We develop a linear element $ds$ in ordinary spacetime which, when held stationary under worldline variations, leads to the gravitational equations of motion which are extended to include the Lorentz force law. We see that in the presence of an electromagnetic vector potential $A^\mu$, the differential coordinate elements $dx^\mu$ are dilated with a term that depends on the strength of the gauge field, the charge $e$ and mass $m$ of the particle placed in that potential, and the linear element $ds$ itself. This same linear element also leads directly to the canonical form of Dirac’s equation.

Contents

1. Introduction ........................................................................................................... 1
2. Variational derivation of the Lorentz force law from geodesic motion along a linear element in spacetime ...................................................................................... 1
3. Geometric understanding of the Lorentz force law in terms of simple coordinate transformations involving the electromagnetic vector potential ........................................... 5
4. Dirac’s equation in canonical form ........................................................................ 9
5. Conclusion ............................................................................................................. 10
1. Introduction

In this paper we shall show how a spacetime metric equation of the form

\[ ds^2 = g_{\mu\nu}dx^\mu dx^\nu + 2 \frac{e}{m} A_\sigma dx^\sigma ds , \] (1.1)

with the linear element \( ds \) held stationary along a line between two points \( A \) and \( B \) such that:

\[ 0 = \delta \int_A^B ds , \] (1.2)

will yield an equation of motion along geodesic lines given by:

\[ \frac{d^2 x^\mu}{ds^2} = -\Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} + \frac{e}{m} F^\mu_{\alpha} \frac{dx^\alpha}{ds} , \] (1.3)

which naturally contains the Lorentz force law for a particle of charge \( e \) and mass \( m \). In (1.1), \( A = A_\sigma dx^\sigma \) is the electromagnetic vector potential one-form, and in (1.3) \( \Gamma^\nu_{\alpha\beta} \) is the Christoffel connection and \( F^\nu_{\alpha} \) is the mixed electromagnetic field strength tensor. It will be observed that (1.1) itself is a binomial in \( ds \), which means that the linear element may be directly expressed as:

\[ ds = \frac{e}{m} A_\sigma dx^\sigma + \sqrt{\left( \frac{e}{m} A_\sigma dx^\sigma \right)^2 + g_{\mu\nu}dx^\mu dx^\nu} . \] (1.4)

In general, we shall use a metric tensor for which the tangent flat spacetime of Minkowski is \( \eta_{\mu\nu} = \text{diag}(+1,-1,-1,-1) \).

2. Variational derivation of the Lorentz force law from geodesic motion along a linear element in spacetime

To demonstrate that (1.3) is in fact the geodesic equation of motion for (1.1), we first rewrite (1.1) in the form:

\[ ds^2 = -\frac{e}{m} \frac{dx^\sigma}{ds} \frac{dx^\sigma}{ds} + 2 \frac{e}{m} A_\sigma dx^\sigma ds . \] (2.1)

We may then use this “1” to write the variation in (1.2) as:

\[ 0 = \delta \int_A^B ds = \delta \int_A^B ds \sqrt{g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} + 2 \frac{e}{m} A_\sigma \frac{dx^\sigma}{ds}} . \] (2.2)
Now, all that remains is to calculate from (2.2) in detail to show that this yields the geodesic equation of motion (1.3).

It is easily seen that when the variation operates on the integrand in (2.2), and then again using the “1” of (2.1), we obtain:

\[
0 = \int_{A}^{B} ds \delta \left( \frac{dx^{\mu}}{ds} \cdot \frac{dx^{\nu}}{ds} + 2 \frac{e}{m} A^{\delta}_{\sigma} \frac{dx^{\sigma}}{ds} \right) = \frac{1}{2} \int_{A}^{B} ds \delta \left( \frac{dx^{\mu}}{ds} \cdot \frac{dx^{\nu}}{ds} + 2 \frac{e}{m} A^{\delta}_{\sigma} \frac{dx^{\sigma}}{ds} \right). \tag{2.3}
\]

