , the above becomes  $\det \tau_i(\vartheta) = \exp(i2\pi n) = 1$ . In all cases, however,  $\det \tau_i$  will have a *magnitude* of 1, that is,  $|\det \tau_i|^2 = (\det \tau_i)^*(\det \tau_i) = 1$ .

Next, let us obtain the commutators  $[\tau_i, \tau_j]$  for any given  $\tau_i(\vartheta)$ . First, working from  $\tau_i(\vartheta) = \exp(i3\vartheta/2)\exp(i\sigma_i\vartheta/2)$  in (6.11) we construct:

$$[\tau_i, \tau_j] = \exp(i3\vartheta) \left[ \exp\left(i\sigma_i \frac{\vartheta}{2}\right), \exp\left(i\sigma_j \frac{\vartheta}{2}\right) \right]. \quad (6.20)$$

To evaluate this, it helps to also construct the commutators  $[U_i, U_j]$  of the unitary matrices (5.1). This exercise is straightforward and yields:

$$\begin{aligned} [U_1, U_2] &= \left[ \exp\left(i\sigma_1 \frac{\theta_1}{2}\right), \exp\left(i\sigma_2 \frac{\theta_2}{2}\right) \right] = -2i \sin\left(\frac{\theta_1}{2}\right) \sin\left(\frac{\theta_2}{2}\right) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -2i \sin\left(\frac{\theta_1}{2}\right) \sin\left(\frac{\theta_2}{2}\right) \sigma_3 \\ [U_2, U_3] &= \left[ \exp\left(i\sigma_2 \frac{\theta_2}{2}\right), \exp\left(i\sigma_3 \frac{\theta_3}{2}\right) \right] = -2i \sin\left(\frac{\theta_2}{2}\right) \sin\left(\frac{\theta_3}{2}\right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -2i \sin\left(\frac{\theta_2}{2}\right) \sin\left(\frac{\theta_3}{2}\right) \sigma_1 \\ [U_3, U_1] &= \left[ \exp\left(i\sigma_3 \frac{\theta_3}{2}\right), \exp\left(i\sigma_1 \frac{\theta_1}{2}\right) \right] = -2i \sin\left(\frac{\theta_3}{2}\right) \sin\left(\frac{\theta_1}{2}\right) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = -2i \sin\left(\frac{\theta_3}{2}\right) \sin\left(\frac{\theta_1}{2}\right) \sigma_2 \end{aligned} \quad (6.21)$$

If we then set  $\vartheta \equiv \theta_1 = \theta_2 = \theta_3$  and also apply the half angle  $\sin^2(\vartheta/2) = (1 - \cos \vartheta)/2$ , this consolidates to:

$$[U_i, U_j] = \left[ \exp\left(i\sigma_i \frac{\vartheta}{2}\right), \exp\left(i\sigma_j \frac{\vartheta}{2}\right) \right] = -2i \sin^2\left(\frac{\vartheta}{2}\right) \varepsilon_{ijk} \sigma_k = i(\cos \vartheta - 1) \varepsilon_{ijk} \sigma_k. \quad (6.22)$$

Now, if we multiply through by  $\exp(i3\vartheta)$ , we arrive at the desired commutator (6.20):

$$[\tau_i, \tau_j] = \exp(i3\vartheta) \left[ \exp\left(i\sigma_i \frac{\vartheta}{2}\right), \exp\left(i\sigma_j \frac{\vartheta}{2}\right) \right] = i \exp(i3\vartheta) (\cos \vartheta - 1) \varepsilon_{ijk} \sigma_k. \quad (6.23)$$

It is then also possible to isolate  $\sigma_i$  with some simple re-indexing and then revert via  $1 - \cos \vartheta = 2 \sin^2(\vartheta/2)$ . Doing so, we may obtain:

$$\sigma_i = \frac{1}{4} i \exp(-i3\vartheta) \csc^2(\vartheta/2) \varepsilon_{ijk} [\tau_j, \tau_k]. \quad (6.24)$$

We see from (6.23) that the  $\tau_i(\vartheta)$  are not, in general, a closed group under multiplication because their commutation reproduces  $i \varepsilon_{ijk} \sigma_k = \frac{1}{2} [\sigma_i, \sigma_j]$  scaled by the numerical factor

$\exp(i3\vartheta)(\cos\vartheta-1)$ . For  $\cos\vartheta=1$  a.k.a.  $\vartheta=2\pi n$  we have  $[\tau_i, \tau_j]=0$  because in this case  $\tau_i = I_i$ , while for  $\vartheta=\pi(2n-1)$  we have  $\frac{1}{2}[\tau_i, \tau_j]=i\varepsilon_{ijk}\sigma_k$  which, because in this case  $\tau_i = \sigma_i$ , is just the group relation  $\frac{1}{2}[\sigma_i, \sigma_j]=i\varepsilon_{ijk}\sigma_k$  of  $SU(2)$ . In this special case, the  $\tau_i$  group is closed.

## 7. Spinors Transformed by the Root-of-Unity Generators

At this point, let us take the unitarity relation  $\tau_i^\dagger \tau_i = I_i$  of (6.18) and combine that with the determinant  $\det \tau_i(\vartheta) = \exp(i3\vartheta)$  of (6.19) to arrive at a basis for generally considering the operation of these  $\tau_i$  on spinors  $\xi$ . In the derivation to follow, we shall retrace pages 32-34 of [6], but using the unitary  $\tau_i$  with  $\det \tau_i = \exp(i3\vartheta)$  in addition to the unitary  $U_i$  with  $\det U_i = 1$ .

First, we keeping in mind that  $\tau_i \neq \tau_i^\dagger$  except for when  $\vartheta=\pi n$ , we abstractly represent any of the  $\tau_i$  in (6.12) by:

$$\tau_i = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \quad (7.1)$$

However, because  $\tau_i^\dagger \tau_i = I_i$  is unitary, we may multiply from the right by  $\tau_i^{-1}$  to rewrite this in terms of the matrix inverse as  $\tau_i^\dagger = \tau_i^{-1}$ . Then, using the mathematical formula for the inverse of a 2x2 matrix while also using  $1/\det \tau_i = \exp(-i3\vartheta)$  and also forming  $\tau_i^\dagger$  directly from (7.1), this means that:

$$\tau_i^\dagger = \tau_i^{-1} = \frac{1}{\det \tau_i} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \exp(-i3\vartheta) \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}. \quad (7.2)$$

Consequently:

$$d = \exp(i3\vartheta)a^*; \quad c = -\exp(i3\vartheta)b^*, \quad (7.3)$$

which permits us to write (7.1) in terms of  $a$  and  $b$  only, as:

$$\tau_i = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ -\exp(i3\vartheta)b^* & \exp(i3\vartheta)a^* \end{pmatrix}. \quad (7.4)$$

From this we find that  $\det \tau_i = (a^*a + b^*b)\exp(i3\vartheta)$ . But we also know that  $\det \tau_i = \exp(i3\vartheta)$  from (6.19), from which we deduce that  $|a|^2 + |b|^2 = 1$ , as expected, with  $|a|^2 = a^*a$  and  $|b|^2 = b^*b$ .

Now, let us posit a spinor  $\xi^T = (\xi_1, \xi_2)$  for which each of  $\xi_1, \xi_2$  are complex numbers. This of course means that  $\xi^\dagger = (\xi_1^*, \xi_2^*)$ . Using (7.4) to transform  $\xi$  yields:

$$\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \rightarrow \xi' = \begin{pmatrix} \xi'_1 \\ \xi'_2 \end{pmatrix} = \tau_i \xi = \begin{pmatrix} a & b \\ -\exp(i3\vartheta)b^* & \exp(i3\vartheta)a^* \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} a\xi_1 + b\xi_2 \\ \exp(i3\vartheta)(-b^*\xi_1 + a^*\xi_2) \end{pmatrix}. \quad (7.5)$$

This contains the two simultaneous equations:

$$\begin{aligned} \xi_1 &\rightarrow \xi'_1 = a\xi_1 + b\xi_2 \\ \xi_2 &\rightarrow \xi'_2 = \exp(i3\vartheta)(-b^*\xi_1 + a^*\xi_2) \end{aligned} \quad (7.6)$$

Conjugating each of these and then reordering and restructuring somewhat, including multiplying each side of the resulting equations through by  $\exp(i3\vartheta)$  then yields:

$$\begin{aligned} \exp(i3\vartheta)(-\xi_2'^*) &= (a(-\xi_2^*) + b\xi_1^*) \\ \exp(i3\vartheta)\xi_1'^* &= \exp(i3\vartheta)(-b^*(-\xi_2^*) + a^*\xi_1^*) \end{aligned} \quad (7.7)$$

This may then be cast using the exact same matrix as the one appearing in (7.5), as:

$$\exp(i3\vartheta) \begin{pmatrix} -\xi_2'^* \\ \xi_1'^* \end{pmatrix} = \begin{pmatrix} a & b \\ -\exp(i3\vartheta)b^* & \exp(i3\vartheta)a^* \end{pmatrix} \begin{pmatrix} -\xi_2^* \\ \xi_1^* \end{pmatrix}. \quad (7.8)$$

This means that the  $(-\xi_2^*, \xi_1^*)^T$  doublet transforms under root of unity operations by  $\tau_i$  in the exact same manner as does the doublet  $(\xi_1, \xi_2)^T$ . This is a well-known special property of SU(2) related to charge conjugation, for transformations under the  $U_i$  of (5.1). We see here that this special property is maintained for the  $\tau_i$  of (6.12).

Next, we observe that:

$$\begin{pmatrix} -\xi_2^* \\ \xi_1^* \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \xi_1^* \\ \xi_2^* \end{pmatrix} = (-i\sigma_2)\xi^* = \zeta\xi^*, \quad (7.9)$$

Note that  $\zeta \equiv -i\sigma_2$  is a real matrix, with  $\zeta = \zeta^*$  and  $\zeta^T = -\zeta$  and  $\zeta^2 = -I$ . Taking the Hermitian conjugate yields:

$$(\zeta\xi^*)^\dagger = (\zeta\xi)^T = \begin{pmatrix} -\xi_2 & \xi_1 \end{pmatrix}. \quad (7.10)$$

As a result of (7.4) as well as (7.9) written also for  $\zeta\zeta'^*$ , we can compact the main equalities in (7.5) and (7.8) to read as such:

$$\begin{aligned}\xi' &= \tau_i \xi \\ \exp(i3\vartheta)(\zeta\zeta'^*) &= \tau_i (\zeta\zeta^*)\end{aligned}\tag{7.11}$$

The conjugate transpose of the above is then:

$$\begin{aligned}\xi'^{\dagger} &= (\tau_i \xi)^{\dagger} = \xi^{\dagger} \tau_i^{\dagger} \\ \exp(-i3\vartheta)(\zeta\zeta'^*)^{\dagger} &= \exp(-i3\vartheta)(\zeta\zeta')^T = (\tau_i (\zeta\zeta^*))^{\dagger} = (\zeta\zeta^*)^{\dagger} \tau_i^{\dagger} = (\zeta\zeta)^T \tau_i^{\dagger}\end{aligned}\tag{7.12}$$

Comparing the rightmost terms in these equations, we see  $\xi'^{\dagger}$  and  $(\zeta\zeta)^T$  both operated upon, from the right, by the same  $\tau_i^{\dagger}$ . The spinor-prime in the former case is  $\xi'^{\dagger}$  and in the latter, from the second term in the second equation, it is  $\exp(-i3\vartheta)(\zeta\zeta')^T$ . This means that  $\xi'^{\dagger}$  transforms in the same manner as does  $\exp(-i3\vartheta)(\zeta\zeta')^T$ , which we write as  $\xi'^{\dagger} \sim \exp(-i3\vartheta)(\zeta\zeta')^T$ . Then, we simply rename this to the unprimed  $\xi^{\dagger} \sim \exp(-i3\vartheta)(\zeta\zeta)^T$  via the inverse transformation  $\xi' \rightarrow \xi$ . Finally, combining this with (7.10) as well as with the explicit expression  $\xi^{\dagger} = (\xi_1^* \quad \xi_2^*)$ , we may finally write the end result:

$$\xi^{\dagger} = (\xi_1^* \quad \xi_2^*) \sim \exp(-i3\vartheta)(\zeta\zeta)^T = \exp(-i3\vartheta)(-\xi_2 \quad \xi_1).\tag{7.13}$$

In the circumstance where  $\det \tau = \exp(i3\vartheta) = 1$  this reduces to  $\xi^{\dagger} \sim (\zeta\zeta)^T$  which will be recognized as a known SU(2) transformation property, see, e.g., equation [2.45] in [6].

Now we may begin to examine the effects of these root of unity transformations via the unitary root of unity operators  $\tau_i$  on the physical space of spacetime, how these give rise to fractionalized Dirac magnetic monopoles, and how these relate to populating a system with multiple fermions in accordance with the fermion Exclusion Principle, using fermion states with multivalued exclusionary quantum numbers.

## 8. Mapping Root-of-Unity SU(2) Spinor Transformations onto the Physical Space of SO(3)

To examine how the transformations (7.5) act on the observed, three-dimensional physical space of SO(3), let us first multiply the result (7.13) from the right by  $\xi$  to specify the transformation relationship between the outer product matrices  $\xi^{\dagger} \xi$  and  $(\zeta\zeta)^T \xi$ , while also showing each matrix explicitly, as such:



$$\begin{aligned} \xi\xi^\dagger &= \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} (\xi_1^* \quad \xi_2^*) = \begin{pmatrix} |\xi_1|^2 & \xi_1\xi_2^* \\ \xi_2\xi_1^* & |\xi_2|^2 \end{pmatrix} \\ &\sim \exp(-i3\vartheta) \xi (\xi\xi)^T = \exp(-i3\vartheta) \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} (-\xi_2 \quad \xi_1) = \exp(-i3\vartheta) \begin{pmatrix} -\xi_1\xi_2 & \xi_1^2 \\ -\xi_2^2 & \xi_1\xi_2 \end{pmatrix}. \end{aligned} \quad (8.1)$$

Simultaneously, let us introduce the physical space coordinates  $x^i = (x, y, z)$  of SO(3) and contact these with the spin matrices  $\sigma_i$  to form the very-recognizable matrix:

$$\sigma_k x^k = \begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix}. \quad (8.2)$$

This is of course traceless  $\text{Tr}(\sigma_k x^k) = 0$  and Hermitian  $\sigma_k x^k = (\sigma_k x^k)^\dagger$  because its  $\sigma_k$  generators are those of SU(2) for which  $\text{Tr}\sigma_k = 0$  and  $\sigma_k = \sigma_k^\dagger$ . Also, the invariant square radial length of the rotation group SO(3) is  $-\det(\sigma_k x^k) = x^2 + y^2 + z^2 = r^2$ .

Ordinarily, when we use the rotation generators (5.1) with  $\det U_i = 1$ , the analogous result for (8.1) is  $\xi\xi^\dagger \sim \xi(\xi\xi)^T$ . This is because  $1/\det \tau_i = \exp(-i3\vartheta)$  is removed in favor of  $1/\det U_i = 1$ . Then, one sets  $-\xi(\xi\xi)^T = \sigma_k x^k$  (note sign flip) to establish the connection between the spinors  $\xi$  and the space coordinates  $x^i$ , which, for example, is implied by equations [2.47], [2.49] and [2.53] in [6]. So the overall relationship for the  $U_i$  of (5.1) is  $-\xi\xi^\dagger \sim -\xi(\xi\xi)^T = \sigma_k x^k$ . What is different about (8.1) is the new term  $1/\det \tau_i = \exp(-i3\vartheta)$ . Because  $|\det \tau_i|^2 = 1$ , this extra terms does not alter the magnitude of anything in (8.1). But it does alter the direction of the unit vector in the complex plane. So let us inquire about what specific effects this term produces on SO(3).

The matrix  $\xi\xi^\dagger$  on the top line of (8.1) is naturally Hermitian, which is to say that the upper right and lower left matrix entries  $\xi_1\xi_2^*$  and  $\xi_2\xi_1^*$  are naturally, inherently conjugate to one another, or precisely put,  $\xi_1\xi_2^* = (\xi_2\xi_1^*)^*$ , by identity. At the same time, the matrix  $\xi(\xi\xi)^T$  on the bottom line of (8.1) is inherently traceless,  $\text{Tr}(\xi(\xi\xi)^T) = (-\xi_1\xi_2 + \xi_1\xi_2) = 0$ , thus so too is  $\exp(-i3\vartheta)\xi(\xi\xi)^T$ , by identity. At the same time, however, there is nothing in  $\xi\xi^\dagger$  to make this identically *traceless*, nor anything in  $\xi(\xi\xi)^T$  or  $\exp(-i3\vartheta)\xi(\xi\xi)^T$  to make these identically *Hermitian*. Rather, it is the fact that  $\xi\xi^\dagger$  and  $\exp(-i3\vartheta)\xi(\xi\xi)^T$  transform in the same way under SU(2) which requires these each to be both Hermitian and traceless. Thus, the Hermiticity of  $\xi\xi^\dagger$  forces the condition  $-\exp(-i3\vartheta)\xi_2^2 = (\exp(-i3\vartheta)\xi_1^2)^*$  a.k.a.

$\xi_2^2 = -\exp(i6\vartheta)(\xi_1^2)^*$  onto  $\exp(-i3\vartheta)\xi(\zeta\xi)^T$  in the bottom line, while the zero trace of  $\exp(-i3\vartheta)\xi(\zeta\xi)^T$  forces the condition  $|\xi_1|^2 + |\xi_2|^2 = 0$  onto  $\xi\xi^\dagger$  in the top line.

Consequently, with these conditions that come about because  $\xi\xi^\dagger \sim \exp(-i3\vartheta)\xi(\zeta\xi)^T$ , we now equate  $-\exp(-i3\vartheta)\xi(\zeta\xi)^T$  (again, sign flip) to  $\sigma_k x^k$  in (8.2) which is both Hermitian and traceless, and thereby combine both (8.1) and (8.2) into:

$$\begin{aligned} -\xi\xi^\dagger &= \begin{pmatrix} -\xi_1\xi_1^* & -\xi_1\xi_2^* \\ -\xi_2\xi_1^* & -\xi_2\xi_2^* \end{pmatrix} \sim -\exp(-i3\vartheta)\xi(\zeta\xi)^T = \exp(-i3\vartheta) \begin{pmatrix} \xi_1\xi_2 & -\xi_1^2 \\ \xi_2^2 & -\xi_1\xi_2 \end{pmatrix} \\ &= \sigma_k x^k = \begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix}. \end{aligned} \quad (8.3)$$

Immediately, (8.3) enables us to deduce the following relationships:

$$x+iy = \exp(-i3\vartheta)\xi_2^2; \quad x-iy = -\exp(-i3\vartheta)\xi_1^2; \quad z = \exp(-i3\vartheta)\xi_1\xi_2, \quad (8.4)$$

from which it further follows that:

$$x = \frac{1}{2}\exp(-i3\vartheta)(\xi_2^2 - \xi_1^2); \quad y = \frac{1}{2i}\exp(-i3\vartheta)(\xi_2^2 + \xi_1^2). \quad (8.5)$$

For  $\vartheta=0$  (8.3) to (8.5) reproduce the usual relations given in [2.40], [2.47] and [2.53] of [6].