We may remove the overall coefficient of \(\frac{1}{2}\) and use the product rule to apply this variation, thus:

\[
0 = \int_{A}^{B} ds \left( \delta g^{\mu\nu} \frac{dx^{\mu}}{ds} \frac{dx^{\nu}}{ds} + g^{\mu\nu} \frac{d \delta x^{\mu}}{ds} \frac{dx^{\nu}}{ds} + g^{\mu\nu} \frac{dx^{\mu}}{ds} \frac{d \delta x^{\nu}}{ds} + 2 \frac{e}{m} A^{\delta}_{\sigma} \frac{dx^{\sigma}}{ds} + 2 \frac{e}{m} A^{\delta}_{\sigma} \frac{d \delta x^{\sigma}}{ds} \right), \tag{2.4}
\]

where we assume there no variation in the charge-to-mass ratio, i.e., that \(\delta (e/m) = 0\), over the path from \(A\) to \(B\). As to the variations \(\delta g_{\mu\nu}\) and \(\delta A_{\nu}\), it is easily shown in the small variation \(\delta \to 0\) limit, using the chain rule, that:

\[
\delta g_{\mu\nu} = \delta \frac{\partial x^{\alpha}}{\partial x^{\mu}} g_{\mu\nu} = \frac{\partial \delta x^{\alpha}}{\partial x^{\mu}} g_{\mu\nu} = \frac{\partial}{\partial x^{\alpha}} g_{\mu\nu} \delta x^{\alpha}, \tag{2.5}
\]

\[
\delta A_{\sigma} = \delta \frac{\partial x^{\alpha}}{\partial x^{\sigma}} A_{\sigma} = \frac{\partial \delta x^{\alpha}}{\partial x^{\sigma}} A_{\sigma} = \frac{\partial}{\partial x^{\alpha}} A_{\sigma} \delta x^{\alpha} = \partial_{\alpha} A_{\sigma} \delta x^{\alpha}. \tag{2.6}
\]

So, using (2.5) and (2.6) in (2.4), and also using a renaming of indexes and the symmetry of \(g_{\mu\nu}\) to combine the second and third terms in (2.4), we next obtain:

\[
0 = \int_{A}^{B} ds \left( \delta x^{\alpha} \partial_{\alpha} g_{\mu\nu} \frac{dx^{\mu}}{ds} \frac{dx^{\nu}}{ds} + 2 g_{\mu\nu} \frac{d \delta x^{\mu}}{ds} \frac{dx^{\nu}}{ds} + 2 \delta x^{\alpha} \frac{e}{m} \partial_{\alpha} A_{\sigma} \frac{dx^{\sigma}}{ds} + 2 \frac{e}{m} A_{\sigma} \frac{d \delta x^{\sigma}}{ds} \right). \tag{2.7}
\]

At this point, let us integrate by parts for the second and fourth terms above. The first step is to use the product rule to write:

\[
\frac{d}{ds} \left( \delta x^{\alpha} g_{\mu\nu} \frac{dx^{\nu}}{ds} \right) = g_{\mu\nu} \frac{d \delta x^{\mu}}{ds} \frac{dx^{\nu}}{ds} + \delta x^{\alpha} \frac{d}{ds} \left( g_{\mu\nu} \frac{dx^{\nu}}{ds} \right), \tag{2.8}
\]
\[
\frac{d}{ds}(A_\sigma \delta x^\sigma) = \frac{d}{ds} A_\sigma \delta x^\sigma + A_\sigma \frac{d \delta x^\sigma}{ds}.
\]  
(2.9)  

We may then use (2.8) and (2.9) to replace the second on fourth terms in (2.7), yielding:

\[
0 = \int_A^B ds \left\{ \delta x^\alpha \partial_\tau g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{dx^\tau} + 2 \frac{d}{ds} \left( \delta x^\alpha g_{\mu\nu} \frac{dx^\nu}{ds} \right) - 2 \delta x^\alpha \frac{d}{ds} \left( g_{\mu\nu} \frac{dx^\nu}{ds} \right) \right. \\
\left. \quad + 2 \delta x^\alpha \frac{e}{m} \partial_\tau A_\sigma \frac{dx^\sigma}{ds} + 2 \frac{e}{m} \frac{d}{ds} \left( A_\sigma \delta x^\sigma \right) - 2 \delta x^\alpha \frac{e}{m} \frac{d}{ds} A_\sigma \right\}.  
\]  
(2.10)  

The terms involving the total derivatives, namely:

\[
\int_A^B ds \frac{d}{ds} \left( \delta x^\alpha g_{\mu\nu} \frac{dx^\nu}{ds} \right) = \delta x^\alpha g_{\mu\nu} \frac{dx^\nu}{ds} \bigg|_A^B = 0,  
\]  
(2.11)  

\[
\int_A^B \frac{e}{m} ds \frac{d}{ds} \left( \delta x^\alpha A_\sigma \right) = \delta x^\alpha \frac{e}{m} A_\sigma \bigg|_A^B = 0  
\]  
(2.12)

are equal to zero at the boundaries, and so can be removed. This is because is one has two worldliness which coincide at points A and B but have a slight variational difference between A and B, then \( \delta x^\alpha (A) = \delta x^\alpha (B) = 0 \) while \( \delta x^\alpha \neq 0 \) elsewhere. Thus, (2.10) simplifies to:

\[
0 = \int_A^B ds \left\{ \delta x^\alpha \partial_\tau g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{dx^\tau} - 2 \delta x^\alpha \frac{d}{ds} \left( g_{\mu\nu} \frac{dx^\nu}{ds} \right) \right. \\
\left. \quad + 2 \delta x^\alpha \frac{e}{m} \partial_\tau A_\sigma \frac{dx^\sigma}{ds} - 2 \delta x^\alpha \frac{e}{m} \frac{d}{ds} A_\sigma \right\}.  
\]  
(2.13)  

Applying the \( d/ds \) derivative contained in the second term above then yields:

\[
0 = \int_A^B ds \left\{ \delta x^\alpha \partial_\tau g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} - 2 \delta x^\alpha \frac{d}{ds} g_{\mu\nu} \frac{dx^\nu}{ds} - 2 \delta x^\alpha g_{\mu\nu} \frac{d^2 x^\tau}{ds^2} \right. \\
\left. \quad + 2 \delta x^\alpha \frac{e}{m} \partial_\tau A_\sigma \frac{dx^\sigma}{ds} - 2 \delta x^\alpha \frac{e}{m} \frac{d}{ds} A_\sigma \right\},  
\]  
(2.14)  

for the first time revealing the acceleration term \( d^2 x^\tau / ds^2 \).

Next, let us make use of the chain rule in the forms:

\[
\frac{d}{ds} g_{\mu\nu} = \frac{dx^\alpha}{ds} \frac{\partial}{\partial x^\alpha} g_{\mu\nu} = \partial_\alpha g_{\mu\nu} \frac{dx^\alpha}{ds},  
\]  
(2.15)
\[
\frac{d}{ds} A_\sigma = \frac{dx^\sigma}{ds} \frac{\partial}{\partial x^\sigma} A_\sigma = \partial_\alpha A_\sigma \frac{dx^\alpha}{ds}
\]  \hspace{1cm} (2.16)

to rewrite the second and fifth terms in (2.14), thus obtaining:

\[
0 = \int_A^B ds \left\{ \delta x^\alpha \partial_\alpha g_{\mu \nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} - 2\delta x^\mu \partial_\alpha g_{\mu \nu} \frac{dx^\alpha}{ds} \frac{dx^\nu}{ds} - 2\delta x^\mu g_{\mu \nu} \frac{d^2 x^\nu}{ds^2} \right\}.
\]  \hspace{1cm} (2.17)