Now let us calculate how transformations under the root of unity matrices  $\tau_i(\vartheta)$  of (6.12) act upon the space coordinates (8.4), (8.5). The transformations upon the spinor components  $\xi^T = (\xi_1, \xi_2)$  are given by (7.6). If we square each of these and also take their product we obtain:

$$\begin{aligned} \xi_1^2 &\rightarrow \xi_1'^2 = a^2\xi_1^2 + b^2\xi_2^2 + 2ab\xi_1\xi_2 \\ \xi_2^2 &\rightarrow \xi_2'^2 = \exp(i6\vartheta)(b^* \xi_1^2 + a^* \xi_2^2 - 2a^*b^* \xi_1\xi_2). \\ \xi_1'\xi_2' &= \exp(i3\vartheta)\left(-ab^* \xi_1^2 + (|a|^2 - |b|^2)\xi_1\xi_2 + a^*b\xi_2^2\right) \end{aligned} \quad (8.6)$$

Then we substitute from (8.4) and (8.5) into (8.6) to first obtain:

$$\begin{aligned}
x &= \frac{1}{2} \exp(-i3\vartheta) (\xi_2^2 - \xi_1^2) \rightarrow x' = \frac{1}{2} \exp(-i3\vartheta) (\xi_2'^2 - \xi_1'^2) \\
&= \frac{1}{2} \exp(i3\vartheta) (b^{*2} \xi_1^2 + a^{*2} \xi_2^2 - 2a^* b^* \xi_1 \xi_2) - \frac{1}{2} \exp(-i3\vartheta) (a^2 \xi_1^2 + b^2 \xi_2^2 + 2ab \xi_1 \xi_2) \\
y &= \frac{1}{2i} \exp(-i3\vartheta) (\xi_2^2 + \xi_1^2) \rightarrow y' = \frac{1}{2i} \exp(-i3\vartheta) (\xi_2'^2 + \xi_1'^2) \\
&= \frac{1}{2i} \exp(i3\vartheta) (b^{*2} \xi_1^2 + a^{*2} \xi_2^2 - 2a^* b^* \xi_1 \xi_2) + \frac{1}{2i} \exp(-i3\vartheta) (a^2 \xi_1^2 + b^2 \xi_2^2 + 2ab \xi_1 \xi_2) \\
z &= \exp(-i3\vartheta) \xi_1 \xi_2 \rightarrow z' = \exp(-i3\vartheta) \xi_1' \xi_2' = -ab^* \xi_1^2 + a^* b \xi_2^2 + (|a|^2 - |b|^2) \xi_1 \xi_2
\end{aligned} \tag{8.7}$$

Then we further substitute the spinors from (8.4) into the above, then reduce, to obtain:

$$\begin{aligned}
x &\rightarrow x' = \exp(i6\vartheta) \left( \frac{1}{2} (a^{*2} - b^{*2}) x + \frac{1}{2} (a^{*2} + b^{*2}) iy - a^* b^* z \right) \\
&\quad + \frac{1}{2} (a^2 - b^2) x - \frac{1}{2} (a^2 + b^2) iy - abz \\
y &\rightarrow y' = \exp(i6\vartheta) \left( -\frac{1}{2} (a^{*2} - b^{*2}) ix + \frac{1}{2} (a^{*2} + b^{*2}) y + a^* b^* iz \right) \\
&\quad + \frac{1}{2} i (a^2 - b^2) x + \frac{1}{2} (a^2 + b^2) y - iabz \\
z &\rightarrow z' = \exp(i3\vartheta) \left( (a^* b + ab^*) x + (a^* b - ab^*) iy + (|a|^2 - |b|^2) z \right)
\end{aligned} \tag{8.8}$$

The above of course is still an abstracted expression based on (7.4) for any of the  $\tau_i$  in (6.12). But it is even more general than that; for example, it can also be applied to the rotation matrices (5.1). While the unitary matrices are  $\tau_i^\dagger \tau_i = I_i$  per (6.18) with  $\det \tau_i(\vartheta) = \exp(i3\vartheta)$  per (6.19), the rotation matrices (5.1) have  $U_i^\dagger U_i = I_i$  and  $\det U_i = 1$ . So, to apply these to the  $U_i$  of (5.1), one would set all of the exponentials in (8.8) to 1, yielding [2.54] of [6]:

$$\begin{aligned}
x &\rightarrow x' = \frac{1}{2} (a^2 + a^{*2} - b^2 - b^{*2}) x - \frac{1}{2} (a^2 - a^{*2} + b^2 - b^{*2}) iy - (ab + a^* b^*) z \\
y &\rightarrow y' = \frac{1}{2} (a^2 - a^{*2} - b^2 + b^{*2}) ix + \frac{1}{2} (a^2 + a^{*2} + b^2 + b^{*2}) y - (ab - a^* b^*) iz \\
z &\rightarrow z' = (a^* b + ab^*) x + (a^* b - ab^*) iy + (|a|^2 - |b|^2) z
\end{aligned} \tag{8.9}$$

This would also apply to  $\tau_i$  any time  $\det \tau_i = \exp(i3\vartheta) = 1$  i.e., for  $\vartheta = 2\pi/3$  and the successive  $\vartheta = 2\pi(3n-2)/3$  which differ from this by an integer multiple of  $2\pi$ . Then we obtain  $a$  and  $b$  from (5.1) and plug those into (8.8). Doing exactly that using  $U_1$  from (5.1), reducing using the double-angle formulae  $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$  and  $\sin 2\theta = 2 \sin \theta \cos \theta$ , then putting the results into the form of rotation matrices  $R_k$ , we obtain the x-axis rotation:

$$x^i = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_1 & \sin \theta_1 \\ 0 & -\sin \theta_1 & \cos \theta_1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = R_1 x^i. \quad (8.10)$$

Using  $U_2$  in like manner yields the y-axis rotation:

$$x^i = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos \theta_2 & 0 & -\sin \theta_2 \\ 0 & 1 & 0 \\ \sin \theta_2 & 0 & \cos \theta_2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = R_2 x^i. \quad (8.11)$$

From  $U_3$ , we likewise find the z-axis rotation:

$$x^i = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos \theta_3 & \sin \theta_3 & 0 \\ -\sin \theta_3 & \cos \theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = R_3 x^i. \quad (8.12)$$

Comparing the half-angles  $\theta_i$  in (5.1) with the whole angles  $\theta_i$  in (8.10) through (8.12), this demonstrates how it is that a rotation through  $\theta_i/2$  on  $SU(2)$  projects onto a rotation of twice the magnitude,  $\theta_i$  on  $SO(3)$ . This is often summarized by the projective mapping  $\pi: SU(2) \rightarrow SO(3)$ . As reviewed in section 5, this is why an electron wavefunction is not restored to its original “version” after a  $2\pi$  rotation on  $SU(2)$ , but requires a  $4\pi$  circuit to return to version, as is known. This in turn is why for a tidally-locked electron, the phase change goes half a revolution off cycle  $\Lambda = \pi(2n-1)$  as in (5.11) and therefore requires the half integer Dirac charges  $2eg = n - \frac{1}{2}$  of (5.13) in order to avoid observable singularities. This does not appear to have been previously pointed out in the monopole literature.

The above, (8.10) through (8.12) provide a check on the correctness of (8.8). But they also demonstrate explicitly why it is that the  $2 \times 2$   $U_i$  matrices in (5.1) formed from the  $SU(2)$  generators  $\sigma_i$  are said to be the generators of rotations  $R_k$  on  $O(3)$  which preserve as an invariant, the radial length  $-\det(\sigma_k x^k) = x^2 + y^2 + z^2 = r^2$  in (8.2). Specifically, because  $R_k$  are real matrices and  $x^i$  contains real space coordinates,  $R_k^\dagger = R_k^T$  and  $x^{i\dagger} = x^{iT}$ . Further, it is easily seen that  $R_k^T R_k = I_{(3)}$  for each  $R_k$ , where  $I_{(3)}$  is the  $3 \times 3$  identity matrix, which is to say that these  $R_k$  rotations are unitary. As a result, using  $x^i = (x, y, z)^T$  and  $x^{iT} = (x, y, z)$  with an explicit transpose that is often left implicit, and summing over the space index  $i$  via  $\Sigma_i$ , the invariance of the length  $r^2 = x^2 + y^2 + z^2$  may be written as:

$$r^2 = \Sigma_i x^{iT} x^i \rightarrow \Sigma_i x'^{iT} x'^i = r'^2 = \Sigma_i x^{iT} R_k^T R_k x^i = \Sigma_i x^{iT} x^i = r^2, \quad (8.13)$$

that is,  $r^2 \rightarrow r'^2 = r^2$ .

To develop an algebraic matrix-free formulation of (8.13), we may use the Minkowski metric tensor  $\text{diag}(\eta_{\mu\nu}) = (1, -1, -1, -1)$  for which the inverse  $\eta_{\mu\sigma}\eta^{\sigma\nu} = \delta_\mu^\nu$  to define the covariant (lower-indexed)  $x_i \equiv \eta_{ij}x^j$  in the usual manner, and generally to raise and lower indexes. Then, representing each of the  $R$  in (8.10) to (8.12) as  $R_{ij}$  with matrix indexes which are the space indexes  $i, j = 1, 2, 3$  such that  $R^{ik}R_{kj} = \delta^i_j$  (with the transpose of  $R_{kj}$  represented by the reverse indexing of  $R^{ik}$ ), we may rewrite (8.13) algebraically, free of any explicit showing of transposes  $T$  or sums  $\Sigma_i$  or matrix multiplications, as:

$$-r^2 = x_i x^i \rightarrow x'_i x'^i = -r'^2 = x_i R^{ik} R_{kj} x^j = x_i \delta^i_j x^j = x_i x^i = -r^2. \quad (8.14)$$

Let us now apply the abstracted result (8.8) to each of the root of unity generators  $\tau_i(\vartheta)$  obtained in (6.12), then form the results into three matrices designated  $P_k$  with the uppercase Greek letter Rho for ‘‘root’’ of unity. Applied to the root of unity generator  $\tau_1(\vartheta)$ , (8.8) yields:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \exp(i3\vartheta) \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \vartheta & \sin \vartheta \\ 0 & -\sin \vartheta & \cos \vartheta \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \equiv P_1 x^i. \quad (8.15)$$

Likewise for  $\tau_2(\vartheta)$  we obtain:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \exp(i3\vartheta) \begin{pmatrix} \cos \vartheta & 0 & -\sin \vartheta \\ 0 & 1 & 0 \\ \sin \vartheta & 0 & \cos \vartheta \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \equiv P_2 x^i, \quad (8.16)$$

while for  $\tau_3(\vartheta)$  the transformation is:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \exp(i3\vartheta) \begin{pmatrix} \cos \vartheta & \sin \vartheta & 0 \\ -\sin \vartheta & \cos \vartheta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \equiv P_3 x^i. \quad (8.17)$$

To obtain (8.15) to (8.17) one first calculates the various  $a$  and  $b$ -based coefficients of  $x, y, z$  in (8.8), then reduces including using the double-angle formulae  $\cos(\vartheta) = \cos^2(\vartheta/2) - \sin^2(\vartheta/2)$  and  $\sin(\vartheta) = 2\sin(\vartheta/2)\cos(\vartheta/2)$ . This shows the effects on the physical space  $SO(3)$ , of the

transformation  $\xi \rightarrow \xi' = \tau_i \xi$  of (7.5), using the explicit  $\tau_i$  of (6.12). It is easily seen that  $\mathbf{P}_k^\dagger \mathbf{P}_k = I_{(3)}$ , thus each  $\mathbf{P}_k$  is unitary.

Comparing (8.10) through (8.12) with (8.15) through (8.17), it will be seen that the root of unity Euler angle  $\vartheta = 2\pi n/m$  transforms the space coordinates in exactly the same way as do the rotation angles  $\theta_i$ , with the sole exception of the factor  $\exp(i3\vartheta) = \det \tau_i(\vartheta)$  (see (6.19)) which is an overall coefficient for each of the matrices in (8.15) through (8.17). So there are really two transformations embedded in (8.15) through (8.17): an ordinary rotation-like transformation based on  $\vartheta$  in lieu of  $\theta_i$ , and a transformation resulting from  $\exp(i3\vartheta) = \cos(3\vartheta) + i \sin(3\vartheta)$  which is a complex number of magnitude  $|\exp(i3\vartheta)|^2 = 1$  that acts equally on all three space coordinates but introduces a complex number in the Euler plane. Consequently, we may segregate the 3x3 matrices themselves which have the effect of a rotation, from  $\exp(i3\vartheta)$ , and examine the separate operation of  $\exp(i3\vartheta)$  on each of the space coordinates  $x^i$ . For all of (8.15) through (8.17), this effect is:

$$x^i \rightarrow x'^i = \exp(i3\vartheta) x^i = \cos(3\vartheta) x^i + i \sin(3\vartheta) x^i = \cos(3\vartheta) \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \sin(3\vartheta) \begin{pmatrix} ix \\ iy \\ iz \end{pmatrix}. \quad (8.18)$$

In the final expression above, we associate the imaginary  $i = \sqrt{-1}$  with the coordinates rather than the sin function from which it originates. So the 3x3 matrices in (8.15) through (8.17) perform the usual type of rotations among  $x, y, z$ . But in addition – and what does not happen in the ordinary rotations of (8.10) to (8.12) – they also simultaneously rotate into and among what are seen to be imaginary space coordinates  $ix, iy, iz$ .

This means that in general, the space coordinates  $x^i$  are not always real, but can become complex depending upon the particular  $\vartheta = 2\pi n/m$  used in any given circumstance. Further, because they all contain the complex number  $\exp(i3\vartheta)$  of magnitude 1, each of the  $\mathbf{P}_k$  in (8.15) through (8.17) contains complex elements. As a consequence of all of this, the invariant length element must now be defined using the Hermitian conjugate relation:

$$r^2 = \sum_i x^{i\dagger} x^i \rightarrow \sum_i x'^{i\dagger} x'^i = r'^2 = \sum_i x^{i\dagger} \mathbf{P}_k^\dagger \mathbf{P}_k x^i = \sum_i x^{i\dagger} x^i = r^2, \quad (8.19)$$

which also uses unitary relation  $\mathbf{P}_k^\dagger \mathbf{P}_k = I_{(3)}$ . In (8.13) we have  $R_k^\dagger = R_k^T$  and  $x^{i\dagger} = x^{iT}$  because  $R_k = R_k^*$  and  $x^i = x^{i*}$ , but in (8.19) we have  $\mathbf{P}_k^\dagger \neq \mathbf{P}_k^T$  and  $x^{i\dagger} \neq x^{iT}$  in general because  $\mathbf{P}_k \neq \mathbf{P}_k^*$  and  $x^i \neq x^{i*}$  in general. Thus, (8.19) is exactly the same as (8.13), with the exception that the  $T$  operation is replaced by the  $\dagger$  operation throughout.

The one change required to represent (8.19) algebraically in the form of (8.14) without any explicit showing of conjugate transposes  $\dagger$  or sums  $\sum_i$  or matrix multiplications, is for the

lower-indexed space coordinates  $x_i$  to be defined as  $x_i \equiv \eta_{ij} x^j *$ , so as to include the conjugation of  $x, y, z$  coordinates which may become complex as seen from (8.18). Likewise, because the  $P_k$  in (8.15) through (8.17) which we now index contravariantly as  $P^{lm}$  are also complex, we define the lower indexed  $P_{kj} \equiv \eta_{kl} \eta_{jm} P^{lm} *$  so as to also include conjugation. As a result, the unitary relation  $P_k^\dagger P_k = I_{(3)}$  becomes written as  $P^{ik} P_{kj} = \delta^i_j$  using the algebraic, matrix-independent notation. Consequently, we may rewrite (8.19) algebraically as:

$$-r^2 = x_i x^i \rightarrow x'_i x'^i = r'^2 = x_i P^{ik} P_{kj} x^j = x_i \delta^i_j x^j = x_i x^i = -r^2. \quad (8.20)$$

Having developed the indexed tensor notations tensor needed to summarize the complex space coordinates (8.18), we now come to a very important question: what does it actually mean, physically, to have space coordinates  $x^i$  which start out as real, but may then be transformed by a unitary factor  $\exp(i3\vartheta) = \exp(i6\pi n / m)$  (see (6.2)) into complex coordinates?

## 9. How Root of Unity Transformations on SO(3) Correspond to the Unitary Time Evolution of Quantum Heisenberg Matrix Operators

We found in the last section that when a spinor is multiplied by a root of the 2x2 identity matrix as in (7.5) and this is projected onto SO(3), the Euler angle for these roots  $\vartheta = 2\pi n / m$  not only rotates the  $x, y, z$  coordinates through an angle  $\vartheta$ , but also transforms each coordinate equally by  $x^i \rightarrow x'^i = \exp(i3\vartheta) x^i$ , as is seen in (8.15) to (8.18). The invariant  $r^2$  of SO(3) is still preserved as seen in (8.20), but the space coordinates on SO(3) become imaginary. Because SO(3) is the physical space of direct material observation, it behooves us to find out what this might mean physically.

Of course imaginary space coordinates, or at least an imaginary  $y$  coordinate, do appear any time we write this invariant length as  $-\det(\sigma_k x^k) = x^2 + y^2 + z^2 = r^2$ , see, e.g., the coordinate  $iy$  appearing in (8.2). But this imaginary coordinate appears on SU(2), and by the time a projection  $\pi: SU(2) \rightarrow SO(3)$  is made onto SO(3) these imaginary coordinates terms have cancelled out via the  $i*i=1$  multiplications which are endemic to multiplication operations involving the Pauli matrices  $\sigma_k$ . For example, these no longer appear in (8.10) to (8.12) which describe rotations on SO(3). So the  $iy$  of SU(2) is not pertinent to understanding the meaning of the imaginary space coordinates in (8.18) and we discard this line of approach.

One might seek precedent for imaginary space coordinates found in Minkowski's original work [7] in which by treating time as an imaginary space coordinate, it became possible to understand a Lorentz transformation as one which preserves the invariant interval  $t^2 - r^2$ . So one might think to define  $ix^i \equiv ct^i$  in (8.18) whereby the imaginary space coordinates are interpreted as real time coordinates. However, beyond all the reasons given when Misner, Thorne and Wheeler famously bade "farewell to  $ict$ " at page 51 of [4], there are several further problems with this. The most important problem is that the operation shown in (8.18) mixes

space in time in a *Euclidean* rotation rather than the type of hyperbolic mixing of Lorentz transformations, see (5.2). So even if one entertains the thought that perhaps this is some previously-unknown space and time transformation, the implication of these Euclidean transformations leads to consequences which simply are not observed. Specifically, when fully developed, these transformations would have the form  $t' = \cos(3\vartheta)t + \sin(3\vartheta)r$  and  $r' = \cos(3\vartheta)r - \sin(3\vartheta)t$ , again, with  $\vartheta = 2\pi n / m$ . Now, it is well known that all atomic radii are within an order of magnitude of the Bohr radius  $a_0 = \hbar / m_e c \alpha$ . And it is also well known that for the worldlines of non-relativistic or even mildly-relativistic material bodies such as electrons with  $v/c \sim \alpha = 1/137.036\dots$ , the spatial length  $r$  traversed over a given time  $t$  is exceedingly less than the time elapsed as represented by the ratio  $r/ct \ll 1$ . So these Euclidean transformations – if they were physically real – would result in atomic radii that are orders of magnitude larger than the Bohr radius owing to the fact that  $ct \gg r$  along the worldliness of material bodies such as electrons, whereby even a small component of  $t$  being converted into  $r$  via a Euclidean rotation would greatly increase the atomic radii in a manner that has no observed support. Consequently, we discard  $ix^i \equiv ct^i$  as a viable interpretation of these imaginary space coordinates.

This now brings us to Heisenberg matrix mechanics. Not only is this another place in physics where imaginary space coordinates can appear, but the appearance of the imaginary space coordinates  $x^i$  in (8.18) precisely parallels the form in which the Heisenberg matrices  $X^i$  evolve over time in the classical limit elucidated by Ehrenfest. Specifically, let us promote the  $x^i$  to the Heisenberg position matrices  $X^i = \frac{1}{2}(A^{\dagger i} + A^i)$  which commute with the momentum matrices  $P^i = \frac{1}{2}i(A^{\dagger i} - A^i)$  according to the canonical commutation relation  $[X^i, P^j] = \delta^{ij}\hbar I_{(\infty)}$  where  $A^{\dagger i}$  and  $A^i$  are the creation and annihilation matrices and  $I_{(\infty)}$  is an infinity x infinity square identity matrix, which relation of course leads to the uncertainty principle in a well-known fashion. With this promotion, we focus on a given coordinate matrix, say,  $X$ , which we write as with the matrix indexes  $X_{ab}$ , where  $a, b = 1, 2, 3 \dots \infty = \mathbb{Z}$  are integers. Then, also using (6.2) which gives discrete values  $\vartheta = 2\pi n / m = 2\pi\mathbb{Q}$  to  $\vartheta$ , we may write (8.18) as:

$$X_{ab} \rightarrow X'_{ab} = \exp(i3\vartheta) X_{ab} = \exp(i6\pi n / m) X_{ab}. \quad (9.1)$$

Independently of (9.1), both  $X$  and  $P$  are constructed from Fourier coefficients of real quantities, and so their time evolution, see the helpful article at [8], is given by:

$$X_{ab}(0) \rightarrow X_{ab}(t) = \exp(i2\pi(E_a - E_b)t / \hbar) X_{ab}(0), \quad (9.2)$$

$$P_{ab}(0) \rightarrow P_{ab}(t) = \exp(i2\pi(E_a - E_b)t / \hbar) P_{ab}(0). \quad (9.3)$$

The above (9.2) in which  $X_{ab}(0)$  evolves over time as a Fourier component was in fact the original form of the Heisenberg equation of motion. The correspondence principle as denoted by



“ $\cong$ ” likewise informs us that the classical frequencies  $f = 1/t$  based on the classical period  $t$  are given by Planck’s law:

$$E_a - E_b \cong h \frac{a-b}{t} = h \frac{n}{t} = nhf \quad (9.4)$$

with the definition  $n \equiv a - b$ , for radiation emissions as between classical orbits  $a$  and  $b$ . This correspondence holds for  $n = a - b \ll a, b$ , that is, for a system which is emitting or absorbing an energy  $E_a - E_b$  which is only a small portion of its total energy, because the period  $t_a \cong t_b$  can then be regarded as approximately equal for either the classical  $a$  or  $b$  “orbits.” Conversely, for  $n = a - b \sim a, b$ , the frequencies are no longer integer multiples of any single frequency because  $t_a \cong t_b$  and the difference in period can no longer be neglected. This is what requires the position and momentum operators to be specified by the Heisenberg matrices  $X_{ab}$  and  $P_{ab}$  which reproduce the Fourier coefficients in the classical limit. By Ehrenfest, the expected values  $\langle X \rangle$  and  $\langle P \rangle$  also satisfy the classical equations of motion. So, if we now substitute (9.4) into (9.2) in the classical  $a - b \ll a, b$  limit of  $t_a \cong t_b$ , we obtain:

$$X_{ab}(0) \rightarrow X_{ab}(t) = \exp(i2\pi(E_a - E_b)t/h) X_{ab}(0) \cong \exp(i2\pi(a-b)) X_{ab}(0) = \exp(i2\pi n) X_{ab}(0). \quad (9.5)$$

At the same time, the ordinary Dirac Quantization Condition  $2eg = n$  obtained in (4.7) is implicitly for the root of unity  $m=1$ ; the whole point of this paper is to show that there are physically-admissible roots other than  $m=1$  as we already did for  $m=2$  in (5.14), and as we shall show in the next section for other  $m$  as well. So, working with the  $m=1$  trivial root of unity which characterizes the standard DQC  $2eg = n$ , (9.1) specializes with the conventional  $m=1$  to:

$$X_{ab} \rightarrow X'_{ab} = \exp(i3\vartheta) X_{ab} = \exp(i6\pi n) X_{ab} = \exp(i2\pi n) \exp(i2\pi n) \exp(i2\pi n) X_{ab}. \quad (9.6)$$

Comparing, we see that (9.5) and (9.6) have the same form, whereby  $X_{ab}$  is operated upon by  $\exp(i2\pi n)$ . The only difference is that in (9.6) this operation happens three times in succession because of  $\exp(i6\pi n)$ , which in turn emanates from the factor of 3 in  $\det \tau_i(\vartheta) = \exp(i3\vartheta)$  of (6.19) which originated in the  $\exp(i3\vartheta/2)$  of (6.11) and later progressed to (8.15) through (8.17). So if we now associate  $X_{ab}$  in (9.6) with  $X_{ab}(0)$  in (9.5) by defining  $X_{ab} \equiv X_{ab}(0)$ , and if we regard  $X'_{ab}$  in (9.6) to be the same as  $X_{ab}(t)$  in (9.5) in the classical limit such that  $X'_{ab} \equiv X_{ab}(t)$ , then (9.5) and (9.6) can be combined to find that:

$$\begin{aligned} X_{ab} \rightarrow X'_{ab} &= \exp(i3\vartheta) X_{ab}(0) = \exp(i2\pi n) \exp(i2\pi n) \exp(i2\pi n) X_{ab}(0) \\ &\cong \exp(i2\pi n) \exp(i2\pi n) X_{ab}(t) \cong \exp(i2\pi n) X_{ab}(2t) \cong X_{ab}(3t) \end{aligned} \quad (9.7)$$

That is, for  $m=1$  which corresponds to the conventional DQC  $2eg = n$ , we find that  $X'_{ab} = \exp(i3\vartheta) X_{ab}(0) \cong X_{ab}(3t)$ . So now the factor of 3 in  $\det \tau_i(\vartheta) = \exp(i3\vartheta)$  at (6.19) has now migrated into a factor of 3 in the time evolution, which is to say, the elapsed time has become multiplied by this same factor of 3.

So with (9.7) being the specialization to the  $m=1$  trivial root of unity, let us now consider (9.1) generally for *all* roots of unity  $m$ . By the very same analysis that brought us to (9.7), the more general result of which (9.7) is the  $m=1$  specialization, is:

$$X_{ab} \rightarrow X'_{ab} = \exp(i3\vartheta) X_{ab}(0) = \exp(i6\pi n/m) X_{ab}(0) \cong X_{ab}(3t/m). \quad (9.8)$$

Then, let us scale  $t \rightarrow 3t/m$  in (9.2), so that (9.2) becomes:

$$X_{ab}(0) \rightarrow X_{ab}(3t/m) = \exp(i6\pi(E_a - E_b)t/hm) X_{ab}(0). \quad (9.9)$$

Now it is possible to combine with (9.8) with (9.9), being attentive to where to place the  $\cong$  sign which designates classical correspondence and where to place equal signs denoting exact equalities. The result is:

$$\begin{aligned} X_{ab} \rightarrow X_{ab}(3t/m) &= \exp(i6\pi(E_a - E_b)t/hm) X_{ab}(0) \\ &\cong X'_{ab} = \exp(i3\vartheta) X_{ab}(0) = \exp(i6\pi n/m) X_{ab}(0). \end{aligned} \quad (9.10)$$

So the upshot is that  $X'_{ab}$  in (9.1) corresponds with the time-evolved  $X_{ab}(3t/m)$  in the classical limit, that is,  $X'_{ab} \cong X_{ab}(3t/m)$  in the classical limit. The exact quantum relationship is the one on the top line, as expressed in terms of energies.

We now return to (8.15) through (8.17) and promote all coordinates  $x, y, z$  to Heisenberg position operator matrices  $X, Y, Z$ . Using  $\exp(i3\vartheta) X(0) \cong X(3t/m)$  from (9.8) and (9.10), we may then write the effect of these root of unity transformations on  $SO(3)$  as:

$$\begin{pmatrix} X(0) \\ Y(0) \\ Z(0) \end{pmatrix} \rightarrow \begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix} = P_1 X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \vartheta & \sin \vartheta \\ 0 & -\sin \vartheta & \cos \vartheta \end{pmatrix} \exp(i3\vartheta) \begin{pmatrix} X(0) \\ Y(0) \\ Z(0) \end{pmatrix} \cong \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \vartheta & \sin \vartheta \\ 0 & -\sin \vartheta & \cos \vartheta \end{pmatrix} \begin{pmatrix} X(3t/m) \\ Y(3t/m) \\ Z(3t/m) \end{pmatrix}, \quad (9.11)$$

$$\begin{pmatrix} X(0) \\ Y(0) \\ Z(0) \end{pmatrix} \rightarrow \begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix} = P_2 X = \begin{pmatrix} \cos \vartheta & 0 & -\sin \vartheta \\ 0 & 1 & 0 \\ \sin \vartheta & 0 & \cos \vartheta \end{pmatrix} \exp(i3\vartheta) \begin{pmatrix} X(0) \\ Y(0) \\ Z(0) \end{pmatrix} \cong \begin{pmatrix} \cos \vartheta & 0 & -\sin \vartheta \\ 0 & 1 & 0 \\ \sin \vartheta & 0 & \cos \vartheta \end{pmatrix} \begin{pmatrix} X(3t/m) \\ Y(3t/m) \\ Z(3t/m) \end{pmatrix}, \quad (9.12)$$

$$\begin{pmatrix} X(0) \\ Y(0) \\ Z(0) \end{pmatrix} \rightarrow \begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix} = P_3 X = \begin{pmatrix} \cos \vartheta & \sin \vartheta & 0 \\ -\sin \vartheta & \cos \vartheta & 0 \\ 0 & 0 & 1 \end{pmatrix} \exp(i3\vartheta) \begin{pmatrix} X(0) \\ Y(0) \\ Z(0) \end{pmatrix} \equiv \begin{pmatrix} \cos \vartheta & \sin \vartheta & 0 \\ -\sin \vartheta & \cos \vartheta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X(3t/m) \\ Y(3t/m) \\ Z(3t/m) \end{pmatrix}. \quad (9.13)$$

So the root of unity angle  $\vartheta = 2\pi n/m$  generates a rotation of the matrices  $X, Y, Z$  through an angle  $\vartheta$  in the real, physical space of  $SO(3)$ . But simultaneously, *in the classical correspondence*, this root of unity angle also generates a time evolution from 0 to  $3t/m$ . And it does so precisely because  $\vartheta = 2\pi n/m = 2\pi\mathbb{Q}$  is equal to  $2\pi$  times a *rational number*, which means that the allowed values of  $\vartheta$  excludes  $2\pi$  times any irrational numbers  $\overline{\mathbb{Q}}$ ,  $\vartheta \neq 2\pi\overline{\mathbb{Q}}$ .

Now, let us take the exact quantum relation in the top line of (9.10) and set  $\hbar = h/2\pi = 1$  into natural units. With  $d(3t/m) = (3/m)dt$  and holding  $dX_{ab}(0) = 0$ , let us then take the derivative of each side of (9.10) to obtain:

$$\frac{dX_{ab}(3t/m)}{dt} = i \frac{3}{m} (E_a - E_b) X_{ab}(3t/m). \quad (9.14)$$

Once we introduce a state vector  $|\psi\rangle$  with components  $\psi_a$  and likewise  $a = 1, 2, 3, \dots, \infty$ , such that any operator  $O$  has the expected value  $\langle O \rangle = \sum_{a,b} \psi_a^* O_{ab} \psi_b = \langle \psi | O | \psi \rangle$ , then it becomes possible to rotate matrices into any basis. So by diagonalizing the Hamiltonian  $H$  such that each entry on the diagonal is its own energy eigenvalue, that is,  $H|\psi\rangle = E|\psi\rangle$ , (9.14) may be written:

$$\frac{dX(3t/m)}{dt} = i \frac{3}{m} (HX(3t/m) - X(3t/m)H) = i \left[ H, \frac{3}{m} X(3t/m) \right]. \quad (9.15)$$

This is a matrix equation that may hold in any basis. We may then reverse the scaling  $t \rightarrow 3t/m$  that was used at (9.9), that is, we may now set  $3t/m \rightarrow t$  and  $3/m \rightarrow 1$ . And we may then write  $X(t)$  simply as  $X$ . Now the above becomes:

$$\frac{dX}{dt} = i[H, X]. \quad (9.16)$$

This is modern form of Heisenberg's equation of motion for  $X(0)$  with no intrinsic time dependence, that is, with the partial derivative  $\partial X(0)/\partial t = 0$ . We see that in this light,  $m$  merely serves to scale the elapsed time, and it is noteworthy that for the special case of  $m=3$  (9.14) reduces to  $dX_{ab}(t)/dt = i(E_a - E_b)X_{ab}(t)$  and (9.15) reduces to  $dX/dt = i[H, X]$  in (9.16) in their usual forms without any rescaling of the time coordinate. For other  $m$ , the equation of motion still takes on the same form; one simply has dilated or contracted time intervals.

So, if we establish  $3t$  for  $m=1$  which corresponds to the conventional DQC  $2eg = n$  as a “baseline” time  $t_{m=1} \equiv t_1 \equiv 3t$  against which to measure time evolution, then we will have  $t_m = 3t / m = t_1 / m$  generally. Thus, for the  $m^{\text{th}}$  root of unity the time interval is contracted, i.e., shortened by a factor of  $m$ . Then, in circumstances where (9.4) applies because  $n = a - b \ll a, b$  and so the  $t$  appearing in (9.14) may be regarded as a classical period with  $t_a \cong t_b$  as between any two classical orbits, the frequency will vary as  $f_m = 1/t_m = m/t_1 = hf_1$ , and so the energy will vary as:

$$(E_a - E_b)_m \cong h \frac{a-b}{t_m} = h \frac{n}{t_m} = nhf_m = mnhf_1 = m(E_a - E_b)_1. \quad (9.17)$$

This is to say that in classical limit, we expect that energy transitions involving the  $m^{\text{th}}$  roots of unity (which we shall further seek to physically interpret in the next few sections) will have energies generally varying in proportion to  $m$ , so that higher- $m^{\text{th}}$  root transitions are generally (i.e. statistically) more energetic than lower ones.

It at least an item for curiosity that  $m=1$  which corresponds to the conventional DQC  $2eg = n$ , puts a  $3t$  rather than just a  $t$  into the Heisenberg evolution equations. And this originates in the fact that  $\exp(i3\vartheta)$  with the same factor of 3 acts uniformly on all three space coordinates as seen in (8.15) to (8.18) and (9.11) to (9.13). So one may allocate one of these three time intervals in  $3t$  to each of the three space coordinates, and contrast this with the natural curiosity that there are three space coordinates but only one time coordinate by writing the coordinates and matrices in (9.11) for  $m=1$  as  $(3t, X, Y, Z)$ . Whether the concurrence of these curiosities has a deeper meaning is not apparent at the moment, but we do point this out for reflection.

Continuing, as is well-known, the formal solution of (9.16) is:

$$X(t) = \exp(iHt) X(0) \exp(-iHt), \quad (9.18)$$

and if a  $\partial X(0) / \partial t \neq 0$  is admitted, then we may differentiate (9.18) in a well-known way to:

$$\frac{dX}{dt} = i[H, X] + \exp(iHt) \left( \frac{\partial X(0)}{\partial t} \right) \exp(-iHt). \quad (9.19)$$

Also, as is well known, (9.16) applies to any matrix operator  $X \rightarrow O$ , that is, the time derivative  $i dO / dt = [O, H]$  is obtained by commuting the operator with the Hamiltonian. Therefore, using a non-relativistic Hamiltonian  $H = p^2 / 2m + V(x)$  and taking expectation values, one can shown in a well-known way via Ehrenfest that  $\langle P \rangle = m d \langle X \rangle / dt$  and  $\langle \nabla V \rangle = -d \langle P \rangle / dt$ , which reproduces the classical equations of motion.

So, we have now established that the imaginary coordinates which are introduced in (8.15) through (8.18) lead to no more and no less than the Heisenberg equations of motion, albeit from a different approach point, when these coordinates are promoted to Heisenberg matrix operators. And, we see how the roots of unity not only rotate the space coordinates as in (9.11) to (9.13), but also advance these coordinates through  $3t/m$  units of time. With this, as well as the previous development in sections 1 through 8 we have all the necessary tools to prove, based on these root of unity transformations: a) that fractional Dirac magnetic monopole charges (in addition to the half-integer charges already established in section 5) may exist for  $m > 2$  without observable singularities (section 10); that the *even*  $m$  charges other than  $m=2$  are excluded by Lorentz symmetry from physical existence without singularity, so that the physically admissible charge fractions, in view of section 5, happen to correspond precisely to the  $m = 1, 2, 3, 5, 7, 9...$  fractions observed the Fractional Quantum Hall Effect (FQHE) near 0K (section 11); c) that these very same odd  $m$  from the roots of unity also are synonymous via  $m = 2j$  with the observed Casimir numbers  $j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}...$  for the total conserved angular momentum states in atomic shells; d) that the tidally-locked  $m=2$  state is related to (Cooper) pairing of electrons near 0K; and e) how all of these results coact to provide a basis for confirmation via proposed experiments which would correlate the fractional charges observed in the FQHE to their angular momentum states.

## **PART IV: FRACTIONAL DIRAC CHARGES, THE FRACTIONAL QUANTUM HALL EFFECT, AND ATOMIC ORBITAL SHELL STRUCTURE**

### **10. Fractional Dirac Magnetic Monopole Charges without Observable Singularities**

In this section we are finally prepared to demonstrate how fractional Dirac magnetic monopole charges  $2eg = n/m$  with  $m > 2$  may exist without observable singularities. Before we do this it is important to state that this does not necessarily mean that these charges *do* exist, or if they do, that *all* charge fractions  $m = 1, 2, 3, 4, 5...$  exist as opposed to only some fractions existing such as the charges with denominators  $2, 3, 5, 7, 9...$  which are observed in the Fractional Quantum Hall Effect (FQHE) [5]. The purpose of this section is only to show that these charges can exist *without observable singularities* and therefore cannot be precluded from existing on the basis of giving rise to observable singularities. But this does not mean that there might not be other reasons why some charge fractions are excluded from existing, again for example, charges with  $4, 6, 8, 10...$  denominators which are not observed in the FQHE. As we shall see in the next section, the  $m = 4, 6, 8, 10...$  even charge fractions are indeed excluded from physical existence, not because of any observable singularities, but because such even charge fractions would permit the charge to impermissibly change its fraction merely by overtaking that charge through a change in relativistic reference frame, and would thus violate Lorentz symmetry.

The last two sections fundamentally focused on how the spinor transformations of (7.5) using the root of unity generators  $\tau_i(\vartheta)$  of (6.12) map onto the SO(3) space of physical existence and observation, and bring about not only spatial rotations through the Euler angle

$\vartheta = 2\pi n / m = 2\pi\mathbb{Q}$  with is discrete because it *excludes* the angles  $\vartheta = 2\pi\overline{\mathbb{Q}}$ , but also bring about a unitary time evolution through a time  $t_m = 3t / m$  based on the Heisenberg equations. Now we shall return to working on SU(2) to study the effect of the transformation (7.5) on electron wavefunctions  $\psi$  making a  $\varphi = 2\pi$  non-relativistic azimuthal circuit about a Dirac monopole. Because these electrons are posited to be non-relativistic, their boost parameter  $\chi \rightarrow 0$  in (5.3), and so any azimuthal rotation which occurs during this circuit will be governed by (5.4). Therefore, each of the two-component spinors  $\xi, \eta$  in  $\psi^T = (\xi^T, \eta^T)$  will transform identically under this azimuthal rotation, and as a result, we may apply (7.5) to both  $\xi$  and  $\eta$  in  $\psi$ , and therefore, to  $\psi$  overall.

Now, as we have already shown, it is the unitary matrices  $U_i$  of (5.1) which act on SU(2) spinors according to  $\xi \rightarrow \xi' = U_i \xi$  via (8.8) and its  $\det U_i = 1$  specialization (8.9) to generate the rotations (8.10) through (8.12) of the space coordinates on SO(3). Likewise, the root of unity matrices (6.12) acting on spinors via  $\xi \rightarrow \xi' = \tau_i \xi$  in (7.5) which, via the more general (8.8), generate the root of unity transformations (8.15) through (8.17) on SO(3). As then shown in (9.11) through (9.13) these include a time evolution through  $t_m = 3t / m$ . So it is natural to inquire what would be the transformation matrices of a rotation via (5.1) *combined* with a root of unity transformation via (6.12). Finding this out is a straightforward proposition: we simply multiply the unitary  $U_i$  of (5.1) by the  $\tau_i$  of (6.12), to arrive at:

$$\begin{aligned} \tau_1 U_1 &= \exp\left(i \frac{3\vartheta}{2}\right) \begin{pmatrix} \cos(\vartheta/2) & i \sin(\vartheta/2) \\ i \sin(\vartheta/2) & \cos(\vartheta/2) \end{pmatrix} \begin{pmatrix} \cos(\theta_1/2) & i \sin(\theta_1/2) \\ i \sin(\theta_1/2) & \cos(\theta_1/2) \end{pmatrix} \\ &= \exp\left(i \frac{3\vartheta}{2}\right) \begin{pmatrix} \cos((\theta_1 + \vartheta)/2) & i \sin((\theta_1 + \vartheta)/2) \\ i \sin((\theta_1 + \vartheta)/2) & \cos((\theta_1 + \vartheta)/2) \end{pmatrix} = \exp\left(i \frac{3\vartheta}{2}\right) \exp\left(i \sigma_2 \frac{\theta_1 + \vartheta}{2}\right), \end{aligned} \quad (10.1)$$

$$\begin{aligned} \tau_2 U_2 &= \exp\left(i \frac{3\vartheta}{2}\right) \begin{pmatrix} \cos(\vartheta/2) & \sin(\vartheta/2) \\ -\sin(\vartheta/2) & \cos(\vartheta/2) \end{pmatrix} \begin{pmatrix} \cos(\theta_2/2) & \sin(\theta_2/2) \\ -\sin(\theta_2/2) & \cos(\theta_2/2) \end{pmatrix} \\ &= \exp\left(i \frac{3\vartheta}{2}\right) \begin{pmatrix} \cos((\theta_2 + \vartheta)/2) & \sin((\theta_2 + \vartheta)/2) \\ -\sin((\theta_2 + \vartheta)/2) & \cos((\theta_2 + \vartheta)/2) \end{pmatrix} = \exp\left(i \frac{3\vartheta}{2}\right) \exp\left(i \sigma_2 \frac{\theta_2 + \vartheta}{2}\right), \end{aligned} \quad (10.2)$$

$$\begin{aligned} \tau_3 U_3 &= \exp\left(i \frac{3\vartheta}{2}\right) \begin{pmatrix} \exp(i\vartheta/2) & 0 \\ 0 & \exp(-i\vartheta/2) \end{pmatrix} \begin{pmatrix} \exp(i\theta_3/2) & 0 \\ 0 & \exp(-i\theta_3/2) \end{pmatrix} \\ &= \exp\left(i \frac{3\vartheta}{2}\right) \begin{pmatrix} \exp(i(\theta_3 + \vartheta)/2) & 0 \\ 0 & \exp(-i(\theta_3 + \vartheta)/2) \end{pmatrix} = \exp\left(i \frac{3\vartheta}{2}\right) \exp\left(i \sigma_3 \frac{\theta_3 + \vartheta}{2}\right). \end{aligned} \quad (10.3)$$

To obtain  $\tau_1 U_1$  and  $\tau_2 U_2$  one makes use of  $\cos \alpha \cos \beta - \sin \alpha \sin \beta = \cos(\alpha + \beta)$  and  $\cos \alpha \sin \beta + \sin \alpha \cos \beta = \sin(\alpha + \beta)$  which are the angle addition formulae. And  $\tau_3 U_3$  is obtained simply by multiplying exponentials. So the effect of combining these two transformations is that the Euler angle  $\vartheta$  is simply added to the rotation angles  $\theta_i$  in the form of  $\theta_i + \vartheta$ , while additionally the Euler angle introduces the overall factor  $\sqrt{\det(\tau_i)} = \exp(i3\vartheta/2)$  which as we saw in (9.10) to (9.12) also produces a time evolution through  $t_m = 3t/m$ . This may all be consolidated into:

$$\tau_i U_i = \exp\left(i \frac{3\vartheta}{2}\right) \exp\left(i \sigma_i \frac{\theta_i + \vartheta}{2}\right) = \exp i \left( \sigma_i \frac{\theta_i + \vartheta}{2} + \frac{3\vartheta}{2} \right). \quad (10.4)$$

We also note that the order of operation does not matter,  $[\tau_i, U_i] = 0$ , so the eigenvalues of both  $\tau_i$  and  $U_i$  are simultaneous observables.