In the bottom line above, we have two terms of identical form, but with different indexes attached to the coordinates \(x\). So if we rename indexes \(\alpha \rightarrow \sigma\) in the last term, we find that the last line (times \(\frac{1}{2}\)) may be rewritten as:

\[
0 = \int_A^B ds \left\{ \delta x^\alpha \partial_\alpha A_\sigma \frac{dx^\sigma}{ds} - 2\delta x^\mu \partial_\alpha A_\sigma \frac{dx^\alpha}{ds} - \frac{e}{m} \partial_\alpha A_\sigma \frac{dx^\alpha}{ds} \right\}.
\]  \hspace{1cm} (2.18)

where the electromagnetic field strength tensor

\[
F_{\alpha \sigma} = \partial_\alpha A_\sigma - \partial_\sigma A_\alpha
\]  \hspace{1cm} (2.19)

has now appeared as a result of the variation. Therefore, we may use (2.18) to simplify (2.17) to:

\[
0 = \int_A^B ds \left\{ \delta x^\alpha \partial_\alpha g_{\mu \nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} - 2\delta x^\mu \partial_\alpha g_{\mu \nu} \frac{dx^\alpha}{ds} \frac{dx^\nu}{ds} - \frac{e}{m} F_{\alpha \sigma} \frac{dx^\sigma}{ds} \right\}.
\]  \hspace{1cm} (2.20)

Now we wish to factor out all of the \(\delta x\) terms. To do so, we rename indexes so that this term is always \(\delta x^\alpha\) with an \(\alpha\) index, and we then factor this out. The result is:

\[
0 = \int_A^B ds \delta x^\alpha \left( \partial_\alpha g_{\mu \nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} - 2\partial_\mu g_{\alpha \nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} - g_{\alpha \nu} \frac{d^2 x^\nu}{ds^2} + 2 \frac{e}{m} F_{\alpha \sigma} \frac{dx^\sigma}{ds} \right).
\]  \hspace{1cm} (2.21)

For this to be equal to zero, the integrand must be zero, that is, we must have:

\[
0 = \frac{1}{2} \partial_\alpha g_{\mu \nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} - \partial_\mu g_{\alpha \nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} - g_{\alpha \nu} \frac{d^2 x^\nu}{ds^2} + \frac{e}{m} F_{\alpha \sigma} \frac{dx^\sigma}{ds},
\]  \hspace{1cm} (2.22)

where we have also multiplied through by \(\frac{1}{2}\). All that is left is some final restructuring.
First, we move the acceleration term to the left, split the term with \( \partial_{\mu}g_{\alpha\nu} = \frac{1}{2} \partial_{\mu}g_{\alpha\nu} + \frac{1}{2} \partial_{\mu}g_{\alpha\nu} \) into two halves, and rename some indexes while using the symmetry of \( g_{\alpha\nu} \) to write:

\[
g_{\alpha\nu} \frac{d^2 x^\nu}{ds^2} = \frac{1}{2} \left( \partial_{\alpha}g_{\mu\nu} - \partial_{\mu}g_{\nu\alpha} - \partial_{\nu}g_{\alpha\mu} \right) \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} + \frac{e}{m} F_{\alpha\sigma} \frac{dx^\sigma}{ds}.
\] (2.23)

Then we multiply by \( g^{\beta\alpha} \) throughout and then raise indexes, yielding:

\[
\frac{d^2 x^\beta}{ds^2} = \frac{1}{2} g^{\beta\alpha} \left( \partial_{\alpha}g_{\mu\nu} - \partial_{\mu}g_{\nu\alpha} - \partial_{\nu}g_{\alpha\mu} \right) \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} + \frac{e}{m} F_{\beta\sigma} \frac{dx^\sigma}{ds}.
\] (2.24)

But of course, we recognize that:

\[
-\Gamma^\beta_{\mu\nu} = \frac{1}{2} g^{\beta\alpha} \left( \partial_{\alpha}g_{\mu\nu} - \partial_{\mu}g_{\nu\alpha} - \partial_{\nu}g_{\alpha\mu} \right).
\] (2.25)

As a consequence that (2.24) reduces to:

\[
\frac{d^2 x^\beta}{ds^2} = -\Gamma^\beta_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} + \frac{e}{m} F_{\beta\sigma} \frac{dx^\sigma}{ds}.
\] (2.26)

This is indeed equation (1.3) following some simple index renaming, and it does in fact contain the Lorentz force law as a consequence of the geodesic line motion specified by the variation (1.2) using the linear element (1.1).

Consequently, we have proved that charged particles moving according to the Lorentz force law are in fact simply following geodesic paths in spacetime, for a squared linear element given by (1.1), or, via the binomial theorem, by the linear element (1.4).

3. Geometric understanding of the Lorentz force law in terms of simple coordinate transformations involving the electromagnetic vector potential

Knowing that (1.1) will yield the Lorentz force upon application of the variation \( 0 = \delta \int ds \) of (1.2), let us see if there is some way to interpret this strictly in terms of the four-dimensional spacetime geometry without resort to any additional dimensions. We start by using \( A_{\sigma}dx^\sigma = g_{\mu\nu}A^\mu dx^\nu \) to rewrite (1.1) as:

\[
ds^2 = g_{\mu\nu}dx^\mu dx^\nu + 2g_{\mu\nu} \frac{e}{m} A^\mu dx^\nu ds = g_{\mu\nu} \left( dx^\mu + 2ds \frac{e}{m} A^\mu \right) dx^\nu = g_{\mu\nu}dx^\mu dx^\nu.
\] (3.1)
So we see that we may think of (1.1) in terms of the ordinary metric equation, in the situation where the presence of an electromagnetic potential has re-scaled the coordinate element $dx^\mu$ according to:

$$dx^\mu \rightarrow dx'^\mu = dx^\mu + 2ds \frac{e}{m} A^\mu.$$  \hspace{1cm} (3.2)

In order to explicitly display the symmetry of (3.1) under $\mu \leftrightarrow \nu$ index interchange, we may split (3.1) into two halves, and use index renaming together with the symmetry of $g_{\mu\nu}$ to rewrite (3.1), also using (3.2), as:

$$ds^2 = \frac{1}{2} g_{\mu\nu} dx^\mu dx^\nu + g_{\mu\nu} ds \frac{e}{m} A^\mu dx^\nu + \frac{1}{2} g_{\mu\nu} dx^\mu dx^\nu + g_{\mu\nu} dx^\mu ds \frac{e}{m} A^\nu \equiv \frac{1}{2} \left( dx^\mu dx^\nu + dx^\nu dx^\mu \right).$$

And this, in turn, raises the question whether the $\frac{1}{2} \left( dx^\mu dx^\nu + dx^\nu dx^\mu \right)$ can instead be expressed in terms of some further rescaled $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$, so that we do not rescale one or the other $dx^\mu$ and $dx^\nu$ but not both in the presence of a potential. Rather, since there is nothing physical to distinguish $dx^\mu$ from $dx^\nu$, it seems that a rescaling of one should necessarily rescale the other, which is to say, if $dx^\mu \rightarrow dx'^\mu$ is rescales in some way, then likewise, $dx^\nu \rightarrow dx'^\nu$ should similarly rescale.

With this in mind, let us now rewrite (3.2) with the factor of 2 removed from the second term, as:

$$dx^\mu \rightarrow dx'^\mu = dx^\mu + ds \frac{e}{m} A^\mu.$$  \hspace{1cm} (3.4)

It is easily seen using (3.4) that:

$$g_{\mu\nu} dx^\mu dx^\nu = g_{\mu\nu} \left( dx^\mu + ds \frac{e}{m} A^\mu \right) \left( dx^\nu + ds \frac{e}{m} A^\nu \right)$$

$$= g_{\mu\nu} dx^\mu dx^\nu + g_{\mu\nu} ds \frac{e}{m} A^\mu dx^\nu + g_{\mu\nu} dx^\mu ds \frac{e}{m} A^\nu + g_{\mu\nu} ds A^\mu ds \frac{e}{m} A^\nu.$$  \hspace{1cm} (3.5)