The next question is, what is the effect of (10.1) through (10.3) on  $O(3)$ ? As we did previously, we may simply apply (8.8) to (10.1) to (10.3). The result, compare (8.10) through (8.12) and (8.15) through (8.17), turns out simply to be:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \exp(i3\vartheta) \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_1 + \vartheta) & \sin(\theta_1 + \vartheta) \\ 0 & -\sin(\theta_1 + \vartheta) & \cos(\theta_1 + \vartheta) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \equiv P_1 R_1 x^i, \quad (10.5)$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \exp(i3\vartheta) \begin{pmatrix} \cos(\theta_2 + \vartheta) & 0 & -\sin(\theta_2 + \vartheta) \\ 0 & 1 & 0 \\ \sin(\theta_2 + \vartheta) & 0 & \cos(\theta_2 + \vartheta) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \equiv P_2 R_2 x^i, \quad (10.6)$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \exp(i3\vartheta) \begin{pmatrix} \cos(\theta_3 + \vartheta) & \sin(\theta_3 + \vartheta) & 0 \\ -\sin(\theta_3 + \vartheta) & \cos(\theta_3 + \vartheta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \equiv P_3 R_3 x^i. \quad (10.7)$$

As usual, the angles on  $SU(2)$  are always equal to half of the angles on  $SO(3)$ . And as before, we still have the factor  $\det \tau_i = \exp(i3\vartheta)$  which spawns the  $t_m = 3t/m$  time evolution. Here too the operation order does not matter,  $[P_i, R_i] = 0$ , so these too yield simultaneous observables.

Now, we update (5.7) to include not only the possibility of an electron rotation under  $U_i(\theta_i)$  and a phase change  $\Lambda_\Delta$ , *but also a root-of-unity transformation under  $\tau_i(\vartheta)$* . So rather than  $\psi_+ \rightarrow \psi'_+ = I_{(2)} \otimes U_3 \exp(i\Lambda) \psi_+$  in (5.7), we now have  $\psi_+ \rightarrow \psi'_+ = I_{(2)} \otimes \tau_3 U_3 \exp(i\Lambda) \psi_+$ ,

with the additional  $\tau_3$  placed in either order relative to  $U_3$  because  $[\tau_3, U_3] = 0$ . In the event that this occurs, then using  $\theta_3 = \varphi$  in (10.3), the explicit wavefunction transformation of (5.7) must now be updated to:

$$\begin{aligned}\psi_+ \rightarrow \psi'_+ &= I_{(2)} \otimes \tau_3 U_3 \exp(i\Lambda) \psi_+ = I_{(2)} \otimes \exp\left(i\frac{3\vartheta}{2}\right) \exp\left(i\sigma_3 \frac{\varphi_{\text{rot}} + \vartheta}{2}\right) \exp(i\Lambda) \psi_+ \\ &= I_{(2)} \otimes \exp i\left(\sigma_3 \frac{\varphi_{\text{rot}} + \vartheta}{2} + \frac{3\vartheta}{2} + \Lambda\right) \psi_+\end{aligned}\quad (10.8)$$

where we have renamed  $\varphi \rightarrow \varphi_{\text{rot}}$  to make clear that this represents the *rotation* of the wavefunction as it makes a circuit about the monopole  $\psi$ . It is easily seen that for  $\vartheta=0$  this reverts to (5.7). Now we repeat the analysis of section 5 subsequent to (5.7), but using (10.8) instead.

To avoid an observable singularity under the operation (10.8) after a  $2\pi$  azimuthal circuit about the monopole, we must have a single valued  $\psi_+ \rightarrow \psi'_+ = \psi_+$  in (10.8). And it may be seen that (10.8) will yield  $\psi_+ \rightarrow \psi'_+ = \psi_+$  under the condition that:

$$\psi'_+ = I_{(2)} \otimes \exp i\left(\sigma_3 \frac{\varphi_{\text{rot}} + \vartheta}{2} + \frac{3\vartheta}{2} + \Lambda\right) \psi_+ = \psi_+ = I_{(4)} \psi_+ = I_{(4)} \exp(i2\pi n) \psi_+, \quad (10.9)$$

where we have introduced a 4x4 identity matrix  $I_{(4)}$  operating on  $\psi_+$  and then used  $1 = \exp(i2\pi n)$ . We see that this is an eigenvalue equation:

$$\left[ I_{(2)} \otimes \exp i\left(\sigma_3 \frac{\varphi_{\text{rot}} + \vartheta}{2} + I_{(2)} \frac{3\vartheta}{2} + I_{(2)} \Lambda\right) - I_{(4)} \exp(i2\pi n) \right] \psi_+ = 0. \quad (10.10)$$

for the phase difference  $\Lambda = \Lambda_\Delta$ , where we explicitly show all the identity matrices.

Now, at (5.8) we also obtained  $(I_{(2)} \otimes \exp i(\sigma_3 \pi + \Lambda) - I_{(4)} \exp(i2\pi n)) \psi_+ = 0$  to which the above reduces for  $\vartheta=0$  and a tidally-locked  $\varphi_{\text{rot}} = 2\pi$ . We then used  $\exp(i\sigma_3 \pi) = -I$  to simplify the reduction leading to  $\Lambda = \pi(2n-1)$  in (5.11) and finally to the half integer Dirac charges  $2eg = n - \frac{1}{2}$  in (5.13). Consequently, we could avoid having to fully treat the eigenvalue equation. Now, in (10.10), we can no longer do this. Now, we must carefully use (10.10) to understand the  $\Lambda$  eigenvalues that it permits, and then relate these to  $4\pi eg$  via (4.5).

To solve the eigenvalue equation, we first distribute the lead  $I_{(2)}$  to write (10.10) more explicitly as:



$$\begin{pmatrix} \exp i \left( \sigma_3 \frac{\varphi_{\text{rot}} + \vartheta}{2} + \frac{3\vartheta}{2} + \Lambda \right) - I_{(2)} \exp(i2\pi n) & 0 \\ 0 & \exp i \left( \sigma_3 \frac{\varphi_{\text{rot}} + \vartheta}{2} + \frac{3\vartheta}{2} + \Lambda \right) - I_{(2)} \exp(i2\pi n) \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}_+ . \quad (10.11)$$

Because the operation on  $\xi_+$  and  $\eta_+$  is identical, let us simply focus on the equation for  $\xi$ . And while so doing, we note that for a non-relativistic electron  $\psi_{+\uparrow}^T = (1, 0, 0, 0)$  i.e.  $\xi_{+\uparrow}^T = (1, 0)$  is the eigenvector of a spin up ( $\uparrow$ ) electron state and  $\psi_{+\downarrow}^T = (0, 1, 0, 0)$  i.e.  $\xi_{+\downarrow}^T = (0, 1)$  is that for a spin down ( $\downarrow$ ) electron state. So now, we extract the equation for  $\xi_+$  from the above, namely:

$$\left[ \exp i \left( \sigma_3 \frac{\varphi_{\text{rot}} + \vartheta}{2} + \frac{3\vartheta}{2} + \Lambda \right) - I_{(2)} \exp(i2\pi n) \right] \xi_+ = 0, \quad (10.12)$$

while recognizing that  $\eta_+$  has precisely the same equation.

Next, we may use (10.4) for  $i=3$  thus  $\varphi = \theta_3$ , together with the explicit form of (10.3), to ascertain that:

$$\begin{aligned} \exp(i\Lambda) \tau_3 U_3 &= \exp(i\Lambda) \exp\left(i \frac{3\vartheta}{2}\right) \exp\left(i \sigma_3 \frac{\varphi_{\text{rot}} + \vartheta}{2}\right) = \exp i \left( \sigma_i \frac{\varphi_{\text{rot}} + \vartheta}{2} + \frac{3\vartheta}{2} + \Lambda \right) \\ &= \exp i \left( \frac{3\vartheta}{2} + \Lambda \right) \begin{pmatrix} \exp i((\varphi_{\text{rot}} + \vartheta)/2) & 0 \\ 0 & \exp i(-(\varphi_{\text{rot}} + \vartheta)/2) \end{pmatrix} \\ &= \begin{pmatrix} \exp i(\varphi_{\text{rot}}/2 + 2\vartheta + \Lambda) & 0 \\ 0 & \exp i(-\varphi_{\text{rot}}/2 + \vartheta + \Lambda) \end{pmatrix} \end{aligned} \quad (10.13)$$

As a result, the explicit form of the eigenvalue equation (10.12) is:

$$\begin{aligned} &\left[ \exp i \left( \sigma_i \frac{\varphi_{\text{rot}} + \vartheta}{2} + \frac{3\vartheta}{2} + \Lambda \right) - I_{(2)} \exp(i2\pi n) \right] \xi_+ \\ &= \begin{pmatrix} \exp i(\varphi_{\text{rot}}/2 + 2\vartheta + \Lambda) - \exp(i2\pi n) & 0 \\ 0 & \exp i(-\varphi_{\text{rot}}/2 + \vartheta + \Lambda) - \exp(i2\pi n) \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}_+ = 0 \end{aligned} \quad (10.14)$$

This contains is a diagonal matrix, so it is simple to determine eigenvalues. For a spin up electron with  $\xi_+^T = (1, 0)$  we have:

$$\exp i(\varphi_{\text{rot}}/2 + 2\vartheta + \Lambda) - \exp i(2\pi n) = 0. \quad (10.15)$$

while for spin down with  $\xi_+^T = (1, 0)$  we have:

$$\exp i(-\varphi_{\text{rot}}/2 + \vartheta + \Lambda) - \exp i(2\pi n) = 0. \quad (10.16)$$

Isolating the phase, both of these results may respectively be summarized as:

$$\begin{aligned} \uparrow: \quad \exp(i\Lambda) &= \exp i(2\pi n - 2\vartheta - \varphi_{\text{rot}}/2) \\ \downarrow: \quad \exp(i\Lambda) &= \exp i(2\pi n - \vartheta + \varphi_{\text{rot}}/2) \end{aligned} \quad (10.17)$$

Next, as we earlier did in sections 4 and 5, we again we take the electron wavefunction over a complete circuit through a  $2\pi$  azimuth which we designate by  $\varphi_{\text{orb}} = 2\pi$  to distinguish the azimuthal  $2\pi$  “orbital” circuit of the  $\psi_+$  wavefunction about the monopole from the  $\varphi_{\text{rot}}$  rotation of this wavefunction *which may or may not be tidally-locked* to its circuit. So for a tidally-locked electron we set  $\varphi_{\text{rot}} = 2\pi$  in the above, while for an electron which does not rotate at all during the circuit (which is the implied assumption that leads to the customary DQC  $e\mu = 2\pi n$  of (4.8)) we set  $\varphi_{\text{rot}} = 0$  in the above. The  $2\pi$  associated with the circuit about the monopole is not the  $\varphi_{\text{rot}}$  above; rather it is the  $\varphi$  in the Wu-Yang equation (4.4) which we now rewrite with the renamed  $\varphi_{\text{orb}}$  to distinguish it from  $\varphi_{\text{rot}}$ , as:

$$\psi_+ \rightarrow \psi_- = \psi'_+ = \exp(i\Lambda)\psi_+ = \exp(i2eg\varphi_{\text{orb}})\psi_+. \quad (10.18)$$

For a single circuit about the monopole,  $\varphi_{\text{orb}} = 2\pi$ , and the above becomes:

$$\psi_+ \rightarrow \psi_- = \psi'_+ = \exp(i\Lambda)\psi_+ = \exp(i4\pi eg)\psi_+ \quad (10.19)$$

which is identical with (4.5).

Now, in order to avert any observable singularities, the electron wavefunction after a single  $\varphi_{\text{orb}} = 2\pi$  circuit about the monopole as described by (10.19) above (and the earlier (4.5)) must be single-valued,  $\psi_+(\varphi_{\text{orb}} = 0) \rightarrow \psi'_+(\varphi_{\text{orb}} = 2\pi) = \psi_+(\varphi_{\text{orb}} = 0)$ . The occurrence of this single-valued condition  $\psi_+ \rightarrow \psi'_+ = \psi_+$  is given by (10.9), which after solving its implied eigenvalue equation (10.10) enables us to deduce (10.17). So to ascertain what is required for the wavefunction to be single-valued after a  $\varphi_{\text{orb}} = 2\pi$  circuit, we may equate (10.17) to (10.19). This combination of (10.19) being set equal to (10.17) for each of the spin up and spin down electrons then produces:

$$\begin{aligned} \uparrow: \quad \psi_+ \rightarrow \psi_- = \psi'_+ &= \exp(i\Lambda)\psi_+ = \exp i(2\pi n - 2\vartheta - \varphi_{\text{rot}}/2)\psi_+ = \exp(i4\pi eg)\psi_+ \\ \downarrow: \quad \psi_+ \rightarrow \psi_- = \psi'_+ &= \exp(i\Lambda)\psi_+ = \exp i(2\pi n - \vartheta + \varphi_{\text{rot}}/2)\psi_+ = \exp(i4\pi eg)\psi_+ \end{aligned} \quad (10.20)$$

This has the respective solutions:

$$\begin{aligned} \uparrow: \Lambda_{\Delta} &= 2\pi n - 2\vartheta - \varphi_{\text{rot}} / 2 = 4\pi e g \\ \downarrow: \Lambda_{\Delta} &= 2\pi n - \vartheta + \varphi_{\text{rot}} / 2 = 4\pi e g \end{aligned} \quad (10.21)$$

where we once again use the notation  $\Lambda_{\Delta}$  to make clear that this is a *change in phase* following the  $\varphi_{\text{orb}} = 2\pi$  circuit. This can easily be restructured into:

$$\begin{aligned} \uparrow: 2e g &= \Lambda_{\Delta} / 2\pi = n - \vartheta / \pi - \varphi_{\text{rot}} / 4\pi \\ \downarrow: 2e g &= \Lambda_{\Delta} / 2\pi = n - \vartheta / 2\pi + \varphi_{\text{rot}} / 4\pi \end{aligned} \quad (10.22)$$

We see that for  $\varphi_{\text{rot}} = 0$  (no tidal lock) and  $\vartheta = 0$  these both reduce to  $2e g = n$  which is the standard Dirac Quantization Condition (DQC) of (4.7). We also see that for  $\vartheta = 0$  and  $\varphi_{\text{rot}} = 2\pi$  where the electron is in a tidal lock with the monopole these reduce to  $2e g = n \mp \frac{1}{2}$ . This is equivalent to  $2e g = n - \frac{1}{2}$  in (5.13) because since  $n$  may be any integer, we can simply redefine  $n \rightarrow n + 1$  in  $2e g = n - \frac{1}{2}$  to obtain  $2e g = n + \frac{1}{2}$ . So this provides a check that (10.22) correctly reproduces the both the standard result (4.7) and the half-integer result (5.13). Now let us see what further results may be found in (10.22).

First, we note that  $n$  in (10.22) may be any arbitrary integer, and that this  $n$  was first introduced via  $1 = \exp(i2\pi n)$  at (10.9). Second, we note that the  $n$  in the Euler angle  $\vartheta = 2\pi n / m = 2\pi \mathbb{Q}$  was first introduced at (6.2) to characterize roots of unity. So these are not the same  $n$  but are integers which may vary independently of one another. The only constraint is that each of these  $n$  must be an integer. So let us momentarily use  $n_1$  for the integer in (10.22) and use  $n_2$  for the integer in  $\vartheta$ , thus writing  $\vartheta = 2\pi n_2 / m$ . With these notational adjustments to segregate the two independent integers, we now substitute  $\vartheta = 2\pi n_2 / m$  into (10.22) to write:

$$\begin{aligned} \uparrow: 2e g &= \Lambda_{\Delta} / 2\pi = n_1 - 2n_2 / m - \varphi_{\text{rot}} / 4\pi \\ \downarrow: 2e g &= \Lambda_{\Delta} / 2\pi = n_1 - n_2 / m + \varphi_{\text{rot}} / 4\pi \end{aligned} \quad (10.23)$$

Now let us further set  $\varphi_{\text{rot}} = 0$  to remove any tidal lock, so that the electron travels one circuit about the monopole without any rotation via  $U_3$  in (5.1). This is in fact the way in which the standard DQC is obtained, so now the only difference from the standard DQC result is the presence of the roots of unity fractions  $n_2 / m$ . So with  $\varphi_{\text{rot}} = 0$  the above now becomes:

$$\begin{aligned} \uparrow: 2e g &= \Lambda_{\Delta} / 2\pi = n_1 - 2n_2 / m \\ \downarrow: 2e g &= \Lambda_{\Delta} / 2\pi = n_1 - n_2 / m \end{aligned} \quad (10.24)$$

Clearly, this describes fractional Dirac magnetic monopoles, given that  $2eg$  is equal to an integer  $n_1$ , minus a rational number  $2n_2/m$  for spin up and a different rational number  $n_2/m$  for spin down. Now, turning from the general to the specific, let us study this solution for the first few integer values of  $m$ , paying special attention to the fact that the spin up electron contains the rational number  $2n_2/m$  while the spin down electron contains the different rational number  $n_2/m$ . This correct yet “asymmetric” result originates from  $\text{diag}(\tau_3) = (\exp(i2\vartheta), \exp(i\vartheta))$  in (6.12), and turns out to be what restricts the charge fractions based on the root of unity transformations to being only odd integers. This is what then raises the question whether this is somehow related to the Fractional Quantum Hall Effect (FQHE), which likewise is restricted only to odd fractions, aside from the half-integer fraction  $2$  which correlates to the tidally-locked electron developed in section 5.

## 11. Lorentz Symmetry Restriction of the Fractional Charge Denominators to only Odd Integers

Let us jump right in to studying the eigenvalue solutions (10.24) for each of  $m = 1, 2, 3, 4, 5, 6$ . For  $m=1$ , (10.24) becomes:

$$\begin{aligned} \uparrow: 2eg &= \Lambda_\Delta / 2\pi = n_1 - 2n_2 = n \\ \downarrow: 2eg &= \Lambda_\Delta / 2\pi = n_1 - n_2 = n \end{aligned} \quad (11.1)$$

which is the standard DQC. Because  $n_1$  and  $n_2$  are arbitrary integers, we can set  $n_1 - 2n_2 = n_3$  and  $n_1 - n_2 = n_4$ , and then rename each of  $n_3$  and  $n_4$  to  $n$ . So both the spin up and spin down states will have the standard charge  $2eg = n$ . Also, the phase difference  $\Lambda_\Delta = 2\pi n$  will have its initial orientation restored, i.e., there will be a “return to phase” following a  $\varphi_{\text{orb}} = 2\pi$  circuit about the monopole, see (4.11) and the subsequent discussion.

For  $m=2$ , which represents the square roots of unity, (10.24) becomes:

$$\begin{aligned} \uparrow: 2eg &= \Lambda_\Delta / 2\pi = n_1 - n_2 = n \\ \downarrow: 2eg &= \Lambda_\Delta / 2\pi = n_1 - n_2 / 2 = n / 2 \end{aligned} \quad (11.2)$$

Here, we again recognize that  $n_1 - n_2$  can be renamed to  $n$  because this will always yield an integer for any and all choices of  $n_1$  and  $n_2$ . Meanwhile,  $n_1 - n_2 / 2$  can be renamed to  $n / 2$  because this will always yield a half integer or a whole integer for any and all  $n_1$  and  $n_2$ . This difference between spin up and spin down solutions is a very interesting and fruitful development, which we now examine closely.

Let us suppose that we are observing a spin down electron  $\psi$  interacting with a monopole for which  $n=1$  in (11.2), in some frame of reference. The monopole charge state

detected by this spin down electron could therefore be the half-unit  $2eg = 1/2$ . However, it is well-known that helicity, which is a conserved quantum number that commutes with the Dirac Hamiltonian, is frame-dependent and so can be reversed by a Lorentz transformation which overtakes the electron. So suppose we were to now overtake this electron by moving to a different reference frame. The result in (11.2) for a spin up electron does not permit this  $2eg = 1/2$ , because it requires that  $2eg = n$ , which is the standard DQC. So *the charge condition would have to change to a whole-integer condition simply as a result of changing our reference frame*, which is a physically-dubious proposition. Because the  $m=2$  states of (11.2) would violate Lorenz symmetry because the monopole charge would not be Lorentz invariant, we are required to exclude these from being physically-observable states.

As such, while Dirac charges with  $2eg = n/2$  are *permitted for electrons which move about the monopole in a tidal lock* as developed in section 5, they appear to be *excluded for root of unity transformations*, not because of any observable singular behavior, but because spin up and spin down electrons when interacting with monopoles are predicted to exhibit different monopole charge conditions and one should not expect that a change in reference frame should be able to change these charge conditions. As a result of requiring that the monopole charge condition not be changed just because we change the helicity, we use the requirement for Lorentz symmetry to exclude  $m=2$  as a physically-observable denominator for a  $\varphi_{\text{rot}} = 0$  wavefunction. Again, however,  $m=2$  is still permitted, but because of what we found in section 5 for a tidally-locked  $\varphi_{\text{rot}} = 2\pi$  electron, not because of (10.24) which leads to the unphysical (11.2).