This is in fact the same as the metric equation (1.1), but for the extra term involving the term $g_{\mu\nu} A^\mu A^\nu = A_\sigma A^\sigma$ which is of second order in the gauge potential. Therefore, we ascertain that (1.1) may be rewritten as:
Jay R. Yablon

\[ \begin{align*}
\text{In consolidated form, this is:} \\
\text{(3.7)} \\
\text{This is reminiscent of, but not quite the same as, the metric elements used in Kaluza-Klein theory, namely,} \\
ds^2 \equiv \bar{g}_{ab} \, dx^a \, dx^b = g_{\mu \nu} \, dx^\mu \, dx^\nu + \phi^5 \left( A_\mu \, dx^\mu + dx^5 \right)^2, \quad \text{where} \quad \bar{g}^{55} = \phi^2 \\
\text{and thus (1.1) as is, and which stays in four dimension only. So if we wish to represent the Lorentz force as a geodesic form of motion in only four dimensions with a metric relationship} \\
ds^2 = g_{\mu \nu} \, dx^\mu \, dx^\nu, \quad \text{need to consider the possibility that (1.1) is only an approximation to the geodesic motion in the special case of electromagnetic fields in the presence of a weak gravitational field. More specifically, because we know that the first two terms in the last line of (3.5) will reproduce the Lorentz force law as part of the geodesic equation (2.26), let us now adopt (3.5) as the invariant length element \( ds^2 \) in the presence of gravitational fields which are strong, or weak, or vanishing entirely, such that in the presence of an electromagnetic potential, the spacetime coordinates themselves are indeed rescaled according to (3.4).} \\
\text{This has another benefit as well: by choosing (1.3) a.k.a. (2.26) as the target equation of motion, we are seeking a metric length element which superimposes the gravitational equation of motion with the Lorentz equation of motion, which in turn assumes that there is no electro-gravitational interaction term in the equation of motion. But suppose that in fact there should be such an interaction term that makes its presence known when the gravitational field is strong? By adopting (3.5) as our length element \( ds^2 \), the equation of motion will naturally acquire an additional term stemming from the variation of} \\
\left( e / m \right)^2 g_{\mu \nu} A^\mu A^\nu ds^2. \quad \text{If this additional term vanishes for weak gravitational fields, then we will still have a result that is consistent with observation when electromagnetic interactions are studied in weak gravitational fields, but which predict something new in the presence of strong gravitational field. Let us see how this plays out.} \\
\text{First, we adopt (3.5) as our new invariant metric line element, that is, we now start with:} \\
\end{align*} \]
\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{\mu\nu} \left( dx^\mu + ds \frac{e}{m} A^\mu \right) \left( dx^\nu + ds \frac{e}{m} A^\nu \right) . \] (3.8)

Then, we construct a new “1” according to (contrast (2.1):

\[
1 = \sqrt{g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} + 2 \frac{e}{m} A_\sigma dx^\sigma + \left( \frac{e}{m} \right)^2 g_{\mu\nu} A^\mu A^\nu} . \] (3.9)

And we then use this to write the variation (1.2) as (contrast (2.2)):

\[
0 = \delta \int_A^B ds \sqrt{g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} + 2 \frac{e}{m} A_\sigma dx^\sigma + \left( \frac{e}{m} \right)^2 g_{\mu\nu} A^\mu A^\nu} . \] (3.10)

Now, all we need to do is calculate, and we can make use of many of the results already obtained so as to focus on the new term \((e/m)^2 g_{\mu\nu} A^\mu A^\nu\).

First, as in (2.3), we apply the variation in (3.10) to the integrand, and further apply the “1” in (3.9), to obtain:

\[
0 = \delta \int_A^B ds \frac{\delta g_{\mu\nu} \frac{dx^\mu}{ds} dx^\nu + 2 \frac{e}{m} A_\sigma dx^\sigma + \left( \frac{e}{m} \right)^2 g_{\mu\nu} A^\mu A^\nu}{2 \sqrt{g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} + 2 \frac{e}{m} A_\sigma dx^\sigma + \left( \frac{e}{m} \right)^2 g_{\mu\nu} A^\mu A^\nu}} . \] (3.11)

Dropping the \(1/2\) and using the product rule we now have:

\[
0 = \int_A^B ds \left( \delta g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} + \frac{e}{m} A_\sigma \frac{dx^\sigma}{ds} + \left( \frac{e}{m} \right)^2 g_{\mu\nu} A^\mu A^\nu \right) . \] (3.12)

The top line is identical to that in (2.4), and we know that these will lead to the usual equation of motion (2.26). The bottom line contains all the new terms. Now, let’s make short work of this calculation by seeing what effect the bottom terms have on the overall equation of motion.

First, using (2.5) and (2.6), we may rewrite the bottom line of the above, to obtain:
0 = \int_\alpha^\beta ds \left( \delta g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} + g_{\mu\nu} \frac{d\delta x^\mu}{ds} \frac{dx^\nu}{ds} + g_{\mu\nu} \frac{dx^\mu}{ds} \frac{d\delta x^\nu}{ds} + 2 \frac{e}{m} \delta A_\sigma \frac{dx^\sigma}{ds} + 2 \frac{e}{m} A_\sigma \frac{d\delta x^\sigma}{ds} \right) + \delta x^\alpha \left( \frac{e}{m} \right)^2 \left( \partial_\alpha g_{\mu\nu} A^\mu A^\nu + g_{\mu\nu} \partial_\alpha A^\mu A^\nu + g_{\mu\nu} A^\mu \partial_\alpha A^\nu \right). \tag{3.13}

But we can use the product rule

\partial_\alpha \left( g_{\mu\nu} A^\mu A^\nu \right) = \partial_\alpha g_{\mu\nu} A^\mu A^\nu + g_{\mu\nu} \partial_\alpha A^\mu A^\nu + g_{\mu\nu} A^\mu \partial_\alpha A^\nu. \tag{3.14}

to re-condense the bottom line into:

0 = \int_\alpha^\beta ds \left( \delta g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} + g_{\mu\nu} \frac{d\delta x^\mu}{ds} \frac{dx^\nu}{ds} + g_{\mu\nu} \frac{dx^\mu}{ds} \frac{d\delta x^\nu}{ds} + 2 \frac{e}{m} \delta A_\sigma \frac{dx^\sigma}{ds} + 2 \frac{e}{m} A_\sigma \frac{d\delta x^\sigma}{ds} \right) + \delta x^\alpha \left( \frac{e}{m} \right)^2 \partial_\alpha \left( g_{\mu\nu} A^\mu A^\nu \right). \tag{3.15}

This new term is also a total derivative, just like the ones seen earlier in (2.11) and (2.12). Here, this new term integrates as:

\int_\alpha^\beta \delta x^\alpha \left( \frac{e}{m} \right)^2 \frac{\partial}{\partial x^\alpha} \left( g_{\mu\nu} A^\mu A^\nu \right) ds = \delta x^\alpha \left( \frac{e}{m} \right)^2 \left. \partial_s \left( g_{\mu\nu} A^\mu A^\nu \right) \right|_\alpha^\beta = 0, \tag{3.16}

again because \( \delta x^\alpha \left( A \right) = \delta x^\alpha \left( B \right) = 0 \) at the boundary ends of the worldlines being varied. Consequently, the term on the bottom line drops out, and (3.15) simplifies to:

0 = \int_\alpha^\beta ds \left( \delta g_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} + g_{\mu\nu} \frac{d\delta x^\mu}{ds} \frac{dx^\nu}{ds} + g_{\mu\nu} \frac{dx^\mu}{ds} \frac{d\delta x^\nu}{ds} + 2 \frac{e}{m} \delta A_\sigma \frac{dx^\sigma}{ds} + 2 \frac{e}{m} A_\sigma \frac{d\delta x^\sigma}{ds} \right). \tag{3.15}

This is identical to (2.4), which means that it will yield the exact same equation of motion:

\frac{d^2 x^\beta}{ds^2} = -\Gamma^\beta_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} + \frac{e}{m} F^\beta_{\sigma} \frac{dx^\sigma}{ds} \tag{3.16}

including the Lorentz force law, that was found from (2.4), in (2.26). As a result, we see that (3.8) will also yield the Lorentz force law together with the law for gravitational geodesic motion, just as does (1.1).