For  $m=3$ , i.e., for the cubed roots of unity with  $\vartheta = 2\pi n/3$  (10.24) becomes:

$$\begin{aligned} \uparrow: 2eg &= \Lambda_{\Delta} / 2\pi = n_1 - 2n_2 / 3 = n/3 \\ \downarrow: 2eg &= \Lambda_{\Delta} / 2\pi = n_1 - n_2 / 3 = n/3 \end{aligned} \quad (11.3)$$

Here, by suitable choices of  $n_1$  and  $n_2$  we can generate charges with  $2eg = n/3$ , thus one-third of a charge and integer multiples thereof. Because we have the same net result for both spin up and spin down, a change in helicity owing to a change in reference frame will not change the charge condition. Because these do not produce observable singularities and the same charge conditions apply even if the electron is overtaken by a change in reference frame, this  $1/3$  charge fraction appears to be a physically-permitted state. In the above, the change in phase for a single  $2\pi$  circuit about the monopole is  $\Lambda_{\Delta} = 2\pi n/3$ , which means also that the electron will return to phase,  $\Lambda_{\Delta} = 2\pi n$ , after a  $\varphi_{\text{orb}} = 6\pi = 3 \cdot 2\pi$  circuit about the monopole. So (11.3), and specifically  $2eg = n/3$ , is the smallest charge fraction permitted for a  $\varphi_{\text{rot}} = 0$  electron. It is worth being reminded that for  $m=3$ ,  $\det \tau_i = \exp(i3\vartheta) = \exp(i6\pi n/3) = \exp(i2\pi n) = 1$ , which is a special and unique case. So it would be possible for this special case to set  $\exp(i3\vartheta) = 1$  in (9.11) through (9.13) and miss the fact that keeping the  $\exp(i2\pi n)$  in place next to  $X^i(0)$  in the form of  $\exp(i2\pi n) X^i(0)$  is tantamount, in the classical correspondence, see (9.2) and (9.4), to generating a unitary time evolution through  $t_3 = 3t/3 = t$ .

This is also a good place to point out how, in the standard Dirac monopole theory, it is believed that  $\Lambda_{\Delta} = 2\pi n$  must hold after each  $\varphi_{\text{orb}} = 2\pi$  circuit about the monopole. But this is because the only root of unity that is considered in the standard theory is the trivial identity  $I_{(2)}$  and not any of its roots of unity. In reality, (10.19) is the fundamental condition which *must be obeyed in order to avoid observable singularities*, whereby the electron wavefunction  $\psi$  must remain single valued after each and every  $\varphi_{\text{orb}} = 2\pi$  circuit. We learn from (10.17) that absent a root of unity transformation ( $\vartheta=0$ ) and absent a rotation ( $\varphi_{\text{rot}}=0$ ) the phase factor will be given by  $\exp(i\Lambda) = \exp(i2\pi n) = 1$  so that the single-valued wavefunction mandate of (10.19) will become  $\exp(i\Lambda)\psi_{+} = \exp(i4\pi eg)\psi_{+} = \exp(i2\pi n)\psi_{+}$ , which recovers the standard DQC  $2eg = n$ . But otherwise, if there is either a rotation ( $\varphi_{\text{rot}} \neq 0$ ) or a root of unity besides  $I_{(2)}$  operating on the spinors ( $\vartheta \neq 0$ ), the factor  $\exp(i\Lambda)$  in (10.17) will no longer be  $\exp(i\Lambda) = \exp(i2\pi n) = 1$ , but will come a complex number with a magnitude equal to 1 but a different orientation. So as seen in (10.21), the phase difference  $\Lambda_{\Delta}$  will be something other than  $2\pi n$ , which will in turn force the Dirac monopole charge condition to be something other than  $2eg = n$ , and specifically, will force it to become (10.22). *This demonstrates how and why fractional Dirac monopole charges may indeed exist without observable singularities, which is the fundamental thesis of this paper.*

For  $m=4$ , i.e. for the fourth roots of unity, (10.24) becomes:

$$\begin{aligned} \uparrow: 2eg &= \Lambda_{\Delta} / 2\pi = n_1 - n_2 / 2 = n / 2 \\ \downarrow: 2eg &= \Lambda_{\Delta} / 2\pi = n_1 - n_2 / 4 = n / 4 \end{aligned} \quad (11.4)$$

Here, we have the same problem as in (11.2) wherein a change in reference frame which flips helicity would cause the charge condition to change. So for the same reason, requiring that the monopole charges must be Lorentz invariants, we exclude quarter-unit charges.

For  $m=5$ , the fifth roots of unity with  $\vartheta = 2\pi n / 5$ , (10.24) yields:

$$\begin{aligned} \uparrow: 2eg &= \Lambda_{\Delta} / 2\pi = n_1 - 2n_2 / 5 = n / 5 \\ \downarrow: 2eg &= \Lambda_{\Delta} / 2\pi = n_1 - n_2 / 5 = n / 5 \end{aligned} \quad (11.5)$$

This does not have the helicity problem of  $m=2$  and  $m=4$ , and yields no observable singularities. So we regard this  $1/5$  charge fraction to be a physically-permitted state. The phase change is  $\Lambda_{\Delta} = 2\pi n / 5$  so this will return to phase after a after a  $\varphi_{\text{orb}} = 10\pi = 5 \cdot 2\pi$  circuit about the monopole circuit. Again, while it takes five  $2\pi$  circuits for the phase to regain its orientation, the electron wavefunction does remain single-valued after each and every  $2\pi$  circuit. Therefore, these  $2eg = n / 5$  fractions may also exist without observable singularity.

Finally, let's look at (10.24) for  $m=6$ . Here we have:

$$\begin{aligned} \uparrow: 2eg &= \Lambda_{\Delta} / 2\pi = n_1 - n_2 / 3 = n / 3 \\ \downarrow: 2eg &= \Lambda_{\Delta} / 2\pi = n_1 - n_2 / 6 = n / 6 \end{aligned} \quad (11.6)$$

We once again have the helicity problem of  $m=2$  and  $m=4$ , and so also need to exclude  $m=6$ .

In general, it will be seen that for odd  $m = 2l - 1 = 1, 3, 5, 7, \dots$  it is possible for the  $\uparrow$  and  $\downarrow$  values of  $2eg$  to be equal,  $2eg = n / m = n / (2l - 1)$ . So there are no observed singularities because the wavefunction is single valued following a  $2\pi$  circuit, and also, there is no change in the charge condition following a helicity flip so the Lorentz symmetry of the charges is preserved. As a result, we regard these odd-fractions to be physically viable states which might be observable under some set of conditions. In contrast, for any even  $m = 2l = 2, 4, 6, 8, \dots$  fraction (absent a tidal lock), the spin down values will always be  $2eg = n / m$  but the spin up value will be more tightly restricted to  $2eg = 2n / m$ . Thus a spin down unit charge with, say,  $2eg = 1 / m$ , if overtaken, would be required to change its charge to some  $2eg = 2n / m$ , which we take to be physically impossible. As a result, for a  $\varphi_{\text{rot}} = 0$  wavefunction, *Lorentz symmetry appears to exclude all even-integer monopole charge denominators from being physically-permitted states*, with the exception of  $m=2$  which arises not based on roots of unity, but on a tidal lock between the electron and the monopole.

So, we see that the odd-charge fractions are permitted without observable singularity, but the even charge fractions are excluded as a result of the Lorentz Symmetry of Special Relativity. Based on section 5, however, see for different reasons that  $m=2$  is permitted as the only even-denominator charge, not because of the root-of-unity transformations, but because of the electron being in a tidal lock as it circuits about the monopole. To summarize: the permitted fractions which are not observably singular *and* which do not raise violate Lorentz symmetry are  $2eg = n / 2$  for a tidally-locked  $\varphi_{\text{rot}} = 2\pi$  electron based on its rotation, and  $2eg = n / 3, n / 5, n / 7, n / 9, \dots$  for a  $\varphi_{\text{rot}} = 0$  electron with no tidal lock based on a root of unity transformation. Taken together, the permitted Dirac fractions which are non-singular and Lorentz invariant, including the conventional  $2eg = n$ , are summarized by:

$$2eg = \frac{\Lambda_{\Delta}}{2\pi} = \frac{n}{m}; \quad n = 1, 2, 3, 4, 5, \dots; \quad m = 1, 2, 3, 5, 7, 9, \dots \quad (11.7)$$

Consequently, the only permitted charge fractions are those with odd denominators, with the sole exception of the even denominator 2. The odd denominators arising from root of unity transformations, and the even denominator 2 arises from a tidal lock of the electron with the monopole. It is impossible not to appreciate that these are precisely the same charge fractions experimentally observed in the Fractional Quantum Hall Effect (FQHE), which raises the question whether these are related to the result in (11.7).

In general, (11.7) also states that after a  $\varphi_{\text{orb}} = 2\pi$  circuit the phase change will be:

$$\Lambda_{\Delta} = 2\pi n / m. \quad (11.8)$$

Therefore, the electron will only return to phase after a  $\varphi_{\text{orb}} = 2\pi m$  circuit, that is, only after making  $m$  circuits about the monopole. The  $m=2$  charge for an electron in tidal lock returns to phase after  $\varphi_{\text{orb}} = 4\pi n$  which is due to the flipped-sign wavefunction version that occurs following  $2\pi$  rotations which sign does not get restored until after a  $4\pi$  rotation. This circumstance, as noted, is often described in relation to orientation / entanglement, again, see the discussion after (5.14) here and section 41.5 of [4]. The  $m = 1, 3, 5, 7, 9, \dots$  states which have odd denominators based on roots of unity return to phase after  $\varphi_{\text{orb}} = 2\pi(2l-1) = 2\pi, 6\pi, 10\pi, 14\pi, \dots$  and thus return to phase after circuits which *differ* from one another by  $\Delta\varphi_{\text{orb}} = 4\pi n$ .

Finally, we may use (11.7) in (4.12) to write the monopole potentials for all the permitted fractional charge states, contrast (4.12), as:

$$\begin{aligned} eA_{+} &\equiv \frac{1}{2} \frac{n}{m} (\cos \theta - 1) d\varphi \\ eA_{-} &\equiv \frac{1}{2} \frac{n}{m} (\cos \theta + 1) d\varphi \end{aligned} \quad (11.9)$$

with  $n = 1, 2, 3, 4, 5, \dots$  and  $m = 1, 2, 3, 5, 7, 9, \dots$

**NOTE: THESE NEXT TWO SECTIONS HAVE NOT BEEN REVIEWED FOLLOWING MY DISCOVERY THAT THE EXPONENTIAL  $\exp(i3\vartheta)$  GENERATES TIME EVOLUTION AS SEEN IN (9.11) THROUGH (9.13). I AM PLANNING SOME MAJOR RESTURTING OF THESE NEXT TWO SECTIONS TO REFLECT THIS NEW UNDERSTANDING. BUT EVERYTHING ABOVE THIS PLACE, I.E. SECTIONS 1 THROUGH 11, ARE PRETTY WELL SETTLED AND I EXPECT THESE TO REMAIN AS IS WITH RELATIVELY MINOR CHAGES FROM HERE.**

## **12. Might Root of Unity Transformations be Synonymous with Wavefunction Transformations into Different Orbital Angular Momentum States of Atomic Shell Structure?**

It is of interest that the permitted fractional charges  $2eg = n/m$  of (11.7) arising from the root of unity generator  $\tau_3$  for (6.12) are naturally restricted by helicity considerations to the odd integer denominators  $m = 1, 3, 5, 7, 9, \dots$ , and that the only permitted even denominator is  $m=2$  as shown in (5.14). First – whether a real connection or merely a coincidence, which must be studied theoretically and experimentally – it just so happens that these are *precisely the same fractional charge denominators empirically observed in the Fractional Quantum Hall Effect (FQHE) at ultra-low temperatures near 0K*. These are typically represented by the fill factors  $\nu = n/m$  with  $n = 1, 2, 3, 5, 7, 9, \dots$ . Second, the exclusion of even denominators other than 2 is not a



condition imposed in order to fit the FQHE. Quite the converse, it is the requirement that the charge condition at any given root of unity  $m$  remain invariant with respect to the flipping of helicity, which theoretically compels the exclusion of the even denominator states other than the tidally-locked  $m=2$ . So in a very basic sense, *even-numbered denominators are excluded by the Lorentz symmetry of special relativity*. Third, the even denominator 2 arises when an electron wavefunction completes a  $2\pi$  circuit about the monopole in a tidal lock with the monopole, which is a *different theoretical cause* than the root-of-unity transformations that yield the odd-integer fractional denominators. Fourth, see, e.g., Figure 3 at [9], for the odd-integer fractions the empirical curves mapping the Hall resistance  $R_H$  against the strengths of the applied perpendicular magnetic field are deep and narrow with  $R_H$  becoming very small at any given fractional plateau, while for the even-fraction  $m=2$  the curve is wider and shallower, with  $R_H$  remaining substantially non-zero. This qualitative difference in the empirical data suggests that the odd fractions have a *different empirical cause* than the half-integer fractions. This is consistent with the different theoretical causes whereby the odd fractions arise from root of unity transformations and the even fraction  $m=2$  arises from a tidal lock.

Given the foregoing, it is difficult not to at least suspect a possible physical connection between fractional Dirac monopoles and FQHE charge fractions. If such a connection can be established theoretically and confirmed experimentally, this would mean that *the FQHE provides direct albeit heretofore-unrecognized experimental evidence that  $U(1)_{em}$  magnetic monopoles do exist in nature*, at least in the ultra-low temperature environment. Consequently, it behooves us to gain a better understanding of what it really means – physically, not mathematically – to subject a spinor  $\xi$  – and by implication a wavefunction  $\psi$  – to a root of unity transformation  $\xi \rightarrow \xi' = \tau_3 \xi$  about the z axis. As we shall now see, these root of unity transformations exhibit behaviors which map very directly to the quantized behaviors of electrons in atomic shells, suggesting that atomic structure is the prime suspect to provide a physical understanding of root of unity transformations.

Mathematically, roots of unity (6.1) are multivalued numbers all with the same magnitude 1, but with an Euler angle  $\vartheta = 2\pi n / m = 2\pi \mathbb{Q}$  from (6.2) which points each of these roots in a unique *direction* in the complex plane. So, for example, the first root of unity  $(1)^{n/1} = 1^n = +1$  has a single value  $+1$ , at an angle  $\vartheta = 360^\circ$ , which remains the same for all iterative multiples of  $360^\circ$ . The square roots of unity  $(1)^{n/2} = \pm 1$  have two values,  $\mp 1$  for  $n=1,2$  respectively at angles  $\vartheta = 180^\circ, 360^\circ$ , with iterative cycling for  $n > 2$ . The cubed roots of unity  $(1)^{n/3}$  have the three values  $\frac{1}{2}(-1 \mp i\sqrt{3})$  and  $+1$ , for  $n=1,2,3$  and  $\vartheta = 120^\circ, 240^\circ, 360^\circ$  respectively, with iterative cycling for  $n > 3$ . Indeed, the primitive, non-trivial roots  $\frac{1}{2}(-1 \mp i\sqrt{3})$ , although complex numbers, have a magnitude  $\frac{1}{2}(-1 - i\sqrt{3})\frac{1}{2}(-1 + i\sqrt{3}) = 1$ , and so may easily be thought about together with the trivial root  $+1$  as a  $\pm$  sign with the three values, not two. The fourth roots of unity  $(1)^{n/4}$  have the four values  $i, -1, -i, 1$  for  $n=1,2,3,4$  and  $\vartheta = 90^\circ, 180^\circ, 270^\circ, 360^\circ$ , respectively, with iteration thereafter, and so are a four-valued analogy to a  $\pm$  sign. And so on for all other roots. So any root of unity  $(1)^{n/m}$  has precisely  $m$  distinct values which then recycle after each  $2\pi$  cycle in the complex plane, and these roots may be

thought of as an  $m$ -valued  $\pm$  sign which consists of complex numbers with magnitude 1. They are distinguished from one another solely by their Euler angle  $\vartheta = 2\pi n/m$ .

Equivalently, and more physically, we can think of the roots of unity as providing a “*root of unity degree of freedom*,” whereby the number 1 can take on a total of exactly  $m$  distinct values for any given unity root  $(1)^{n/m}$ . Each  $(1)^{n/m}$  with  $1 \leq n \leq m$  for a given  $m$  can be thought of as a generalization of electron “versions” from the two versions discussed in section 5, to  $m$  versions for which the two-versioned  $\pm$  sign is simply represented by the special case  $\pm 1 = (1)^{n/2}$ . The reason we wish to think about this root of unity multi-valuedness as being a *degree of freedom*, aside from this being a step closer from mathematics to physics, is because when we start to talk about fermions – and electrons are quintessential fermions – the Exclusion Principle demands that only way one can assemble a system of fermions containing more than one fermion, is to provide each fermion with a unique set of quantum numbers that distinguish it from all the other fermions in that system. But, at bottom, these exclusive quantum numbers are simply multi-values permitted by some degree or degrees of freedom. So the question we shall now consider is whether the multi-valuedness of the roots of unity – when generalized to the 2x2 root of unity matrices developed in section 6 – is in fact related to the degrees of freedom and quantum-numbered-values that electrons must have in order to coexist in the shells of the same atom consistently with Exclusion. If so, then we can acquire a more direct physical understanding of what it means to subject a fermion to what we have all along called a “root of unity transformation,” and further, can relate the fractionalized Dirac monopoles not only to low-temperature FQHE physics, but also to atomic structure, with opportunities for direct experimental validation of the results presented here. So let us now proceed.

The smallest fractional charge permitted by (11.7) based on the root of unity degree of freedom, beyond the  $m=1$  state  $2eg = n$  of the standard DQC, is the  $m=3$  fraction  $2eg = n/3$  for which  $\vartheta = 2\pi n/3 = n \cdot 120^\circ$ . (Again,  $m=2$  has a different genesis in the tidal lock reviewed in section 5.) As discussed near the end of section 9, when  $m=3$  the discretized space and time transformation (9.18) with the  $\Sigma_j$  and  $\delta^{ij}$  suppressed becomes:

$$\mathcal{X}^M = \begin{pmatrix} t^i \\ x^i \end{pmatrix} \rightarrow \mathcal{X}'^M = \begin{pmatrix} t'^i \\ x'^i \end{pmatrix} = \begin{pmatrix} \cos(2\pi n) & \sin(2\pi n) \\ -\sin(2\pi n) & \cos(2\pi n) \end{pmatrix} \begin{pmatrix} t^j \\ x^j \end{pmatrix} = \mathcal{X}^M \quad (12.1)$$

So in sum, for  $m=3$ , and indeed uniquely to only  $m=1$  and  $m=3$ , there is no Euclidean space and time mixing. Let us now study  $m=3$  in further depth, as an important example.

The  $m=3, n=1$  generators for which  $\vartheta = 2\pi/3 = 120^\circ$ , using  $\exp(i\pi) = -1$  as well as  $\cos(\pi/3) = 1/2$  and  $\sin(\pi/3) = \sqrt{3}/2$  and the sign behaviors of sin and cos in the four quadrants of the  $0 \leq \theta_i \leq 2\pi$  domain, are ascertained from (6.12) to be:

$$\tau_1\left(\frac{2\pi}{3}\right) = -\frac{1}{2}\begin{pmatrix} 1 & i\sqrt{3} \\ i\sqrt{3} & 1 \end{pmatrix}; \quad \tau_2\left(\frac{2\pi}{3}\right) = -\frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}; \quad \tau_3\left(\frac{2\pi}{3}\right) = -\frac{1}{2}\begin{pmatrix} 1+i\sqrt{3} & 0 \\ 0 & 1-i\sqrt{3} \end{pmatrix}. \quad (12.2)$$

For  $m=3$  and  $n=2$  we have  $\vartheta = 4\pi/3 = 240^\circ$  which flips the sign of the sin but not of the cos. These  $\tau_i$  generators are the squares  $\tau_i^2$  of each of (12.2) as is easily checked, and are:

$$\tau_1\left(\frac{4\pi}{3}\right) = -\frac{1}{2}\begin{pmatrix} 1 & -i\sqrt{3} \\ -i\sqrt{3} & 1 \end{pmatrix}; \quad \tau_2\left(\frac{4\pi}{3}\right) = -\frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}; \quad \tau_3\left(\frac{4\pi}{3}\right) = -\frac{1}{2}\begin{pmatrix} 1-i\sqrt{3} & 0 \\ 0 & 1+i\sqrt{3} \end{pmatrix}. \quad (12.3)$$

Of course for  $m=n=3$  we have  $\vartheta = 2\pi = 360^\circ$  and all three  $\tau_i(2\pi) = I_i$ , the 2x2 identity triplet.