4. Dirac’s equation in canonical form

Although electrodynamics may not be typically described using the metric relationship (3.8), we can readily see how (3.8) leads to other familiar results in addition to the Lorentz force law.
and gravitational geodesics. Specifically, it is simple to rewrite the top line of (3.8) as \( l^2 = 1 \), then multiply through by \( m^2 \), to obtain:

\[
m^2 = g_{\mu\nu} \left( m \frac{dx^\mu}{ds} + eA^\mu \right) \left( m \frac{dx^\nu}{ds} + eA^\nu \right) = g_{\mu\nu} \left( p^\mu + eA^\mu \right) \left( p^\nu + eA^\nu \right) = g_{\mu\nu} \pi^\mu \pi^\nu = \pi_\mu \pi^\mu.
\]

(4.1)

Specifically, with the particle kinetic momentum \( p^\mu = m \frac{dx^\mu}{ds} \) and the canonical momentum \( \pi^\mu = p^\mu + eA^\mu \), we see that (3.8) is just the metric form of \( m^2 = p_\sigma \pi^\sigma \) canonically extended to \( m^2 = \pi_\sigma \pi^\sigma \) through the usual quantum mechanical substitution \( p^\mu \to \pi^\mu = p^\mu + eA^\mu \). As this is the momentum space version of the gauge-covariant derivative \( D^\mu = \partial^\mu + i eA^\mu \) in configuration space, and using the Dirac matrices \( \{ \gamma^\mu, \gamma^\nu \} = \eta^{\mu\nu} \) in flat spacetime, it should be readily apparent that in first order, this will yield the Dirac equation:

\[
(i \gamma^\nu D_\mu - m)\psi = 0
\]

(4.1)

for an electron wavefunction in an electromagnetic potential. As a result, (3.8) not only yields the Lorentz force, but it also yields the canonical form of Dirac’s equation.

5. Conclusion

Apparently, the observed physics of moving particles does not care whether we start with (1.1) or (3.8): the observational result are unchanged. Although we used the linear element (3.8) in lieu of (1.1), the equation of motion remained unchanged, and is still (2.26) a.k.a. (3.16) a.k.a. (1.3). Under variation, the added term with \( \left( \frac{e}{m} \right)^2 g_{\mu\nu} A^\mu A^\nu ds^2 \) has no effect whatsoever on the particle motion.

However, very importantly, (3.8), which again is:

\[
ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{\mu\nu} \left( dx^\mu + \frac{e}{m} A^\mu \right) \left( dx^\nu + \frac{e}{m} A^\nu \right),
\]

(5.1)

provides a geometric understanding of electrodynamics, not on the basis of any change whatsoever to the metric tensor or of the addition of a fifth dimension. Rather, just as space and time coordinates transform generally according to \( dx^\mu \to dx^\nu = \left( \frac{\partial x^\nu}{\partial x^\mu} \right) dx^\nu \) including the special case of relativistic motion in flat spacetime for which \( a^\mu = \frac{\partial x^\mu}{\partial x^\nu} \partial x^\nu \) represents a Lorentz transformation, we see that in the presence of an electromagnetic potential, the coordinates likewise undergo a physically-observable transformation / dilation of the form (3.4), namely:

\[
dx^\mu \to dx'^\mu = dx^\mu + ds \frac{e}{m} A^\mu.
\]

(5.2)
When varied according to $0 = \delta \int_A^B ds$ as between any two points $A$ and $B$ at which the worldlines meet so that clocks and measuring rods can be coordinated at the outset and then compared at the conclusion, the geodesics of motion will be those of gravitation in curved spacetime together with those of the Lorentz force, that is, the equation of motion will be (3.16). And, when (5.1) is manipulated as in (4.1), it also yields Dirac equation for an electron in an electromagnetic field.

Because the metric length (5.1) under variation simultaneously provides a geodesic description of motion in a gravitational field and in an electromagnetic field, and also yields the canonical form of Dirac’s equation, this may fairly be regarded as a unification of classical electrodynamics with classical gravitation, using four spacetime dimensions only.