So now, focusing on the azimuthal generator  $\tau_3$  about the  $z$  axis, let us transform a spinor via  $\xi \rightarrow \xi' = \tau_3\xi$ . As already reviewed near (5.4) and (5.5) and again near the start of section 10, if we take the electron  $\psi$  to be non-relativistic (no boost), then the entire Dirac wavefunction  $\psi$  will transform in the same manner as  $\xi \rightarrow \xi' = \tau_3\xi$  acting on the spinor and so we may use the spinor  $\xi$  as a proxy for the complete  $\psi$ . So we take  $\xi^T = (\xi_1, \xi_2)$  as in section 7, and as observed at (10.2) we recognize that  $\xi^T = (1, 0)$  corresponds with a spin up  $\psi^T = (1, 0, 0, 0)$  while  $\xi^T = (0, 1)$  corresponds with a spin down  $\psi^T = (0, 1, 0, 0)$ . (At this point, we no longer need the “+” subscript introduced at (2.1).) So to directly and symbolically remind us of these spin correspondences, let us now denote this spinor as  $\xi^T = (\uparrow, \downarrow)$  rather than  $\xi^T = (\xi_1, \xi_2)$ . Consequently, the transformation  $\xi \rightarrow \xi' = \tau_3(2\pi/3)\xi$  is explicitly given using (12.2) by:

$$\xi = \begin{pmatrix} \uparrow \\ \downarrow \end{pmatrix} \rightarrow \xi' = \begin{pmatrix} \uparrow' \\ \downarrow' \end{pmatrix} = \tau_3\left(+\frac{1}{3}2\pi\right)\xi = \frac{1}{2}\begin{pmatrix} -1-i\sqrt{3} & 0 \\ 0 & -1+i\sqrt{3} \end{pmatrix}\begin{pmatrix} \uparrow \\ \downarrow \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}(1+i\sqrt{3})\uparrow \\ -\frac{1}{2}(1-i\sqrt{3})\downarrow \end{pmatrix} = \begin{pmatrix} (1)^{-1/3}\uparrow \\ (1)^{+1/3}\downarrow \end{pmatrix}. \quad (12.4)$$

Likewise we turn to the transformation  $\xi \rightarrow \xi'' = \tau_3(4\pi/3)\xi$ . But before we do this, because of the cyclical nature of  $\vartheta = 2\pi n/m = 2\pi\mathbb{Q}$ , let us instead write the root of unity generator as  $\tau_3(4\pi/3) = \tau_3(-2\pi/3)$  using a *negative angle*. In short, we make use of the cyclical nature of trigonometric functions to work in the domain  $-\pi \leq \vartheta \leq +\pi$  rather than  $0 \leq \vartheta \leq 2\pi$ . The results are the same, but the domain  $-\pi \leq \vartheta \leq +\pi$  displays certain symmetries of interest that  $0 \leq \vartheta \leq 2\pi$  does not. Thus, we use (12.3) represented by  $\tau_3(4\pi/3) = \tau_3(-2\pi/3)$  to write:

$$\xi = \begin{pmatrix} \uparrow \\ \downarrow \end{pmatrix} \rightarrow \xi'' = \begin{pmatrix} \uparrow'' \\ \downarrow'' \end{pmatrix} = \tau_3\left(-\frac{1}{3}2\pi\right)\xi = \frac{1}{2}\begin{pmatrix} -1+i\sqrt{3} & 0 \\ 0 & -1-i\sqrt{3} \end{pmatrix}\begin{pmatrix} \uparrow \\ \downarrow \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}(1-i\sqrt{3})\uparrow \\ -\frac{1}{2}(1+i\sqrt{3})\downarrow \end{pmatrix} = \begin{pmatrix} (1)^{+1/3}\uparrow \\ (1)^{-1/3}\downarrow \end{pmatrix}. \quad (12.5)$$

And of course if we use  $\tau_3(0) = I_{(2)}$  now that our domain is  $-\pi \leq \vartheta \leq +\pi$ , then for the trivial root at  $\vartheta = 0$  we have merely  $\xi \rightarrow \xi''' = \xi$ .

Reviewing the primitive root transformations (12.4) and (12.5) together with the trivial  $\xi \rightarrow \xi''' = \xi$ , we see that there are a total of six (6) distinct states for  $m=3$ . Over the domain  $-\pi \leq \vartheta \leq +\pi$ , first, for what we write as  $\vartheta = 0 \cdot 2\pi/3$  there are the untransformed  $\uparrow''' = (1)^0 \uparrow$  and  $\downarrow''' = (1)^0 \downarrow$ . Second, also referring to (6.1), for what we write as  $\vartheta = +1 \cdot 2\pi/3$  there are the transformed  $\uparrow' = -\frac{1}{2}(1+i\sqrt{3})\uparrow = (1)^{-1/3} \uparrow$  and  $\downarrow' = -\frac{1}{2}(1-i\sqrt{3})\downarrow = (1)^{+1/3} \downarrow$ . Finally, for what we write as  $\vartheta = -1 \cdot 2\pi/3$  there are the transformed  $\uparrow'' = -\frac{1}{2}(1-i\sqrt{3})\uparrow = (1)^{+1/3} \uparrow$  and  $\downarrow'' = -\frac{1}{2}(1+i\sqrt{3})\downarrow = (1)^{-1/3} \downarrow$ . So, if we now define a first quantum number  $-1 \leq l_z \leq +1$  so we may summarize these three roots of unity as  $(1)^{l_z/3}$ , and if we define a second quantum number  $s_z = \pm \frac{1}{2}$  which corresponds to  $\uparrow$  and  $\downarrow$ , and if we define a third quantum number  $j_z \equiv l_z + s_z$ , then we may summarize these six states as follows:

$$\begin{aligned}
 (1)^{+1/3} \uparrow &\equiv |l_z = +1, s_z = +\frac{1}{2}, j_z = \frac{3}{2}, \vartheta = -1 \cdot 2\pi/3\rangle \\
 (1)^{+1/3} \downarrow &\equiv |l_z = +1, s_z = -\frac{1}{2}, j_z = \frac{1}{2}, \vartheta = +1 \cdot 2\pi/3\rangle \\
 (1)^0 \uparrow &\equiv |l_z = 0, s_z = +\frac{1}{2}, j_z = +\frac{1}{2}, \vartheta = 0 \cdot 2\pi/3\rangle \\
 (1)^0 \downarrow &\equiv |l_z = 0, s_z = -\frac{1}{2}, j_z = -\frac{1}{2}, \vartheta = 0 \cdot 2\pi/3\rangle \\
 (1)^{-1/3} \uparrow &\equiv |l_z = -1, s_z = +\frac{1}{2}, j_z = -\frac{1}{2}, \vartheta = +1 \cdot 2\pi/3\rangle \\
 (1)^{-1/3} \downarrow &\equiv |l_z = -1, s_z = -\frac{1}{2}, j_z = -\frac{3}{2}, \vartheta = -1 \cdot 2\pi/3\rangle
 \end{aligned} \tag{12.6}$$

This is of high interest, because the root of unity generators effectively take each of the two  $\uparrow$  and  $\downarrow$  states and multiply them with a cubed root of unity that has a magnitude of 1 but variable direction, triplicating  $\uparrow$  and  $\downarrow$  into a total of six states which all together may be characterized over the  $-\pi \leq \vartheta \leq +\pi$  domain by *azimuthal quantum numbers* that correspond precisely to six permitted exclusionary electron states in the  $p$  shells of atoms which are likewise characterized by *azimuthal quantum numbers*. And in particular, the degree of freedom provided by the multi-valuedness of the root of unity generators – now characterized by  $l_z = -1, 0, +1$  and obtained by a rotation in the complex plane based on a  $z$ -axis generator – maps precisely to the degrees of freedom provided by the  $z$ -axis component of *orbital angular momentum*, often denoted by the same symbol  $l_z = m$  (a different “ $m$ ” from the one we have been using as the fractional root of unity denominator.).

Additionally, if we do treat these  $l_z, s_z, j_z$  quantum numbers as the third component of operators  $\mathbf{L} = l_i, \mathbf{S} = s_i, \mathbf{J} = j_i$  with Casimir numbers  $l, s, j$  defined respectively in the usual way

by  $\mathbf{L}^2|\xi\rangle = l(l+1)|\xi\rangle$ ,  $\mathbf{S}^2|\xi\rangle = s(s+1)|\xi\rangle$  and  $\mathbf{J}^2|\xi\rangle = j(j+1)|\xi\rangle$  thus  $j = l + s$ , then all of the  $m=3$  states in (12.6) have the common set of Casimir numbers  $l=1$ ,  $s=\frac{1}{2}$  and  $j=\frac{3}{2}$ . So not only is there a one-to-one, *azimuth-to-azimuth* mapping between the six states of (12.6) and the six states of a  $p$ -shell electron all of which represent operations through an azimuth  $\varphi$  about the  $z$  axis, but the mapping is structurally isomorphic because these each arise from *two* degrees of freedom, one being the three *cubed roots of unity*, the other the two *intrinsic spin* states. To summarize this mapping of these  $6 = 2 \cdot m = 2 \cdot 3$  states, we may write:

$$6: \left\{ (1)^{-1/3}, (1)^0, (1)^{+1/3} \right\} \otimes \left\{ \uparrow, \downarrow \right\} \Leftrightarrow \left\{ l_z \right\} \otimes \left\{ s_z \right\} \Leftrightarrow p: \left| l=1, s=\frac{1}{2}, j=\frac{3}{2} \right\rangle, \quad (12.7)$$

using the set of cubed roots of unity  $\left\{ (1)^{+1/3}, (1)^0, (1)^{-1/3} \right\}$  composed with the set of up and down spins  $\left\{ \uparrow, \downarrow \right\}$ . And, of course, because  $m=3$ , via (11.7), these are also the states for which the fractional Dirac charges are given by  $2eg = n/3$ , that is, these are the  $1/3$ -unit Dirac monopole charge states. This would suggest, if these mappings are meaningful and there is a connection between fractional Dirac monopoles and the observed FQHE charges, that one should be able to observe six (6) distinct spin states for the  $1/3$  charge fractions, namely those of (12.6), and that the electrons (presently thought of as quasi-particles) connected with these observed  $1/3$  fractional charge states should correlate to  $p: \left| l=1, s=\frac{1}{2}, j=\frac{3}{2} \right\rangle$  state electrons.

This raises two questions: First, *does this pattern generalize* to other atomic shell structures? Second, if it does generalize, does it make physical sense to entertain the possibility that *the root of unity degree of freedom is simply the orbital angular momentum degree of freedom in a different guise*? We take these two questions in succession.

As to generalization, saving the (here, unnecessary) difficulty of solving the polynomial  $\sum_{i=0}^{m-1} x^i = 0$  in (6.14), we first use (6.1) to write  $\tau_3$  from (6.12) directly terms of unity roots as:

$$\tau_3(\vartheta) = \begin{pmatrix} \exp(i2\vartheta) & 0 \\ 0 & \exp(i\vartheta) \end{pmatrix} = \begin{pmatrix} \exp(i2\pi(2n/m)) & 0 \\ 0 & \exp(i2\pi n/m) \end{pmatrix} = \begin{pmatrix} (1)^{2n/m} & 0 \\ 0 & (1)^{n/m} \end{pmatrix}. \quad (12.8)$$

Therefore, the general transformation  $\xi \rightarrow \xi' = \tau_3 \xi$  is now:

$$\xi = \begin{pmatrix} \uparrow \\ \downarrow \end{pmatrix} \rightarrow \xi' = \begin{pmatrix} \uparrow' \\ \downarrow' \end{pmatrix} = \tau_3 \xi = \begin{pmatrix} (1)^{2n/m} & 0 \\ 0 & (1)^{n/m} \end{pmatrix} \begin{pmatrix} \uparrow \\ \downarrow \end{pmatrix} = \begin{pmatrix} (1)^{2n/m} \uparrow \\ (1)^{n/m} \downarrow \end{pmatrix}. \quad (12.9)$$

So now let's sample  $m=5$ , the fifth root of unity, which is the next higher fraction permitted by (11.7). The five distinct states have  $n=1, 2, 3, 4, 5$ , although  $n=5$  is the trivial  $\xi \rightarrow \xi' = \xi$  while the other four states are primitive roots. Working with the domain  $-\pi \leq \vartheta \leq +\pi$  for which the

Euler angles are  $\varphi = 0^\circ, \pm 72^\circ, \pm 144^\circ$  and given the recycling of roots of unity, we may also represent these recycling states as  $n = -2, -1, 0, +1, +2$  which as we see sets up an  $l=2$  Casimir number for the orbital quantum number. So setting  $m=5$  in (12.9) and inserting each of  $n = -2, -1, 0, +1, +2$  and using recycling, and also indicating the Euler angle  $\vartheta = 2\pi n/5$ , we obtain the five transformations:

$$\begin{aligned}
 \vartheta = -2 \cdot (2\pi/5) : \begin{pmatrix} \uparrow' \\ \downarrow' \end{pmatrix} &= \begin{pmatrix} (1)^{-4/5} \uparrow \\ (1)^{-2/5} \downarrow \end{pmatrix} = \begin{pmatrix} (1)^{+1/5} \uparrow \\ (1)^{-2/5} \downarrow \end{pmatrix} \\
 \vartheta = -1 \cdot (2\pi/5) : \begin{pmatrix} \uparrow' \\ \downarrow' \end{pmatrix} &= \begin{pmatrix} (1)^{-2/5} \uparrow \\ (1)^{-1/5} \downarrow \end{pmatrix} \\
 \vartheta = 0 \cdot (2\pi/5) : \begin{pmatrix} \uparrow' \\ \downarrow' \end{pmatrix} &= \begin{pmatrix} (1)^0 \uparrow \\ (1)^0 \downarrow \end{pmatrix} \\
 \vartheta = +1 \cdot (2\pi/5) : \begin{pmatrix} \uparrow' \\ \downarrow' \end{pmatrix} &= \begin{pmatrix} (1)^{+2/5} \uparrow \\ (1)^{+1/5} \downarrow \end{pmatrix} \\
 \vartheta = +2 \cdot (2\pi/5) : \begin{pmatrix} \uparrow' \\ \downarrow' \end{pmatrix} &= \begin{pmatrix} (1)^{+4/5} \uparrow \\ (1)^{+2/5} \downarrow \end{pmatrix} = \begin{pmatrix} (1)^{-1/5} \uparrow \\ (1)^{+2/5} \downarrow \end{pmatrix}
 \end{aligned} \tag{12.10}$$

We then reassemble these into the form of (12.6) to enumerate the ten (10) distinct states:

$$\begin{aligned}
 (1)^{+2/5} \uparrow &\equiv |l_z = +2, s_z = +\frac{1}{2}, j_z = +\frac{5}{2}, \vartheta = +1 \cdot (2\pi/5)\rangle \\
 (1)^{+2/5} \downarrow &\equiv |l_z = +2, s_z = -\frac{1}{2}, j_z = +\frac{3}{2}, \vartheta = +2 \cdot (2\pi/5)\rangle \\
 (1)^{+1/5} \uparrow &\equiv |l_z = +1, s_z = +\frac{1}{2}, j_z = +\frac{3}{2}, \vartheta = -2 \cdot (2\pi/5)\rangle \\
 (1)^{+1/5} \downarrow &\equiv |l_z = +1, s_z = -\frac{1}{2}, j_z = +\frac{1}{2}, \vartheta = +1 \cdot (2\pi/5)\rangle \\
 (1)^0 \uparrow &\equiv |l_z = 0, s_z = +\frac{1}{2}, j_z = +\frac{1}{2}, \vartheta = 0 \cdot (2\pi/5)\rangle \\
 (1)^0 \downarrow &\equiv |l_z = 0, s_z = -\frac{1}{2}, j_z = -\frac{1}{2}, \vartheta = 0 \cdot (2\pi/5)\rangle \\
 (1)^{-1/5} \uparrow &\equiv |l_z = -1, s_z = +\frac{1}{2}, j_z = -\frac{1}{2}, \vartheta = +2 \cdot (2\pi/5)\rangle \\
 (1)^{-1/5} \downarrow &\equiv |l_z = -1, s_z = -\frac{1}{2}, j_z = -\frac{3}{2}, \vartheta = -1 \cdot (2\pi/5)\rangle \\
 (1)^{-2/5} \uparrow &\equiv |l_z = -2, s_z = +\frac{1}{2}, j_z = -\frac{3}{2}, \vartheta = -1 \cdot (2\pi/5)\rangle \\
 (1)^{-2/5} \downarrow &\equiv |l_z = -2, s_z = -\frac{1}{2}, j_z = -\frac{5}{2}, \vartheta = -2 \cdot (2\pi/5)\rangle
 \end{aligned} \tag{12.11}$$

This provides a different view of the results at (11.1) through (11.6) which led to the inclusion of odd-integer fractions and the exclusion of even-integer fractions. When we write the

transformation  $\xi \rightarrow \xi' = \tau_3 \xi$  as in (12.9), we see very clearly that the eigenvalues for  $\uparrow$  are  $(1)^{2n/m}$  while those for  $\downarrow$  are  $(1)^{n/m}$ . More symbolically,  $\uparrow = \left| \tau_3 = (1)^{2n/m} \right\rangle$  and  $\downarrow = \left| \tau_3 = (1)^{n/m} \right\rangle$ , where the factor of 2 is an essential aspect. This is why in (12.11) the  $\downarrow$  states cycle the coefficient of  $2\pi/5$  in  $\vartheta$  in harmony with  $l_z$ , but the  $\uparrow$  states cycle at twice the rate, see also (12.11). Nonetheless, *for odd m only*, over the full domain  $-\pi \leq \vartheta \leq +\pi$ , we end up with all the roots of unity being distributed to both spin up and spin down in the manner of (12.7), which for (12.11) has  $10 = 2 \cdot m = 2 \cdot 5$  states we may write as:

$$10: \left\{ (1)^{-2/5}, (1)^{-1/5}, (1)^0, (1)^{+1/5}, (1)^{+2/5} \right\} \otimes \left\{ \uparrow, \downarrow \right\} \Leftrightarrow \left\{ l_z \right\} \otimes \left\{ s_z \right\} \Leftrightarrow p: \left| l = 2, s = \frac{1}{2}, j = \frac{5}{2} \right\rangle. \quad (12.12)$$

So this does indeed have the ten (10) azimuthal states in which  $d$ -shell electrons may be found. And it can readily be shown that this pattern this will hold for  $f, g, h \dots$  and other electrons permitting 14, 18, 22... states with  $j = \frac{7}{2}, \frac{9}{2}, \frac{11}{2} \dots$ . So as to whether this pattern generalizes, the answer is yes. For  $m=1$  which yields via (11.7) the standard DQC  $2eg = n$  of (4.7), we have  $(1)^0 \uparrow \equiv \left| l_z = 0, s_z = +\frac{1}{2}, j_z = +\frac{1}{2}, \vartheta = 0 \cdot (2\pi) \right\rangle$  and  $(1)^0 \downarrow \equiv \left| l_z = 0, s_z = -\frac{1}{2}, j_z = -\frac{1}{2}, \vartheta = 0 \cdot (2\pi) \right\rangle$ , which are simply the two (2) states permitted for  $s$ -shell electrons.

In contrast, were  $m=2,4,6,8 \dots$  to be an *even* integer, we would lose some states, which is another view of why even integers are excluded but odd integers are permitted. Take  $m=4$ , for example, and let's go back to using the domain  $0 < \varphi \leq 2\pi$  thus  $n=1,2,3,4$ . Because  $\downarrow = \left| \tau_3 = (1)^{n/4} \right\rangle$ , the pattern of the root numerators will be 1,2,3,4. But because  $\uparrow = \left| \tau_3 = (1)^{2n/4} \right\rangle$ , the root numerator pattern here will be 2, 4,  $6 \rightarrow 2$ ,  $8 \rightarrow 4$  when we account for recycling. We would thus duplicate the even numerators, and exclude the odd numerators, but only for spin up and not for spin down. *This is the intrinsic pattern for any and all even m*. But let's look at odd  $m$ , now using  $m=7$  as an example. Here, the numerators for  $\downarrow = \left| \tau_3 = (1)^{n/7} \right\rangle$  in the  $0 < \varphi \leq 2\pi$  domain are 1,2,3,4,5,6,7. And over the same domain the numerators in  $\uparrow = \left| \tau_3 = (1)^{2n/7} \right\rangle$ , including recycling, are 2,4,6,8  $\rightarrow$  1,10  $\rightarrow$  3,12  $\rightarrow$  5,14  $\rightarrow$  7. So all the seventh roots are included because all of the numerators 1,2,3,4,5,6,7 do occur but in the 2,4,6,1,3,5,7 sequence, and there are no excluded states. *This is the intrinsic pattern for any and all odd m*. Again, this is another vantage point on (11.1) to (11.6) which caused us to discard the even-integer  $m$  as unphysical on helicity grounds. So referring to (11.4) and the  $m=4$  example just discussed, the odd numerators are skipped for  $\uparrow$  but not for  $\downarrow$ , which means that the charge fractions for the former are  $2eg = n/2$  and for the latter are  $2eg = n/4$ , which means that if we Lorentz transform so as to flip the helicity we can alter the charge, which is an unphysical result, which is why even-integer charge fractions must be physically excluded (except  $m=2$  based on the tidal lock and not the roots of unity).

These patterns may also be concisely characterized using modular arithmetic. The least residue for  $n \bmod m$  is the set  $0,1,2,3 \dots m-1$ , which we shall denote as  $n \bmod m$  where we

define the symbol  $\underline{\text{mod}}$  with an underbar to mean “*least* residue modulo.” So using this notation,  $n \underline{\text{mod}} \text{ even} \neq 2n \underline{\text{mod}} \text{ even}$  but  $n \underline{\text{mod}} \text{ odd} \neq 2n \underline{\text{mod}} \text{ odd}$  for *even* and *odd* integers. With the same examples used above, we have  $n \underline{\text{mod}} 4 \equiv \{0, 1, 2, 3\} \neq 2n \underline{\text{mod}} 4 = \{0, 2\}$  for *even* = 4, and  $n \underline{\text{mod}} 7 \equiv \{0, 1, 2, 3, 4, 5, 6\} = 2n \underline{\text{mod}} 7 = \{0, 2, 4, 6, 1, 3, 5\}$  for *odd* = 7. The former is missing odd elements from the  $2n \underline{\text{mod}} 4$  set, while the later  $2n \underline{\text{mod}} 7$  contains all the same elements as  $n \underline{\text{mod}} 7$ , but simply generated in a different order. And this pattern applies to *all* integers both odd and even.

Then, to “balance” this least residue symmetrically about zero *which can only be done for odd m*, we need to subtract  $(m-1)/2$  from  $n \underline{\text{mod}} m$ , thus forming  $n \underline{\text{mod}} m - (m-1)/2$ . To simplify we further introduce the notation “ $\underline{\text{mod}}_0$ ” to denote this *least* residue modulo when symmetrized about zero, such that  $n \underline{\text{mod}}_0 m \equiv n \underline{\text{mod}} m - (m-1)/2$ . This “symmetric least residue modulo” exists only for odd integers, which is another reflection of the odd-integer restriction first uncovered at (11.1) through (11.6). But these are the same integers that we have heretofore denoted as  $l_z$ , which is to say that:

$$l_z = n \underline{\text{mod}} m - (m-1)/2 = n \underline{\text{mod}}_0 m. \quad (12.13)$$

And this in turn means that:

$$-(m-1)/2 \leq l_z \leq +(m-1)/2. \quad (12.14)$$

At the same time, the outer bounds on  $l_z$  are given by  $\pm l$ , so the above may be connected to:

$$-l = -(m-1)/2 \leq l_z \leq +(m-1)/2 = +l. \quad (12.15)$$

And from this, given also that  $s = \frac{1}{2}$  and that  $j = l + s$ , we find that:

$$\frac{m}{2} = l + \frac{1}{2} = l + s = j. \quad (12.16)$$

With this final observation that  $2j = 2(l + s) = m$  for all the odd fractions, we return to (11.7) and now write the Fractional Dirac Quantization Condition (FDQC) as:

$$2eg = \frac{n}{m} = \frac{n}{2j} = \frac{n}{2(l+s)}; \quad n = 1, 2, 3, 4, 5, \dots; \quad m = 2j = 2(l+s) = 2(l+\frac{1}{2}) = 1, 2, 3, 5, 7, 9, \dots \quad (12.17)$$

Note that although the even denominator 2 originates from a tidal lock while the odd denominators originate from roots of unity, we have kept all the denominators together to be able



to later examine these different origins from the viewpoint of spins and orbital angular momenta. We may also return to (6.2) and write the Euler angle which drives the unity roots as:

$$\vartheta = 2\pi \frac{n}{m} = \pi \frac{n}{j} = \pi \frac{n}{l+s} = 2\pi\mathbb{Q} \quad (12.18)$$

where the rational number  $\mathbb{Q} = n/2j = n/2(l+s)$ . Then, we find that (12.7) and (12.12), when fully generalized, become

$$2 \cdot m : \left\{ (1)^{\wedge \frac{n \bmod_0 m}{m}} \right\} \otimes \{ \uparrow, \downarrow \} \Leftrightarrow l_z \otimes s_z \Leftrightarrow \left| l = +\frac{m-1}{2}, s = +\frac{1}{2}, j = +\frac{m}{2} \right\rangle. \quad (12.19)$$

So, for example, for  $m=3$  the unity root exponent  $n \bmod_0 3 / 3 = -\frac{1}{3}, 0, +\frac{1}{3}$  as seen in (12.7) while for  $m=5$  the unity root exponent  $n \bmod_0 5 / 5 = -\frac{2}{5}, -\frac{1}{5}, 0, +\frac{1}{5}, +\frac{2}{5}$  as in (12.12). Further, while we have developed the above in the domain  $-\pi \leq \vartheta \leq +\pi$ , the fact that we may only have odd root-of-unity denominators means that we may never have  $\vartheta = \pm\pi$ , because this Euler angle exists for *any and all* even-denominator roots of unity, but for *none* of the odd-denominator roots. *Every* root of unity includes  $\vartheta=0$ , but *only the even roots* include  $\vartheta = \pm\pi$ . Therefore, the physically-permitted domain is really  $-\pi < \vartheta < +\pi$ , where we have removed the equality  $\leq \rightarrow <$  from the domain definition to likewise represent the unphysical character of the even-root denominators.

Consequently, now we have a possible answer to the question raised at the outset, “of what it really means – physically, not mathematically – to subject a spinor  $\xi$  – and by implication a wavefunction  $\psi$  – to a root of unity transformation  $\xi \rightarrow \xi' = \tau_3 \xi$  about the z axis.” When we start with an electron for which  $m=1$ , which designates the first root of unity which is equal to 1 alone, then if these connections are physically valid (and that is the subject of the next section), (12.16) tells us that  $l=0$ . Now, if we subject that starter electron to a “root of unity transformation” (12.9) using a particular odd integer  $m = 3, 5, 7, 9, \dots$ , (12.15) and (12.16) then tell us that this electron has a total angular momentum Casimir number  $l = (m-1)/2 = 1, 2, 3, 4, \dots$  respectively. So each time we increase  $m$  by 2 (and because  $m$  must be odd we must always use increments of 2), we will increase  $l$  by 1. So what is the physical interpretation of transforming a spinor / electron using the root of unity generators (6.12)?

A possible answer is that *applying a root of unity transformation is synonymous with transforming that electron into a different state of orbital angular momentum*. So when we add an electron to an atomic shell, and we need to satisfy the Exclusion Principle, and when in order to do so we need to add some orbital angular momentum, the mathematical operation we use is a root of unity transformation. When we remove an electron from an atom and change its orbital, the mathematical operation we perform is root of unity transformation. This possible connection between roots of unity and the orbital quantum number certainly appears to work based on its isomorphic mapping to degrees of freedom and the available quantum states. The question to

which we now turn is how all of this might be understood and assimilated in relation to pure theoretical and empirical physics.

### 13. How Root of Unity Transformations Generate Orbital Angular Momentum Changes

If the fractional Dirac charges  $2eg = n/m$  found in (11.17) to be restricted to  $m = 1, 2, 3, 5, 7, 9, \dots$  by Lorentz symmetry and by a tidal lock analysis are in reality being empirically observed in the FQHE even though this has not yet been understood or recognized, then because the FQHE is a phenomenon only observed at ultra-low temperatures, this empirical knowledge would require us to regard (11.7) not as a general electrodynamic phenomenon, but rather as a phenomenon of *low-temperature electrodynamics*. This would mean that the appearance of a symmetry between electric and magnetic charges under  $e \leftrightarrow g$  interchange in  $2eg = n/m$  is a low-temperature symmetry of electrodynamics which apparently is not observed, and is likely broken in some fashion, at “ordinary” temperatures sufficiently removed from 0K. Consequently, the question would arise as to how (11.7) migrates from ultra-low temperatures, through ordinary temperatures which are neither ultra-low nor ultra-high, up to ultra-high temperatures GUT associated with what many regard as the conditions in the very early universe. Our present understanding of high temperature monopole physics is laid out in the original works by ‘t Hooft and Polyakov [10], [11] as well as by Weinberg’s clear summarization at 442-443) of [12]. Answering this question as to the migration from 0K all the way up to GUT temperatures would require a direct consideration of the relationship between electrodynamics and thermodynamics, possibly requiring their unification, which is a question we simply point out, but shall not attempt to resolve, in the present paper.

The FQHE is observed in conductive materials which are cooled to ultra-low temperatures and then subjected to large perpendicular magnetic fields. These conductive host materials which exhibit the FQHE are composed of atoms which contain electrons and protons and neutrons, and the fact that we cool these host materials down to ultra-low temperatures does not alter the fact these materials contain electrons and protons and neutrons. In particular, the electrons in these materials are subject to the Exclusion Principle because they are fermions, and this does not change by virtue of cooling the host materials down to low temperatures. In fact, one of the striking features of low-temperature physics which has been amply confirmed is that although low temperatures might *a priori* be thought to remove all energies from a system, the need to maintain Exclusion even at low temperatures means that electrons will maintain certain energies simply because they need to be in elevated energy states to satisfy exclusion.

Thus, an electron state  $|n, l_z, s_z\rangle$  which has a particular set of principal, orbital and spin quantum numbers where are greater than the ground state  $s: |n=1, l_z=0, s_z=\pm\frac{1}{2}\rangle$  of an *s*-shell electron will maintain its elevated quantum numbers even near 0K. How do we know this? We know this by the very fact that the host material still retains its material identity even when it is cooled to near 0K. If cooling to near 0K allowed all the electrons in the material to drop down to the lowest energy  $s: |n=1, l_z=0, s_z=\pm\frac{1}{2}\rangle$  states, then the host material would disintegrate into Hydrogen or Helium, because these are the only elements for which all electrons can be

maintained in an  $s$  shell. For any host material from Lithium all the way up the Periodic Table, there must be some electrons in  $p, d, f, g...$  shells and these necessarily have elevated energies forced by exclusion even near 0K. Because these FQHE host materials are not observed to turn into Hydrogen or Helium near 0K, we know that they contain some  $p, d, f, g...$  electrons, in addition to the  $s$  electrons which they house in their very inner shells. Consequently, we may begin to study what happens to an electron in some  $|n, l_z, s_z\rangle$  state, and particularly, to how outer shell “itinerant” electron behaviors might be observed, when the host material containing that electron is cooled to near 0K and a very strong perpendicular magnetic field is applied.

The electric charge strength  $e$  in  $eg = \frac{1}{2}n$  of the standard DQC (4.7) and by extension of the fractional  $eg = \frac{1}{2}n/m$  of (11.7) is the same one which is related to the running fine structure coupling via  $\alpha = e^2 / 4\pi\hbar c$ , which, at low probe energies / large impact distance, approaches the numeric value  $\alpha = e^2 / 4\pi\hbar c \cong 1/137.036...$  asymptotically, see, e.g., equation [1] in Dirac’s [1] and Witten’s [13], pages 27 and 28. Indeed, *Dirac’s original purpose* for the derivation in [1] was to “give a theoretical value for  $e$ ” and thus the number  $\sim 137$ . However, the DQC left this number “from the theoretical standpoint, completely undetermined,” and to date, despite many efforts to explain this number, this still is an experimentally-derived number with no commonly-accepted theoretical explanation. Dirac perceived it “rather disappointing to find this reciprocity between electricity and magnetism, instead of a purely electronic quantum condition, such as [Dirac’s equation number [1]].” This means that the charge strength  $e$  in (4.7) and (11.7) is the charge strength for an electron. So – notwithstanding the presently-prevailing explanation using quasi-particles and collective excitations – if (11.7) were to be the underlying cause of the FQHE, this would suggest that at low temperatures near 0K, under the influence of a large applied perpendicular magnetic field, electrons in the conductive host material exhibit an electric / magnetic duality including a magnetic monopole charge, and also exhibit charge quantization and fractionalization. This would also suggest that when the temperature is raised above a certain critical temperature related to the host material, this duality symmetry becomes broken, the magnetic monopoles become hidden or transmuted into some other form (perhaps related to the rise in temperature), and the electric charge loses its fractional character and simply becomes quantized in accordance with what is observed at temperatures sufficiently above 0K.

If, however, the fractional denominator  $m$  were to be further related to the total angular momentum Casimir number according to  $j = l + s = \frac{1}{2}m$  as is suggested by (12.16), then because these electronic quantum numbers do not go away near 0K lest the host material disintegrate into Hydrogen or Helium which it does not, the charge fraction being exhibited would be a direct manifestation the total angular momentum, so that each charge fraction should then correlate to a set of definitive orbital and spin states which should be experimentally detectable under the right circumstances. For example, an itinerant  $m=3$  charge of  $n/3$  should exhibit azimuthal momentum characteristics of the  $j = \frac{3}{2}$ ,  $p$ -shell electrons. And an itinerant  $m=5$  or  $m=7$  charge of  $n/5$  or  $n/7$  should exhibit azimuthal momentum characteristics of the  $j = \frac{5}{2}$  or  $j = \frac{7}{2}$ ,  $d$ -shell or  $f$ -shell electrons, and so on. This in turn should provide a range of opportunities for experimental validation via spin state / fractional charge state correlations which will be proposed in detail in the section 15. But at the moment, the question to be addressed is whether on theoretical and physical grounds, it makes sense for these fractional denominators  $m$ , which

originate from the  $m^{\text{th}}$  roots of unity in (6.1), to be so directly connected with the total angular momentum as to be one and the same thing via  $m = 2j = 2(l + s)$  as found in (12.16).

It should be clear that the identification of the quantum number  $s = \frac{1}{2}$  in  $m = 2(l + s)$  with the spin Casimir in  $s(s+1)|\xi\rangle = s_i^2|\xi\rangle$  is fully, deductively supported, because this originated in the  $\uparrow$  and  $\downarrow$  spin states being transformed in (12.9), and these states are the eigenstates of  $s_z = \pm\frac{1}{2}$ , that is,  $s_z|\uparrow\rangle = +\frac{1}{2}|\uparrow\rangle$  and  $s_z|\downarrow\rangle = -\frac{1}{2}|\downarrow\rangle$ . So the answer to this question can be boiled down to the question whether the identification  $l_z = n \bmod_0 m$  in (12.13) can be sustained, because if it can, then this sets the lower and upper bounds on  $l_z$  in (12.15), and this then establishes  $m = 2(l + s)$  in (12.16). So taking  $m=3$  for the cubed roots of unity as an example, does it make sense to associate  $n \bmod_0 3 = \{-1, 0, +1\}$  with  $l_z$  for  $l=1$ ? And for the fifth and seventh roots of unity, does it make sense to associate  $n \bmod_0 5 = \{-2, -1, 0, +1, +2\}$  with  $l_z$  for  $l=2$  and to associate  $n \bmod_0 7 = \{-3, -2, -1, 0, +1, +2, +3\}$ , and so on? Specifically, is there something about a root of unity which is in some way suggestive of – or better yet, which leads directly to – making the general association  $l_z = n \bmod_0 m$  between odd roots of unity  $m$  which are mathematical, and the azimuthal quantum number  $l_z$  which is physical?

At the simplest level, there is certainly a numeric correspondence. The total angular momentum of an electron, times 2, is always an odd number  $2j = 1, 3, 5, 7, 9, \dots$ , while the helicity considerations of Lorentz symmetry reviewed in section 11 forced the observable roots of unity to be the same odd number  $m = 1, 3, 5, 7, 9, \dots$ . And on top of this, setting aside the denominator 2, the fill factor for the FQHE is also an odd integer  $\nu = 1, 3, 5, 7, 9, \dots$ . So at least numerically, we can set  $m = 2j = \nu$  and have these results all fit tightly together. But sometimes an odd integer is just an odd integer, and one should not read anything more into it. So let's dig further.

At the next level, the root of unity denominator  $m = 1, 3, 5, 7, 9, \dots$  is more than just a number. Each root of unity provides a degree of freedom with exactly  $m$  multi-values, so when these  $m$  values are twice replicated for  $\uparrow$  and  $\downarrow$  as is shown in section 12, the number of distinct exclusionary states for any given  $m$  will be  $2m = 2, 6, 10, 14, 18, \dots$ , and this maps perfectly to the number of electrons which can fit into an  $s, p, d, f, g, \dots$  shell, which is fundamentally driven by the available  $-l \leq l_z \leq +l$  times two spin states, i.e., by  $l_z \otimes s_z$ , see (12.7), (12.12) and (12.19). And from (12.13) we have the direct correspondence  $l_z = n \bmod_0 m$  between the permitted  $l_z$  and the symmetric least residue  $n \bmod_0 m$ . So now an odd integer becomes somewhat more than an odd integer. Now we have two degrees of freedom, each degree of freedom permits the same number of exclusive states, the composition of these two degrees of freedom permits the same number of states, and all of the quantum numbers which provide this freedom are identical.

At the next level, let us talk about observables. All of the development through section 10 of this paper fundamentally established that fractional Dirac monopoles could exist without

observable singularities, which means that these fractional monopoles themselves with charge fractions  $m = 1, 2, 3, 4, 5, \dots$  could be observable. Then in section 11 we showed how Lorentz symmetry and specifically helicity considerations physically excluded all of the even-numbered fractions, and thus placed tighter restrictions on what is observable. So now, aside from the charge fraction 2 which arises from the tidal lock discussed in section 5, the observable fractions are the odd numbers  $m = 1, 3, 5, 7, 9, \dots$ . But if these odd fractions  $m$  are directly observable, and if they are also directly related to some Casimir number of angular momentum, then the angular momentum Casimir number to which they are directly related *must itself also be a direct observable*. So, if we are to regard  $m = 2j$  found in (12.16) to be a correct relationship, then it is necessary that  $j$  be a direct observable. Of course,  $j$  is a direct observable, because the total angular momentum  $\mathbf{J} = J_i$  commutes with the Dirac Hamiltonian,  $[H, \mathbf{J}] = 0$ . In fact,  $j$  is the *only* Casimir number of angular momentum to which the fraction  $m$  could be related if it is to be a direct observable, because neither  $\mathbf{L} = L_i$  nor  $\mathbf{S} = S_i$  is separately observable. The orbital  $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ , when commuted with the Dirac Hamiltonian, yields  $[H, \mathbf{L}] = -i(\boldsymbol{\alpha} \times \mathbf{P})$ , while the spin / helicity operator  $\mathbf{S} \equiv \frac{1}{2}\boldsymbol{\Sigma}$  for which  $\text{diag}(\boldsymbol{\Sigma}) = (\boldsymbol{\sigma}, \boldsymbol{\sigma})$  commutes as  $[H, \mathbf{S}] = +2i(\boldsymbol{\alpha} \times \mathbf{P})$ . Only when we form the total angular momentum  $\mathbf{J} = \mathbf{L} + \frac{1}{2}\boldsymbol{\Sigma} = \mathbf{L} + \mathbf{S}$ , do we obtain  $[H, \mathbf{J}] = [H, \mathbf{L} + \mathbf{S}] = 0$  and thus find that the Casimir  $j$  in  $\mathbf{J}^2|\psi\rangle = j(j+1)|\psi\rangle$  is the direct observable and that  $m = 2j = 2(l+s)$  found in (12.16) does indeed directly relate an observable to and observable. Had we found a relationship such as  $m = l$  alone (i.e., had we found in section 11 all charge fractions were permitted), then we would have found an observable  $m$  directly related to a non-observable  $l$ , which would have to be forbidden as unphysical. So from this vantage point, it was the finding in section 11 that only odd-integer charges may be observed (aside from the tidal locked  $m=2$ ) which lead to a physically admissible alignment of  $m$  with  $j$  which is the only observable angular momentum Casimir number.

At the final and perhaps deepest level, we turn to the heart of quantum theory itself. Since the earliest days of quantum theory, scientists have sought to explain the existence of quantization by imposing boundary conditions upon waves. For example, the so-called “particle in a box” problems envision that two ends of a vibrating string are affixed to two walls on the sides of a box, which of course then restricts the string to vibrating with a quantized number of nodes between its two ends. Once de Broglie established wave particle duality, it became possible to improve and extend the original Bohr model of the atom and its treatment of angular momentum by regarding the electron – in wavelike incarnation – to be a closed circular string vibrating in an orbit with a circumference which is an integral multiple of the wavelength  $\lambda$  of vibration, where the electron velocity  $v = \lambda f$  and  $f = 1/t$  is the frequency per time of this vibration measured, for example, in cycles per second. Indeed, from this view, the requirement that (4.4) be single-valued,  $\psi_+(0) \rightarrow \psi_+(2\pi) = \psi_+(0)$ , which led to the standard DQC at (4.8), the half-integer condition at (5.13), (5.14), and the fractional condition at (11.7), is simply an extension of the basic de Broglie approach of “fitting” both ends of a closed loop over  $0 \leq \varphi \leq 2\pi$  with continuity, so that  $\varphi = 0$  and  $\varphi = 2\pi$  effectively become the “boundaries” for

imposing a boundary condition that there be continuity, not only for the function  $f(\varphi)$  which specifies the periodic vibration, but also for its derivatives  $\partial f / \partial \varphi$  and  $\partial^2 f / \partial \varphi^2$ .

What does this have to do with roots of unity and their possible relation  $m = 2j$  to the total angular momentum? The root of unity relationship (6.1) is the quintessential canonical *pure mathematics* foundation for specifying continuity as the boundary condition at the extrema of a  $0 \leq \vartheta \leq 2\pi$  closed loop domain. If we think of each root of unity as representing one periodic cycle, then for any  $m^{\text{th}}$  root of unity, the trivial root 1 will always be spotted at  $\varphi = 0$  and will make its first recycled reappearance at  $\varphi = 2\pi$ . In between, there will be  $m-1$  roots all spotted at equally-spaced orientations along the unit circle over the  $0 \leq \vartheta \leq 2\pi$  domain, at the angles  $\vartheta = 2\pi n / m = 2\pi \mathbb{Q}$  of (6.2). If we then imagine that one might “vibrate” this unit circle so that the first node for each cycle is spotted at one of the unity roots, then what we have effectively done is fitted a closed de Broglie wave into the complex plane and relied upon the pure mathematics of roots of unity to do so by using these roots to spot the start of each cycle in the vibration. In short, roots of unity split up the unit circle with continuity over  $0 \leq \vartheta \leq 2\pi$ , just like closed stationary de Broglie waves split up a  $0 \leq \theta_i \leq 2\pi$  domain in physical space with continuity at each end of the domain.

But of course, the roots of unity exit in a complex plane. So to talk about real, physical angular momentum in the real physical space of SO(3) we need to then map the root of unity points out of the complex plane onto SO(3) and determine the behaviors of azimuthal transformations on SO(3). This mapping onto SO(3) was earlier obtained in (8.15) to (8.17), with (8.17) showing the azimuthal transformation. As we can see, the root of unity angle  $\vartheta = 2\pi n / m$  has *the exact same effect as a rotation* through  $\varphi = \vartheta$  about the z axis, supplemented by the Euclidean space and time transformation (9.18) emanating from the factor  $\det \tau_i(\vartheta) = \exp(i3\vartheta)$ , see (6.19), for which the operator matrix is (9.3). So there will *always* be a rotation  $\varphi = \vartheta$  about the z axis, and further, depending upon the selected  $\vartheta = 2\pi n / m$  in the unitary root of unity generator  $\tau_3$ , there *may* also be a dilation or constriction of the space and time axes owing to the transformation (9.18).

To explore this further, let us start with  $m=3$ , because in this special case, as already discussed toward the end of section 9 and also at (12.1),  $\vartheta = 2\pi n / 3$  and so  $\exp(i3\vartheta) = \exp(i2\pi n) = 1$ . Thus, for  $m=3$ , there is no transformation occurring between space and time, and (8.17) becomes:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos(2\pi n / 3) & \sin(2\pi n / 3) & 0 \\ -\sin(2\pi n / 3) & \cos(2\pi n / 3) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos(n \cdot 120^\circ) & \sin(n \cdot 120^\circ) & 0 \\ -\sin(n \cdot 120^\circ) & \cos(n \cdot 120^\circ) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad (13.1)$$

For the primitive roots  $n=1,2$ ,  $\cos(2\pi n / 3) = -1/2$ , and  $\sin(2\pi n / 3) = \pm\sqrt{3}/2$ , and for the trivial  $n=0,3,6,9\dots$  this becomes a 3x3 unit matrix and there is no rotation at all,  $x^k \rightarrow x'^k = x^k$ ,

but we do not need to use these explicit values here. Rather let's consider the set of all possible closed de Broglie waves, which by definition must have some integer number 1, 2, 3, 4, 5, 6, 7... of cycles per  $0 \leq \varphi \leq 2\pi$  to maintain a continuous closed loop. And let's pose the question: what is the *subset* of such closed waves which will remain invariant under the transformation (13.1)? The answer is evident from the above: any closed wave which completes  $m=3$  full oscillations over  $0 \leq \varphi \leq 2\pi$ , i.e., any closed wave with  $m=3$  cycles as its fundamental (first) harmonic will be invariant under (13.1), because a three-cycle closed wave rotated by some multiple of  $120^\circ$  will be indistinguishable from the original closed wave before its rotation.

We mention the harmonics, because a close wave with higher  $3 \cdot k$ -cycle harmonics with integer  $k = 1, 2, 3, 4, \dots$  will also be rotationally invariant under (13.1). Thus, a 6 or 9 or 12... cycle closed wave will also exhibit symmetry under a  $120^\circ$  azimuthal rotation. However, by the analysis of section 11, the even roots of unity  $m = 6, 12, 18, \dots$  are excluded by Lorentz symmetry, and this means that the de Broglie waves with these same numbers of cycles are also excluded. So in reality the permitted higher harmonics are  $3 \cdot k$  with  $k = 1, 3, 5, 7, \dots$  likewise restricted to being an odd integer. Further, while for  $m=3$  the permitted higher harmonics are 9, 15, 21, 27... cycles over the domain  $0 \leq \varphi \leq 2\pi$ , these same harmonics are also the fundamental harmonics of the roots of unity for which  $m = 9, 15, 21, 27, \dots$ . And, of course, each of 9, 15, 21, 27... is a non-prime number, because each it is a multiple of 3. So we may associate the Lorentz-invariant root of unity  $m=3$  with the fundamental harmonic of an 3-cycle closed de Broglie wave fitted to  $0 \leq \varphi \leq 2\pi$ , and the higher, odd-multiple harmonics may be associated with the fundamental harmonic of a higher odd root which, mathematically, is a non-prime number.

To provide a second example for contrast before we generalize, now let look at  $m=5$ . Here  $\vartheta = 2\pi n/5$  and (8.17) becomes:

$$\begin{aligned} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} &= \exp\left(i \frac{6\pi n}{5}\right) \begin{pmatrix} \cos(2\pi n/5) & \sin(2\pi n/5) & 0 \\ -\sin(2\pi n/5) & \cos(2\pi n/5) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ &= \exp\left(i \frac{6\pi n}{5}\right) \begin{pmatrix} \cos(n \cdot 72^\circ) & \sin(n \cdot 72^\circ) & 0 \\ -\sin(n \cdot 72^\circ) & \cos(n \cdot 72^\circ) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \end{aligned} \quad (13.2)$$

The factor  $\exp(i6\pi n/5) = \cos(6\pi n/5) + i \sin(6\pi n/5)$  now does give rise to a Euclidean space and time transformation. The best way to approach this is using (9.18) with explicit sin and cos values. For  $m=5$  and also setting  $\mathbf{x} = x^i$  and  $\mathbf{t} = t_i$ , the space and time transformation (9.18) is:

$$\begin{pmatrix} t^j \\ x^j \end{pmatrix} \rightarrow \begin{pmatrix} t'^j \\ x'^j \end{pmatrix} = \begin{pmatrix} \cos(6\pi n/5) & \sin(6\pi n/5) \\ -\sin(6\pi n/5) & \cos(6\pi n/5) \end{pmatrix} \begin{pmatrix} t^j \\ x^j \end{pmatrix}. \quad (13.3)$$

For the first cycle of angles  $6\pi n/5$  with primitive roots  $n=1,2,3,4$ , all of the sin and cos functions reduce to those of either  $\pm$  the sin and cos of  $\pi/5=36^\circ$  or of  $2\pi/5=72^\circ$ . These are  $\sin(36^\circ)=\frac{1}{4}\sqrt{10-2\sqrt{5}}=.5878$ ,  $\cos(36^\circ)=\frac{1}{4}(1+\sqrt{5})=.8090$ ,  $\sin(72^\circ)=\frac{1}{4}\sqrt{10+2\sqrt{5}}=.9511$  and  $\cos(72^\circ)=\frac{1}{4}(-1+\sqrt{5})=.3090$ , to four digits. These root expressions may be obtained by solving the polynomial  $\sum_{i=0}^{m-1} x^i = 0$  of (6.14) for  $m=5$ , or geometrically manipulating the angles on a regular pentagon, or they may be looked up. And of course the numeric values of sin and cos may be obtained from any table or any calculator. For the trivial root at  $n=5$  we simply have  $\cos(6\pi n)=1$  and  $\sin(6\pi n)=0$  thus  $\chi^k \rightarrow \chi'^k = \chi^k$  as was the case for all the roots in (13.1).

Using these explicit values of sin and cos, for  $n=\{1,4\}$ , which using  $n \bmod_0 5$  is the set  $n=\{\pm 1\}$  which we are seeking to connect to the angular momentum  $l_z = \pm 1$ , and obtaining the correct sign for sin and cos by subtracting off full cycles through  $2\pi$  then rotating the remainder angle into the upper-right  $0 \leq \vartheta \leq 90^\circ$  quadrant, (13.2) becomes, for  $n = \pm 1$  respectively:

$$\begin{aligned} \begin{pmatrix} t^j \\ x^j \end{pmatrix} \rightarrow \begin{pmatrix} t'^j \\ x'^j \end{pmatrix} &= \begin{pmatrix} -\cos(\pi/5) & \mp \sin(\pi/5) \\ \pm \sin(\pi/5) & -\cos(\pi/5) \end{pmatrix} \begin{pmatrix} t^j \\ x^j \end{pmatrix} = \frac{1}{4} \begin{pmatrix} -(1+\sqrt{5}) & \mp \sqrt{10-2\sqrt{5}} \\ \mp \sqrt{10-2\sqrt{5}} & -(1+\sqrt{5}) \end{pmatrix} \begin{pmatrix} t^j \\ x^j \end{pmatrix} \\ &= \begin{pmatrix} -.8090 & \mp .5878 \\ \pm .5878 & -.8090 \end{pmatrix} \begin{pmatrix} t^j \\ x^j \end{pmatrix} = \begin{pmatrix} -.8090t^j \mp .5878x^j \\ -.8090x^j \pm .5878t^j \end{pmatrix} \end{aligned} \quad (13.4)$$

For  $n=\{2,3\}$  which via  $n \bmod_0 5$  is the set  $n=\{\pm 2\}$  that we seek to connect to  $l_z = \pm 2$ , using the same procedure, (13.2) becomes, for  $n = \pm 2$  respectively:

$$\begin{aligned} \begin{pmatrix} t^j \\ x^j \end{pmatrix} \rightarrow \begin{pmatrix} t'^j \\ x'^j \end{pmatrix} &= \begin{pmatrix} \cos(2\pi/5) & \pm \sin(2\pi/5) \\ \mp \sin(2\pi/5) & \cos(2\pi/5) \end{pmatrix} \begin{pmatrix} t^j \\ x^j \end{pmatrix} = \frac{1}{4} \begin{pmatrix} (-1+\sqrt{5}) & \pm \sqrt{10+2\sqrt{5}} \\ \mp \sqrt{10+2\sqrt{5}} & (-1+\sqrt{5}) \end{pmatrix} \begin{pmatrix} t^j \\ x^j \end{pmatrix} \\ &= \begin{pmatrix} .3090 & \pm .9511 \\ \mp .9511 & .3090 \end{pmatrix} \begin{pmatrix} t^j \\ x^j \end{pmatrix} = \begin{pmatrix} .3090t^j \pm .9511x^j \\ .3090x^j \mp .9511t^j \end{pmatrix} \end{aligned} \quad (13.5)$$

To make the nature of this discrete (quantized) Euclidean space and time transformation for the fifth root  $m=5$  very clear, from (13.4) for  $n = \pm 1$  we may extract:

$$\begin{aligned} t'^j &= -.8090t^j \pm .5878x^j \\ x'^j &= -.8090x^j \mp .5878t^j \end{aligned} \quad (13.6)$$

while from (13.5) for  $n = \pm 2$  we extract:



$$\begin{aligned} t'^j &= .3090t^j \pm .9511x^j \\ x'^j &= .3090x^j \mp .9511t^j \end{aligned} \quad (13.7)$$

We see how these look like Lorentz transformations, but as seen in section 9, these are *Euclidean* rotations between space and time which preserve the invariance of the Minkowski interval  $t^2 - r^2$  in flat spacetime and are generally characterized by the extended metric equation  $ds^2 = g_{\mu\nu}dx^\mu dx^\nu = g_{MN}d\chi^M d\chi^N$  of (9.16) which preserves  $ds^2 = dt^2 - dr^2$  on any geodesic tangent space with  $g_{MN} \rightarrow \eta_{MN}$  and  $g_{\mu\nu} \rightarrow \eta_{\mu\nu}$ .

As with (13.1), any closed stationary de Broglie wave with  $m=5$  cycles over  $0 \leq \varphi \leq 2\pi$  and higher  $5 \cdot k$  harmonics will remain rotationally-invariant under (13.2). To preserve Lorentz symmetry  $k$  must be an odd integer, and of course these waves will simply be the fundamental harmonic of the non-prime root of unity  $m=5 \cdot k$ , which will necessarily include non-primitive roots beyond the number 1 itself. Therefore, any closed wave closed wave with  $m=5$  cycles as its fundamental (first) harmonic will be rotationally invariant under (13.2), because the three-cycle closed wave rotated by some multiple of  $72^\circ$  will be rotationally indistinguishable from the original wave before the transformation rotation. However, the time and space coordinates ascribed to this closed wave will not be invariant under (13.2), but rather will transformed according to (13.3) which leads to the specific, discrete Euclidean space and time mixing of (13.4) to (13.7). This means two things: First, of the closed de Broglie wave has a radius  $r = \sqrt{x^2 + y^2 + z^2}$  before the transformation (13.2) is applied, it will have a different radius  $r' = \sqrt{x'^2 + y'^2 + z'^2}$  following transformation. Second, if the time  $t = 1/f$  associated with the vibrational frequency is  $t = \sqrt{t_x^2 + t_y^2 + t_z^2}$  before transformation, then following transformation it will become  $t' = \sqrt{t_x'^2 + t_y'^2 + t_z'^2}$ . Because each of  $x^j = (x, y, z)^T$  and  $t^j = (t_x, t_y, t_z)^T$  transforms in an identical manner under (13.3) for the fifth roots of unity  $m=5$ , this means (13.6) for  $n = \pm 1$  and (13.7) for  $n = \pm 2$  respectively, may be written directly in terms of  $r$  and  $t$  as:

$$\begin{aligned} t' &= -.8090t \pm .5878r \\ r' &= -.8090r \mp .5878t \end{aligned} \quad (13.8)$$

and:

$$\begin{aligned} t' &= .3090t \pm .9511r \\ r' &= .3090r \mp .9511t \end{aligned} \quad (13.9)$$

Now, restoring natural constants  $\hbar$  and  $c$ , let us the Bohr radius  $a_0 = \hbar / m_e c \alpha$  as a reference length against which to specify the radius  $r$  of the closed de Broglie wave where  $m_e$  is the electron rest mass and  $\alpha = e^2 / 4\pi\hbar c$  is the running electromagnetic coupling which approaches  $1/137.036\dots$  at low probe energies. Let us then posit that the closed de Broglie wave with  $m=5$  cycles over  $0 \leq \varphi \leq 2\pi$ , prior to the transformation (13.2), has a radius

$r = \rho a_0 = \rho \hbar / m_e c \alpha$  where  $\rho$  is some dimensionless factor for which the Bohr radius used as a reference length. That is,  $r$  is  $\rho$  tme the Bohr radius. The circumference of this closed wave is then  $C = 2\pi r = \rho 2\pi a_0 = \rho 2\pi \hbar / m_e c \alpha$ , keeping in mind also that  $h = 2\pi \hbar$ . Because there are  $m=5$  cycles fitted into the circumference, the wavelength of this de Broglie wave will be:

$$\lambda = C / 5 = 2\pi r / 5 = \rho 2\pi a_0 / 5 = \rho 2\pi \hbar / 5 m_e c \alpha = \rho h / 5 m_e c \alpha. \quad (13.10)$$

The momentum using de Broglie's wave-particle duality formula is then  $p = h / \lambda$ , so that at the radial distance  $r = \rho a_0 = \rho \hbar / m_e c \alpha$ , the angular momentum  $pr$  will be:

$$pr = \frac{hr}{\lambda} = h \frac{\rho \hbar}{m_e c \alpha} \frac{5 m_e c \alpha}{\rho 2\pi \hbar} = 5 \frac{h}{2\pi} = 5\hbar. \quad (13.11)$$

$$p = h / \lambda \rightarrow p = h / 2\lambda$$

$$p = mv$$

$$E = \frac{1}{2}mv^2 = \frac{p^2}{2m}$$

This leads to several conclusions. First, the angular momentum of a de Broglie wave with a fixed number of cycles per  $0 \leq \varphi \leq 2\pi$  is invariant with respect to the radius  $r$ . If the radius is increased by a factor  $\rho > 1$ , we wavelength will diminish by the same factor, for a net cancellation as seen in (13.11). Second, the root of unity which in this case is  $m=5$  is in fact synonymous with the angular momentum of this de Broglie wave, because  $\hbar=1$  is the elementary, quantized unit of angular momentum in natural units. Likewise, it is clear that for any of the other odd roots of unity  $m = 1, 3, 5, 7, \dots$  permitted under Lorentz symmetry as reviewed in section 11, the result will be exactly the same, that is, in general, for a de Broglie wave which is rotationally-invariant under the roots of unity transformation (8.17) with  $\vartheta = 2\pi n / m$  for a given  $m$ , the angular momentum of that de Broglie wave, in natural units  $\hbar = 1$ , will be:

$$pr = m = 1, 3, 5, 7, \dots \quad (13.12)$$

Therefore it does indeed make sense to associate roots of unity with angular momentum via  $j = m / 2$  as was found in (12.16), and if we combine (12.16) with (13.2) we find that:

$$j = l + \frac{1}{2} = l + s = \frac{m}{2} = \frac{pr}{2} = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \dots \quad (13.13)$$

The above associates the total angular momentum Casimir number  $j$  with  $pr / 2$  of the de Broglie wave, including the factor  $1/2$  which is required to match up these two results. This factor is a direct indication of the fact that fermions, e.g., electrons, are observed with spins which have

half a unit of  $\hbar$ , but that orbital angular momenta come in whole units of  $\hbar$ . This is most readily seen if we rewrite (13.13) as

$$pr = 2l + 1 = 2l + 2s = 2j. \quad (13.14)$$

With  $l=0$ , this becomes  $pr = 1 = 2s$ . Were we to have set  $pr = m = j = l + s = 1, 3, 5, 7, \dots$  without the factor of  $\frac{1}{2}$ , then for  $l=0$  we would have  $s=1$ , which is not empirically correct, and for non-zero  $l$  we would have had  $j = 3, 5, 7, \dots$  which is also not empirically correct. The relationship (13.13) achieves three correct results: First, the smallest permitted total angular momentum is  $\frac{1}{2}\hbar$ . Second, other permitted angular momenta add whole units of  $\hbar$  to this half unit of angular momentum. Third, the factor of 2 compensates for that fact that  $m = 1, 3, 5, 7, \dots$  must be an odd integer to maintain Lorentz symmetry, and this ensures that the angular momentum increments above  $\frac{1}{2}\hbar$  come in units of  $\hbar$ , rather than units of  $2\hbar$ . In short, (13.13) comports closed de Broglie waves to angular momenta actually observed empirically.

Thus, in answer to the question earlier posed, we have shown that it does make sense on theoretical and physical grounds to directly connect the fractional denominators  $m$ , which originate from the  $m^{\text{th}}$  roots of unity in (6.1), with the total angular momentum via  $m = 2j = 2(l + s)$  found in (12.16). And so, we conclude that *applying a root of unity transformation is indeed synonymous with transforming that electron into a different state of orbital angular momentum*. Now the root of unity transformation (6.12) is not just some abstract mathematical operation; it is *the symmetry operation* which alters the orbital angular momentum of electrons.

## References

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- [1] Dirac, P.A.M., *Quantized Singularities in the Electromagnetic Field*, Proceedings of the Royal Society A 133 (821): 60–72, (September 1931)
  - [2] Wu, T. T. and Yang, C.N., *Concept of non-integrable phase factors and global formulation of gauge fields*, Phys. Rev. D 12 (1975) 3845
  - [3] Wu, T. T. and Yang, C. N., *Dirac Monopole without Strings: Classical Lagrangian theory*, Phys. Rev. D 14 (1976) 437
  - [4] Misner, C. W., Thorne, K. S., and Wheeler, J. A., *Gravitation*, Freeman (1973)
  - [5] Laughlin, R. B. Anomalous Quantum Hall Effect: An Incompressible Quantum Fluid with Fractionally Charged Excitations, *Physical Review Letters* 50 18 (1983)
  - [6] Ryder, L. H., *Quantum Field Theory*, second edition, Cambridge (1996)
  - [7] Minkowski, H., *Space and Time*, Jahresbericht der Deutschen Mathematiker-Vereinigung, 75–88 (1909)
  - [8] [https://en.wikipedia.org/wiki/Matrix\\_mechanics](https://en.wikipedia.org/wiki/Matrix_mechanics)
  - [9] [http://www.nobelprize.org/nobel\\_prizes/physics/laureates/1998/press.html](http://www.nobelprize.org/nobel_prizes/physics/laureates/1998/press.html)
  - [10] 't Hooft, G. (1974). "Magnetic monopoles in unified gauge theories". *Nuclear Physics B* **79** (2): 276–284
  - [11] A.M. Polyakov, Zh. Eksp. Teor. Fiz. Pis'ma. Red. 20, 430, JETP Lett. 20, 194 (1974)
  - [12] Weinberg, S., *The Quantum Theory of Fields, Volume II, Modern Applications*, Cambridge (1996)
  - [13] Witten, E., *Duality, Spacetime and Quantum Mechanics*, Physics Today, 28 to 33 (1997)